Last time

Let K be a field of char 0.

<u>Defn</u> An ell. curve is a proj. variety in $\mathbb{P}^2_{\overline{K}}$ given by

$$y^2 = 4x^3 + ax + b, \qquad a^3 + 27b^2 \neq 0.$$

It is <u>defined over</u> K if $a, b \in K$.

Two structures on ell. curves:

- If $\underline{E/K}$, then G_K acts on E
  "E is defined over K"
- Group law on E, $E(K) \subseteq E$ is a subgroup
  - If $P, Q, R \in E$ are colinear then $P + Q + R = 0$.

Relating complex tori + elliptic curves

Recall the Weierstrass $\wp$-function associated to $\Lambda = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} \subseteq \mathbb{C}$

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

<u>Thm</u> (see CJ's lecture). Let $\wp = \wp_\Lambda$.

(1) $\wp$ is meromorphic and Λ-periodic. Hence, it gives a holo. map of RS's $\wp : \mathbb{C}/\Lambda \longrightarrow \mathbb{P}^1$.

(2) $\wp$'s only pole is a double pole at $0 \in \mathbb{C}/\Lambda$
$\wp'$'s only zeros are simple 0's at the nontrivial 2 torsion pts of $\mathbb{C}/\Lambda$: $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}$.

(3) $\wp$ and $\wp'$ satisfy

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$$

where for $k>2$

$$G_k(\Lambda) = \sum_{w\in\Lambda\setminus\{0\}} \frac{1}{w^k}$$

is the Eisenstein series of wt k, and

$g_2(\Lambda) = 60\,G_4(\Lambda)$ $\quad g_3(\Lambda) = 140\,G_6(\Lambda)$. Also $g_2(\Lambda)^3 - 27g_3(\Lambda)^2 \neq 0$.

(4) Let $E = E_\Lambda$ be the elliptic curve $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$.

The map

$$\Phi : \mathbb{C}/\Lambda \longrightarrow E$$
$$z \longmapsto [\wp(z) : \wp'(z) : 1]$$

is a bijection. So $\mathbb{C}/\Lambda \cong E$ as RS's.

<u>Thm</u> (Uniformization thm) For every $a, b \in \mathbb{C}$ s.t. $a^3 + 27b^2 \neq 0$, $\exists\, \Lambda \subseteq \mathbb{C}$ s.t.

$$g_2(\Lambda) = a \quad g_3(\Lambda) = b.$$

<u>Pf.</u> Later in the course.

<u>Prop.</u> If $z_1, z_2, z_3 \in \mathbb{C}/\Lambda$ with $z_1 + z_2 + z_3 = 0$ then $\Phi(z_1), \Phi(z_2), \Phi(z_3) \in E$ are colinear.

<u>Cor</u> The proposed group law on E is a group law, and $\mathbb{C}/\Lambda \cong E$ as groups.

To prove the prop, need to talk about divisors.

Divisors

Let X be a compact Riemann surface. Write $\mathbb{C}(X) = \{f : X \longrightarrow \mathbb{P}^1\}$ for the field of meromorphic functions on X.

Defn

(1) A <u>divisor</u> on X is a formal integer linear combination of pts $z \in X$:

$$D = \sum_{z \in X} n_z (z) \qquad n_z \in \mathbb{Z},\ n_z = 0 \text{ for almost all } z \in X.$$

The divisors form a group $\underline{\mathrm{Div}(X)}$.

(2) The degree of $D = \sum n_z (z) \in \mathrm{Div}(X)$ is $\deg D = \sum n_z$. So $\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}$ is a homomorphism.

$\mathrm{Div}^0(X) = \{ D \in \mathrm{Div}(X) : \deg D = 0 \}$.

(3) If $f : X \longrightarrow \mathbb{P}^1$ is a nonzero meromorphic fn, then $\forall x \in X$, let $\mathrm{ord}_x f$ be the order of pole or vanishing. Let $\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x f \cdot (x)$. A divisor of this form is called <u>principal</u>.

(4) Let $\mathrm{Pic}(X)$ be the quotient

$$\mathrm{Div}(X) \Big/ \{\text{principal divisors}\}.$$

[Note that $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$.].

<u>Note</u>: $\mathrm{div}(f) = 0 \iff f$ is constant. So we get an exact sequence:

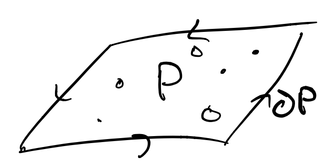
$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathbb{C}(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

<u>Prop</u> If $f \in \mathbb{C}(X)^\times$ then $\deg \mathrm{div}(f) = 0$.

<u>Pf.</u> General proof in pset 4. For $X = \mathbb{C}/\Lambda$, proof is:

Let P be fundamental parallelogram, shifted so that ∂P doesn't have zeros or poles of f.

$$\frac{1}{2\pi i}\int_{\partial P} \frac{f'(z)}{f(z)}\,dz = \sum_{z \in \mathbb{C}/\Lambda} \mathrm{ord}_z(f) = 0$$



$\square$

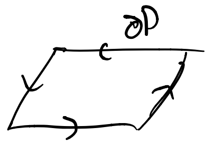
<u>Warning</u>: If $z \in \mathbb{C}/\Lambda$, $(2z) \neq 2(z)$!

<u>Prop</u>: If $f \in \mathbb{C}(\mathbb{C}/\Lambda)^\times$ with $\mathrm{div}(f) = \sum_z n_z(z)$, then $\sum_z n_z z = 0$ in $\mathbb{C}/\Lambda$.

