THE PLECTIC CONJECTURE OVER FUNCTION FIELDS

SIYAN DANIEL LI-HUERTA

ABSTRACT. Let \( \mathbb{Q} \) be a global field of positive characteristic. We prove the plectic conjecture of Nekovář–Scholl \([16]\) for moduli spaces of shtukas over \( \mathbb{Q} \). For example, when the cocharacter is defined over \( \mathbb{Q} \) and the structure group is a Weil restriction from a degree \( d \) separable extension \( F/\mathbb{Q} \) with the same constant field, we construct an action of \( (\mathcal{E}_d \ltimes \text{Weil}(F)^d)^I \) on the \( \ell \)-adic intersection cohomology with compact support of the associated moduli space of shtukas over \( \mathbb{Q} \). This extends the action of \( \text{Weil}(Q)^I \) constructed by Xue along the \( I \)-fold product of the map \( \text{Weil}(Q) \hookrightarrow \mathcal{E}_d \ltimes \text{Weil}(F)^d \). We show that the action of \( (\mathcal{E}_d \ltimes \text{Weil}(F)^d)^I \) commutes with the Hecke action, and we give a moduli-theoretic description of the action of Frobenius elements in \( \text{Weil}(F)^d \times I \).

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INTRODUCTION

The plectic conjecture of Nekovář–Scholl \([16]\) predicts extra symmetries in the cohomology of Shimura varieties when the structure group \( G \) is a Weil restriction. In the \( \ell \)-adic realization, the case of trivial coefficients can be formulated as follows. Suppose that \( G \) is a Weil restriction \( R_{F/\mathbb{Q}} H \) for some connected reductive group \( H \) over a number field \( F \). We have the plectic Galois group \( \Gamma_{F/\mathbb{Q}}^{\text{plec}} := \text{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \), which admits a natural continuous injective homomorphism from the absolute Galois group \( \Gamma_{\mathbb{Q}} \) of \( \mathbb{Q} \). The plectic Galois group acts naturally on the set of conjugacy classes of cocharacters of \( G_\overline{\mathbb{Q}} \), so we can form the stabilizer \( \Gamma_{F/\mathbb{Q}}^{[\mu]} \) of the Hodge cocharacter \( [\mu] \) in \( \Gamma_{F/\mathbb{Q}}^{\text{plec}} \). Note that the reflex field \( E \) is characterized by \( \Gamma_E = \Gamma_{\mathbb{Q}} \cap \Gamma_{F/\mathbb{Q}}^{[\mu]} \). For sufficiently large level \( N \), write \( \overline{\mathcal{S}}_N \) for the minimal compactification of our Shimura variety at level \( N \) over \( E \).

Conjecture (\([16]\) Conjecture 6.1, \([16]\) Conjecture 6.3). The intersection cohomology complex of \( \overline{\mathcal{S}}_N \) with coefficients in \( \overline{Q}_\ell \) canonically lifts from an object in \( D^b(\Gamma_E, \overline{Q}_\ell) \) to an object in \( D^b(\Gamma_{F/\mathbb{Q}}^{[\mu]}, \overline{Q}_\ell) \). In particular, the \( \Gamma_E \)-action on the intersection cohomology groups of \( \overline{\mathcal{S}}_N \) canonically extends to a \( \Gamma_{F/\mathbb{Q}}^{[\mu]} \)-action.
At the time of writing, the plectic conjecture on the level of complexes is widely open. The plectic conjecture on the level of cohomology groups is known in degree 0 by our knowledge of connected components [16 Proposition 6.4]. It is also “known” in all degrees for quaternionic Shimura varieties by directly computing the semisimplification of the cohomology groups as a $\Gamma_E$-representation, observing that this extends to a $\Gamma_{F/Q}$-representation, and then proving the semisimplicity of cohomology [16 Proposition 6.6]. However, the resulting $\Gamma_{F/Q}$-action is noncanonical, and this strategy is limited to specific cases.

The goal of this paper is to systematically prove the plectic conjecture on the level of cohomology groups in the function field setting. More precisely, we prove that an analogous phenomenon holds for the moduli space of shtukas, which is an equi-characteristic analogue of Shimura varieties. However, moduli spaces of shtukas admit richer variants than their number field counterparts: namely, the ability to have multiple legs. This already plays a crucial role in applications to the Langlands program [14, 21], and it also plays a crucial role in this paper.

