Lecture 1

Wednesday, June 24, 2020 12:04 PM

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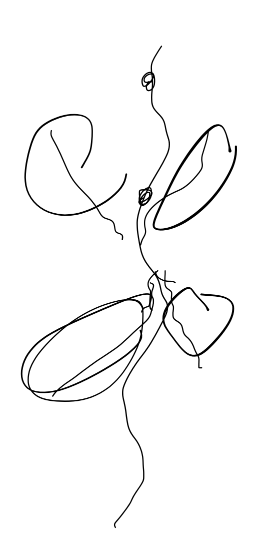
Office hours: TBD

Modular forms

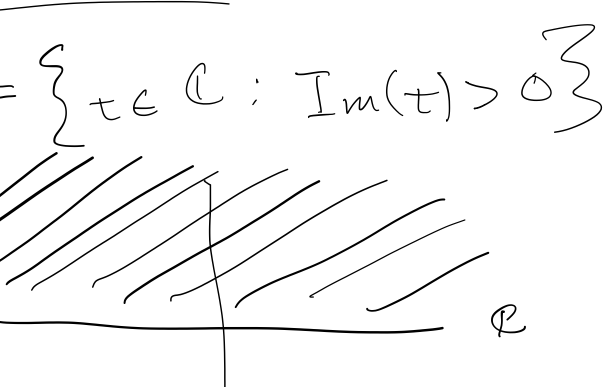
Expectations:
- come to class
- some pset problem
- Give one lecture (1 hour)
- Final paper



Topics:
- Topological grps
- Rep. and its extensions

A bad defn

$\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$



$\mathcal{H}$ has an action by $SL_2(\mathbb{Z})$

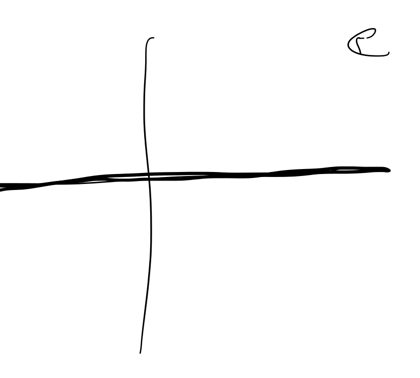
$\gamma\tau = \frac{a\tau+b}{c\tau+d}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in SL_2(\mathbb{Z})$

Well-defined? ✓

$$\mathrm{Im}(\gamma\tau) = \mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\cdot\frac{c\bar\tau+d}{c\bar\tau+d}\right)$$

$$= \mathrm{Im}\left(\frac{* + i(ad-bc)\mathrm{Im}(\tau)}{|c\tau+d|^2}\right)$$

$$= \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2} > 0$$



$SL_2(\mathbb{Z})$ on $\mathbb{P}^1_{\mathbb{C}}$

$SL_2(\mathbb{R})$ fixes $\mathbb{P}^1_{\mathbb{R}}$

$\Rightarrow SL_2(\mathbb{R})$ preserves components of $\mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{P}^1_{\mathbb{R}}$

Defn Let $k \in \mathbb{Z}$.

(1) A meromorphic fn $f : \mathcal{H} \to \mathbb{C}$ is weakly modular of weight k (and level 1) if $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$f(\gamma\tau) = (c\tau+d)^k f(\tau)$$

(2) Such a function is a modular form of wt k, lvl 1, if also
- $f : \mathcal{H} \to \mathbb{C}$ is holomorphic
- $f(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$ (i.e. @ ∞)

(3) A modular form is called a cusp form if $f(\tau) \to 0$ as $\mathrm{Im}(\tau) \to \infty$.

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ $\quad T\tau = \frac{\tau+1}{1} = \tau+1$

$\Rightarrow$ $f(\tau+1) = f(\tau)$ if f is modular of wt. $k \in \mathbb{Z}$.

