We have seen that when using the GSP auction in sponsored search, the bidders sometimes have an incentive to underbid to receive slightly fewer clicks at a deep discount.

The goal of this lecture is to modify the GSP so that bidders are incentivized to bid truthfully.

To do this, I'm going to introduce the notion of the externality that bidder k imposes on the other bidders.

Externality

We are going to look at the externality imposed by bidder 1 on the other bidders in the case of a single-item auction as well as for a multi-slot auction.

Consider a single-item auction

Relabel the bidders so that $v_1 > v_2 > \ldots > v_n$

v is the bidder's private valuation for each click

Q1: In the outcome of the Vickrey auction, what is the total welfare enjoyed by bidders $2, 3, \ldots, n$?

A1: 0 (all of them lose the item to the highest bidder)

Q2: What if we removed bidder 1 and reran the Vickrey auction?

A2: v_2 (bidder 2 wins the item, the other bidders lose the item)

Difference in total welfare of bidders $2, 3, \ldots, n$ with vs. without bidder 1:

$v_2 - 0 = v_2$

The externality imposed by bidder 1 on bidders $2, 3, \ldots, n$ is therefore v_2.

Note: The winner's payment in the Vickrey auction is exactly the externality she imposes (v_2)

Now, let's go through the same thought experiment with sponsored search

Consider a multi-slot auction

n bidders labeled so that $v_1 > v_2 > \dots > v_n$

Multiple items, so suppose we make the usual greedy assignment:

i^{th} highest bidder assigned to the i^{th} best slot $(i = 1, 2, \dots, k)$

Now we want to ask the same questions as in the case of a single-item auction

Q1: In the outcome of the multi-slot auction, what is the total welfare enjoyed by bidders $2, 3, \dots, n$?

A1: each bidder $j = 2, 3, \dots, k$ gets assigned slot j and obtains welfare $v_j \alpha_j$

for a total of: $\displaystyle\sum_{j=2}^{k} v_j \alpha_j$

where v_j is bidder j's private valuation for each click its link receives and α_j is the click-through-rate (CTR) of slot j.

Q2: What if we removed bidder 1 and reran the multi-slot auction?

A2: The total welfare earned by bidders $2, 3, \dots, n$ in this case is:

$\displaystyle\sum_{j=2}^{k+1} v_j \alpha_{j-1}$

Each bidder moves up one slot. $k+1$ because we now have an extra slot to fill up since bidder 1 has been removed.

Difference in total welfare (externality):

The sum with k, we can add $k+1$ and let $\alpha_{k+1} = 0$, so that our two sums are on the same form

$$\sum_{j=2}^{k+1} v_j \alpha_{j-1} - \sum_{j=2}^{k+1} v_j \alpha_j = \boxed{\sum_{j=2}^{k+1} v_j (\alpha_{j-1} - \alpha_j) \text{ per impression, with } \alpha_{k+1} = 0} \quad (1)$$

This is the externality imposed by bidder 1 on bidders $2, 3, \dots, n$

With the Vickrey auction, we have a single item and we charge the winner the second highest bid which is equal to the externality. We know that this leads to truthful bidding. However, we have seen that the externality is not equal to the next-highest bid for the multi-slot auction. What if we want to use the same idea here? What should we charge? If we for the multi-slot auction would charge each occupant the next-highest bid, we have the GSP auction. And we know that GSP auction doesn't lead to truthful bidding. So how can we modify the payment rule?

Let me introduce the VCG auction

The VCG auction

charges each bidder the externality it imposes.

This is how it works:

1. Accept a bid from each bidder.

   Relabel the bidders so that $b_1 \geq b_2 \geq \ldots \geq b_n$

2. For $i = 1, 2, \ldots, k$ assign the i^{th} bidder to the i^{th} slot

3. For $i = 1, 2, \ldots, k$ charge bidder i:

$$\frac{1}{\alpha_i} \sum_{j=i+1}^{k+1} b_j (\alpha_{j-1} - \alpha_j) \text{ per click, with } \alpha_{k+1} = 0 \qquad (2)$$

This expression looks quite similar to the expression we got for the externality

Two differences between (1) and (2)

1. Each valuation v_j in (1) has been replaced by the corresponding bid b_j in (2)

   The mechanism doesn't know the bidders' valuations, only their bids.

   If bidders are truthful, then these will be the same.

2. (1) is per impression $(v_j \alpha_j)$, (2) is per click

   In a sponsored search auction, we can only charge a bidder for a click.

   Since only an α_i fraction of impressions lead to a click for bidder i, the per-click payment is $\frac{1}{\alpha_i}$ times the per-impression externality.

   Therefore, with truthful bids, the expected payment per impression is exactly the bidder's externality (1)

Proving that the VCG auction is truthful:

Suppose that we have bidders $1, 2, \ldots, n$ who are bidding for k slots.

We want to show that bidder i prefers to bid truthfully.

Reorder the bidders so that $b_1 > b_2 > \ldots > b_n$

Then the utility of bidder i is:

$(\alpha_{k+1} = 0)$

$$\underbrace{v_i \alpha_i}_{\text{value per impression}} - \underbrace{\sum_{j=i+1}^{k+1} b_j (\alpha_{j-1} - \alpha_j)}_{\text{price per impression}} = v_i \alpha_i + \sum_{j=i+1}^{k+1} b_j \alpha_j - \sum_{j=i+1}^{k+1} b_j \alpha_{j-1}$$

adding and subtracting the expected welfare of the bidders with bids higher than v_i

$$= \underbrace{\left(\sum_{j=1}^{i-1} b_j \alpha_j + v_i \alpha_i + \sum_{j=i+1}^{k+1} b_j \alpha_j \right)}_{\text{bidder } i \text{ is trying to maximize this term}} - \underbrace{\left(\sum_{j=1}^{i-1} b_j \alpha_j + \sum_{j=i+1}^{k+1} b_j \alpha_{j-1} \right)}_{\text{expected welfare of the other bidders if bidder } i \text{ were to not exist, and it doesn't depend on the actual value of } b_i \text{, so we can ignore it.}}$$

The only effect that the bid b_i has on the first term is by reordering which α_j gets paired with which coefficient.

By the rearrangement inequality, this is maximized when $b_{i-1} > v_i > b_{i+1}$ which they can achieve by bidding $b_i = v_i$

This shows that the VCG auction is truthful, so we have now modified the GSP payment rule to incentivize truthful bidding in a sponsored search auction.