To state our results, we need some notation. Let $Q$ be a global field of positive characteristic, write $k = \mathbb{F}_q$ for its constant field, and assume that $\ell \nmid q$. Henceforth let $F$ be a degree $d$ separable extension of $Q$ with the same constant field, let $H$ be a connected reductive group over $F$, and write $G$ for the Weil restriction $R_{F/Q} H$. Let $I$ be a finite set, and let $\omega = (\omega_i)_{i \in I}$ be an $I$-tuple of conjugacy classes of cocharacters of $G_{\overline{Q}}$ such that each $\omega_i$ is defined over $Q^{[i]}$. Write $X$ for the geometrically connected smooth proper curve over $k$ associated with $Q$, and write $Q_I$ for the generic point of $X^I$. For any finite closed subscheme $N$ of $X$, we get a moduli space of shtukas $\text{Sht}_{G,N,I,\omega} |_{Q_I}$ at level $N$ over $Q_I$. Let $\Xi$ be a discrete cocompact subgroup of $Z(Q) \backslash Z(A_Q)$. Work of Xue [20 Proposition 6.0.10] yields a natural $\text{Weil}(Q)^I$-action on the intersection cohomology groups with compact support of $\text{Sht}_{G,N,I,\omega} |_{Q_I}/\Xi$.

We now turn to the plectic Galois group in our setting. Since the $\omega_i$ are defined over $Q$, they are stabilized by all of $\Gamma_{F/Q}^{\text{plec}}$. By fixing extensions of the $d$ different $Q$-embeddings $F \hookrightarrow \overline{Q}$ to automorphisms of $\overline{Q}$ over $Q$, we can identify $\Gamma_{F/Q}^{\text{plec}}$ with the semidirect product $\mathcal{S}_d \rtimes \Gamma_{Q}^d$, where $\mathcal{S}_d$ denotes the $d$-th symmetric group. Applying similar observations (which hold for any topological group with an index $d$ open subgroup [4, p. 7]) to the Weil group yields a continuous injective homomorphism $\text{Weil}(Q) \hookrightarrow \mathcal{S}_d \rtimes \text{Weil}(F)^d$.

**Theorem A.** The $\text{Weil}(Q)^I$-action on the intersection cohomology groups with compact support of $\text{Sht}_{G,N,I,\omega} |_{Q_I}/\Xi$ canonically extends to a $(\mathcal{S}_d \rtimes \text{Weil}(F)^d)^I$-action via the $I$-fold product of the map $\text{Weil}(Q) \hookrightarrow \mathcal{S}_d \rtimes \text{Weil}(F)^d$.

**Remark.** For $\omega$ not necessarily defined over $Q$, our method also proves a similar result for the intersection cohomology groups with compact support of the union of the Galois translates of $\text{Sht}_{G,N,I,\omega}/\Xi$. In fact, our results all apply at this level of generality. For more details, see Theorem 5.9, Theorem 5.10, and Theorem 5.13.

---

1We can always enlarge $Q$ such that the $\omega_i$ are defined over $Q$. This is entirely analogous to the number field setting, since the field of definition of $[i]$ is precisely the field over which our Shimura variety lives.

2Strictly speaking, we need to choose a parahoric group scheme over $X$ with generic fiber $G$. We also need to choose an ordered partition of $I$, especially when defining partial Frobenius morphisms. However, we will ignore these issues for the rest of the introduction.
The $(\mathcal{G}_d \times \text{Weil}(F)^d)^I$-action we construct enjoys the following compatibility. Write $\mathfrak{H}_{G,N}$ for the Hecke algebra of $G$ at level $N$, which acts naturally on $\text{Sht}_{G,N,I,\omega}|Q_I/\Xi$ via finite étale correspondences and hence on its intersection cohomology groups with compact support.

**Theorem B.** The action of $(\mathcal{G}_d \times \text{Weil}(F)^d)^I$ from Theorem A commutes with the action of $\mathfrak{H}_{G,N}$.

Finally, we can describe the action of Frobenius elements in $(\mathcal{G}_d \times \text{Weil}(F)^d)^I$ in terms of partial Frobenius morphisms, as conjectured in [16] Remark 6.7. Let $k'$ be a degree $r$ extension of $k$, and let $x = (x_i)_{i \in I}$ be a $k'$-point of $X^I$ such that each $x_i$ splits completely in $Y$, i.e. $m^{-1}(x_i)$ is a disjoint union of $k'$-points $(y_{h,i})_{h=1}^d$. For $x_i$ lying in a certain dense open subscheme $U \setminus N$ of $X$, a smoothness result of Xue [20, Theorem 6.0.12] identifies the intersection cohomology groups with compact support of $\text{Sht}_{G,I,\omega}|Q_I/\Xi$ and the intersection cohomology groups with compact support of $\text{Sht}_{G,I,\omega}|y/\Xi$. Write $y$ for the $k'$-point $(y_{h,i})_{(h,i) \in d \times I}$ of $Y^{d \times I}$. Diagrams (6) and (7) below enable us to identify $\text{Sht}_{G,I,\omega}|y$ and $\text{Sht}_{H,d \times I,d \times \omega}|y$ up to universal homeomorphism.