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$

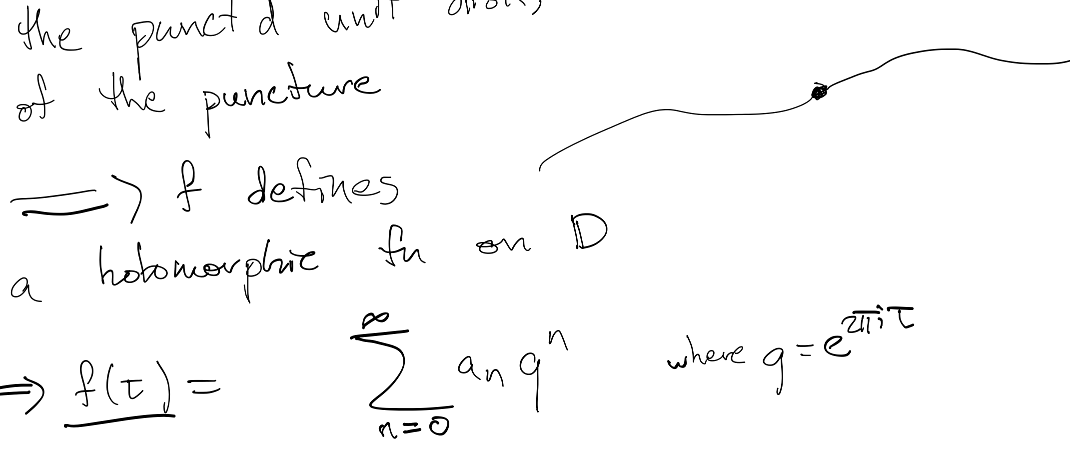
$\Rightarrow$ $f(\frac{-1}{\tau}) = \tau^k f(\tau)$

(In fact T, S generate $SL_2(\mathbb{Z})$.)

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$

$f(\tau) = (-1)^k f(\tau)$

$\Rightarrow f = 0$ if k is odd.



If f is a modular form then f defines a holo. function on the punct'd unit disk, bdd in a nbhd of the puncture

$\Rightarrow$ f defines a holomorphic fn on $\mathbb{D}$

$$\Rightarrow f(\tau) = \sum_{n=0}^{\infty} a_n q^n \quad \text{where } q = e^{2\pi i\tau}$$

Fourier expansion of f.

$f(\tau)$ bdd as $\mathrm{Im}(\tau) \to \infty$
$\iff$ f holo at ∞

$f(\tau) \to 0$ as $\mathrm{Im}(\tau) \to \infty$
$\iff$ f vanishes at ∞.

Some questions

Q1. What do m.f.s have to do with NT?

Wrong ans: $SL_2(\mathbb{Z})$

Correct ans: $SL_2(\mathbb{Z}) \backslash \mathcal{H} \longleftrightarrow$ {iso classes elliptic curves over $\mathbb{C}$}

Q2. How do you produce mod forms?

$$\sum_{\gamma \in SL_2(\mathbb{Z})} (c\tau+d)^{-k} \quad \text{not convergent.}$$

Eisenstein series.

$SL_2\mathbb{Z} \backslash \mathcal{H}$ $\qquad f : (E, \omega) \longrightarrow \mathbb{C}$

$E_\tau : y^2 = 4x^3 - g_4(\tau)x - g_6(\tau)$

Rmk. wt. k mod forms form a $\mathbb{C}$-vector space.

Q3. Is this space fin.-dim'l? What dimension?
Does it have a canonical basis?

Compute dimension w/ Riemann-Roch!

$$Y' \to Y, \quad Y' \to Y = SL_2(\mathbb{Z}) \backslash \mathcal{H}$$

$\rightsquigarrow$ Operator on space of MF's.

Hecke operators

By diagonalizing, get eigenforms.

Q4. What do we learn from Fourier coeffs?

$$f(\tau) = \sum_{n \geq 0} a_n q^n \qquad q = e^{2\pi i\tau}$$

$a_n \in \mathbb{C}$

$a_0 = 0$

f: cusp form, eigenform, get an normalized ($a_1 = 1$)

$a_n \in \mathbb{Z}$.

$a_n a_m = a_{nm}$ $\quad (n, m) = 1$.

Q5. What are mod. forms used for?

§ The Δ

Galois representations

Geometry ⊇ Abelian varieties /$\mathbb{Q}$ ⊇ Elliptic curves /$\mathbb{Q}$

Automorphic forms ⊇ modular forms ⊇ modular forms w/ $\mathbb{Q}$-coeffs.

Abelian varieties /$\mathbb{Q}$ $\longleftarrow$ modular forms

Elliptic curves /$\mathbb{Q}$ $\longleftrightarrow$ modular forms w/ $\mathbb{Q}$-coeffs. (modularity Thm)

Geometry, Abelian varieties, modular forms $\to$ Galois representations; Automorphic forms $\to$ Galois representations (?)

$G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

X: smooth proj variety /k #field

$H^i(X(\mathbb{C}), \mathbb{Q}_\ell)$ $\circlearrowleft$ $G_{\mathbb{Q}}$

Automorphic forms
$n=2$: mod forms.
$n=1$: Hecke characters

$f \longmapsto \rho$
mod form $\quad$ Gal. rep.

a_n

$E \longmapsto \rho$

$a_p = \#E(\mathbb{F}_p) + p - 1$