Intro to Auctions and the Vickrey Auction

Lecture by Claire Zhou

Today: Formalize some foundational ideas and discuss the second price auction.

To begin, we want to know what value each bidder places on the item and what bidders know about each other.

Assumptions

- Each player has a private value v for the auctioned item
  - They won't pay more than v
- 2 outcomes
  - Price paid for item is $p \Rightarrow$ utility is $v - p$
  - Doesn't get the item $\Rightarrow$ utility is 0

Goal: bid to maximize utility

TERMINOLOGY

Def. A (direct) single-item auction A with n bidders is a mapping from any vector of bids $b = (b_1, \ldots, b_n)$ to a winner and set of prices

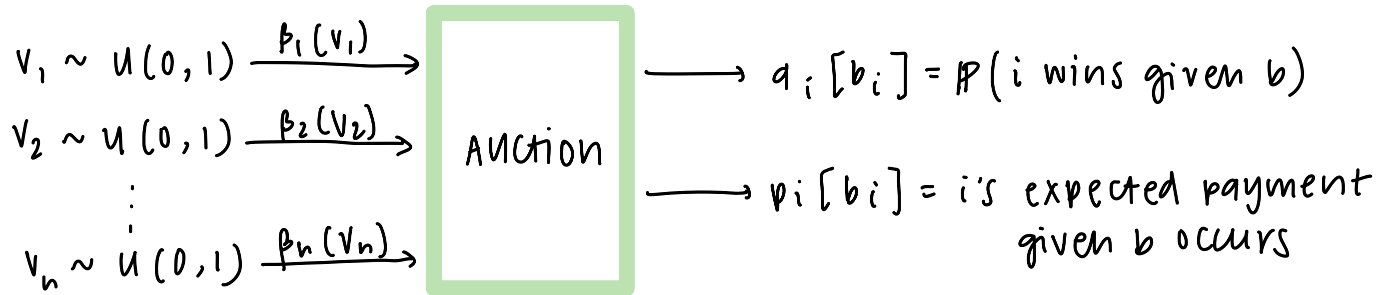
- Allocation rule of A is $\alpha^A[b] = (\alpha_1[b], \ldots, \alpha_n[b])$ where $\alpha_i[b] = 1$ if bidder i gets the item and 0 otherwise
- payment rule of A is $p^A[b] = (p_1[b], \ldots, p_n[b])$ where $p_i[b]$ is the payment of bidder i

A bidding strategy for bidder i is a mapping $\beta_i : [0, \infty) \to [0, \infty)$ that tells us their bid $\beta_i(v_i)$ for any value v_i

Def. Private values Suppose there are n bidders. Their values

$V_1, V_2, \ldots, V_n$ are independent. However, for each i, the distribution F_i of V_i is public knowledge. Fix a bidding strategy β_i for each agent i.

Ex.



utility of bidder i : $u_i[b_i | v_i] = v_i a_i[b_i] - p_i[b_i]$

- Allocation probability is $a_i[b] := \mathbb{P}[\text{bidder } i \text{ wins bidding } b$ when other bids are $\beta_{-i}(V_{-i})] = \mathbb{E}[\alpha_i[b_i, \beta_{-i}(V_{-i})]]$
- Expected payment is $p_i[b] := \mathbb{E}[\text{payment of bidder } i \text{ bidding } b$ when other bids are $\beta_{-i}(V_{-i})] = \mathbb{E}[p_i[b_i, \beta_{-i}(V_{-i})]]$
- Expected utility of bidder i with value v_i bidding b is $u_i[b | v_i] = v_i a_i[b] - p_i[b]$

$$\begin{cases} \text{bidder } i \text{ gets item} & : \quad u_i[b|v_i] = v_i - p_i[b] \\ \text{bidder } i \text{ doesn't get item} & : \quad u_i[b|v_i] = 0 \end{cases}$$

The bidding strategy profile $(\beta_1, \ldots, \beta_n)$ is in Bayes-Nash equilibrium if $\forall i$, $u_i[\beta_i(v_i) | v_i] \geq u_i[b | v_i]$ $\forall v_i, b$. i.e. for each bidder i, the bidding strategy β_i maximizes i's expected utility given other agents also bid by their strategy.

Now that we have covered terminology, we move to discussing second-price auctions, also called a Vickrey auction.

THE VICKREY AUCTION

Sealed-bid auctions

1) Each bidder i privately communicates a bid b_i to the auctioneer
2) The auctioneer decides who gets the item
   - We'll assume this is the highest bidder
3) The auctioneer decides on a selling price
   - First price auction: winner pays what they bid

Second price auction: the highest bidder wins and pays a price equal to the second highest bid

Claim 1: In a second price auction, every bidder has a dominant strategy: set their bid b_i equal to their private valuation v_i. This strategy maximizes the utility of bidder i regardless of what others do.

- Don't need to reason about other players
- In 1st price, if $b_i = v_i$, utility is always 0, so would need to consider how much to underbid.

Proof. Fix bidder i with valuation v_i and the bids b_{-i} of the other bidders. We WTS that bidder i's utility is maximized by setting $b_i = v_i$.

Let $B = \max b_j$ $(j \neq i)$ be the highest bid by another bidder.

If $b_i < B$, utility is 0. If $b_i \geq B$, then i wins and pays B, receiving utility $v_i - B$.

2 cases:

1) $v_i < B$. The highest utility bidder i gets is $\max\{0, v_i - B\} = 0$. This occurs when they bid v_i and lose.
2) $v_i \geq B$. The highest utility bidder i gets is $\max\{0, v_i - B\} = v_i - B$

which also occurs by bidding truthfully, but winning.

Economic jargon: An auction where truthful bidding is a dominant strategy is called truthful or dominant-strategy incentive capable (DSIC)

Claim 2: In a second price auction, every truthtelling bidder is guaranteed non-negative utility.

- Losers get utility 0
- Winner i gets utility $v_i - p$ where $p \le v_i$.

Economic jargon: An auction where truthful bidding guarantees non-negative utility is called individually rational (IR).

Claim 3: In a second price auction, if every bidder bids truthfully, the item is given to the bidder who values it the most.

Economic jargon: An auction with this property is called welfare maximizing.

- A strong form of Pareto optimality - no other allocation has more total value to the bidders