Pf.

$$\frac{1}{2\pi i}\int_{\partial P}\frac{z f'(z)}{f(z)}\,dz = \sum_x \mathrm{res}_x\left(z\,\frac{f'(z)}{f(z)}\right)$$

$$= \sum_x x\cdot \mathrm{res}_x\left(\frac{f'}{f}\right) = \sum_x \mathrm{ord}_x(f)\, x$$



Also,

$$\frac{1}{2\pi i}\int_{\partial P}\frac{z f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\Bigg(\int_0^{\omega_1}\frac{z f'(z)}{f(z)}\,dz + \int_0^{\omega_2}\frac{(z+\omega_1) f'(z)}{f(z)}\,dz$$

$$- \int_0^{\omega_1}\frac{(z+\omega_2) f'(z)}{f(z)}\,dz - \int_0^{\omega_2}\frac{z f'(z)}{f(z)}\,dz\Bigg)$$

$$= \frac{1}{2\pi i}\left(\omega_1\int_0^{\omega_2}\frac{f'(z)}{f(z)}\,dz - \omega_2\int_0^{\omega_1}\frac{f'(z)}{f(z)}\,dz\right)$$

$\gamma(z) = f(z)$ $\qquad \in \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \in \Lambda$

$$2\pi i\mathbb{Z} \ni \int_\gamma \frac{1}{z}\,dz = \int \frac{f'(z)}{f(z)}\,dz \qquad \square$$

Prop. If $z_1, z_2, z_3 \in \mathbb{C}/\Lambda$ with $z_1+z_2+z_3=0$ then $\Phi(z_1), \Phi(z_2), \Phi(z_3) \in E_\Lambda$ are colinear.
($\Phi(z) = [\wp_\Lambda(z) : \wp'_\Lambda(z) : 1]$).

Pf. Let $aX + bY + cZ = 0$ be the line thru $\Phi(z_1), \Phi(z_2)$. Let $f(z) = a\wp(z) + b\wp'(z) + c \in \mathbb{C}(\mathbb{C}/\Lambda)^\times$.
By construction, f has 0's at z_1, z_2. The only poles

of f are a triple pole at 0. By the previous prop,

$$\mathrm{div}(f) = (z_1) + (z_2) - 3(0) + (*)$$

where $* = z_3$, and also

$* = \Phi^{-1}$(other pt on the line and E).

Thus $\Phi(z_3)$ is the other pt on the line. $\square$

Differentials

Again, X is a cmpt RS.

Defn A differential on X is a meromorphic 1-form.

Formally, this is just an elt of the vector space.

$$\Omega^1(X) = \frac{\mathrm{span}_{\mathbb{C}(X)}\{df : f \in \mathbb{C}(X)\}}{\begin{array}{l}(dc = 0 \ : \ c \in \mathbb{C}) + \\ (d(f+g) = df + dg \ : \ f,g \in \mathbb{C}(X)) + \\ (d(fg) = fdg + gdf \ : \ f,g \in \mathbb{C}(X))\end{array}}$$

If $\omega \in \Omega^1(X)$ is a diff'l, locally at each $x \in X$ we can pick a coord. nbhd of x with coord fn z and write

$$\omega = f(z)dz \quad \text{for some } f \text{ meromorphic on the coord nbhd.}$$

Ex. $X = \mathbb{P}^1$. $\omega = dz$. At $\infty \in \mathbb{P}^1$, local coord is $\frac{1}{z}$.

$$d\left(\tfrac{1}{z}z\right) = d(1) = 0$$
$$\|$$
$$\tfrac{1}{z}dz + z\,d\left(\tfrac{1}{z}\right) \implies d\left(\tfrac{1}{z}\right) = -\tfrac{1}{z^2}dz$$

$$\omega = dz = -z^2 d\left(\tfrac{1}{z}\right).$$

__Defn__ If $\omega \in \Omega^1(X)$ is a diff'l, then let for $x \in X$
$$\mathrm{ord}_x(\omega) = \mathrm{ord}_x(f) \quad \text{where} \quad \omega = f(z)dz$$
with a local coord. at x.

Let $\mathrm{div}(\omega) = \sum_x \mathrm{ord}_x \omega \cdot (x)$.

__Ex.__ Let $\omega = dz$ on $\mathbb{P}^1$.

At $x \in \mathbb{C} \subseteq \mathbb{P}^1$, $\mathrm{ord}_x(\omega) = \mathrm{ord}_x(1) = 0$

At $\infty \in \mathbb{P}^1$, $\mathrm{ord}_\infty(\omega) = \mathrm{ord}_\infty(-z^2) = -2$.

$\mathrm{div}(\omega) = -2(\infty)$

__Prop__ $\dim_{\mathbb{C}(X)} \Omega^1(X) = \underline{1}$.

__Pf__ (sketch). Let $\{U_i\}$ be a cover of X with local coords z_i. Then if $\omega, \omega' \in \Omega^1(X)$, then on each U_i, $\omega = f(z_i)dz_i$, $\omega' = g(z_i)dz_i$, so on U_i
$$\omega = \frac{f(z_i)}{g(z_i)}\omega'.$$
Letting $h \in \mathbb{C}(X)$ be
$$h(z) = \frac{f(z_i)}{g(z_i)} \quad \text{if } z \in U_i,$$
get $\omega = h\omega'$, $h \in \mathbb{C}(X)$. $\square$

<u>Cor</u> If $\omega, \omega' \in \Omega^1(X)$ are nonzero, then

$$\mathrm{div}(\omega) = \mathrm{div}(\omega') \text{ in } \mathrm{Pic}(X).$$

<u>Pf.</u> $\omega = h\omega'$, $h \in \mathcal{O}(X)^\times$, so $\mathrm{div}(\omega) = \mathrm{div}(h) + \mathrm{div}(\omega')$.

$\square$