We now introduce partial Frobenius. For any $(h,i)$ in $d \times I$, we have a commutative square

$$
\begin{array}{ccc}
\text{Sht}_{H,d \times I,d \times \omega}\left|(V \setminus M)^{d \times I}\right. & \xrightarrow{\text{Fr}_{(h,i)}} & \text{Sht}_{H,d \times I,d \times \omega}\left|(V \setminus M)^{d \times I}\right. \\
\uparrow{p} & & \uparrow{p} \\
(V \setminus M)^{d \times I} & \xrightarrow{\text{Frob}_{(h,i)}} & (V \setminus M)^{d \times I},
\end{array}
$$

where $\text{Frob}_{(h,i)}$ equals absolute $q$-Frobenius on the $(h,i)$-th factor and the identity on the other factors, and $V$ and $M$ denote the preimages of $U$ and $N$ in $Y$. Therefore $\text{Fr}_{(h,i)}$ induces a $\text{Frob}_{(h,i)}$-semilinear endomorphism $F_{(h,i)}$ of the relative intersection cohomology with compact support of $\text{Sht}_{H,d \times I,d \times \omega}|(V \setminus M)^{d \times I}/\Xi$ over $(V \setminus M)^{d \times I}$. As $\text{Frob}^\prime_{(h,i)}$ fixes $y$, we obtain an action of $F^\prime_{(h,i)}$ on the intersection cohomology groups with compact support of $\text{Sht}_{H,d \times I,d \times \omega}|y/\Xi$.

On the other hand, the $k'$-point $y_{h,i}$ of $Y$ yields a geometric $q'$-Frobenius element $\gamma_{y_{h,i}}$ in $\text{Weil}(F)$. It acts on the intersection cohomology groups with compact support of $\text{Sht}_{G,I,\omega}|Q_I/\Xi$ via the $(h,i)$-th factor of $\text{Weil}(F)^{d \times I}$ in Theorem A.

**Theorem C.** Under these identifications, the action of $\gamma_{y_{h,i}}$ equals the action of $F^\prime_{(h,i)}$.

Let us now discuss the proofs of our main theorems. For ease of notation, assume that $H$ is split, take $N = \emptyset$, and suppose that $F$ is everywhere unramified over $Q$. Then $F/Q$ corresponds to a finite étale morphism $m : Y \to X$ of degree $d$, where $Y$ is also geometrically connected over $k$.

We begin by observing that $G$-bundles on $X$ are naturally equivalent to $H$-bundles on $Y$. Moreover, this equivalence is compatible with replacing $X$ by the punctured curve $X \setminus x$, as long as $Y$ is replaced by $Y \setminus m^{-1}(x)$. Therefore we obtain a Cartesian square

$$
\begin{array}{c}
\text{Sht}_{G,I,\omega} \longrightarrow \text{Sht}_{H,d \times \omega}^{(d)} \\
\downarrow{p} \quad \quad \downarrow{p} \\
X^I \longrightarrow (\text{Div}^d_Y)^I,
\end{array}
$$

\[3\]We treat the general case in the body of the paper, and this simplified case already illustrates the main ideas.
where \( \text{Div}^d_Y \) denotes the space of degree \( d \) divisors of \( Y \), and \( \text{Sht}^{(d)}_{H,d×\omega} \) denotes a symmetrized variant of the moduli space of shtukas that keeps track of an \( I \)-tuple of divisors of \( Y \), instead of just points of \( Y \). Diagram (\( 1 \)) provides one incarnation of the conjectured plectic diagram from [16 (1.3)]. Note that, because \( m \) is étale, the image of the closed immersion \( m^{-1} : X \to \text{Div}^d_Y \) lies in the open subscheme \( \text{Div}_{Y}^{d,\circ} \) of étale divisors.

We can relate \( \text{Sht}^{(d)}_{H,I,d×\omega} \) to a usual, unsymmetrized moduli space of shtukas as follows. By viewing \( \text{Div}^d_Y \) as the scheme-theoretic quotient of \( Y^d \) by \( \mathfrak{S}_d \), we get a commutative square

\[
\begin{array}{ccc}
\text{Sht}^{(d)}_{H,I,d×\omega} & \xleftarrow{p} & \text{Sht}_{H,d×I,d×\omega} \\
\downarrow & & \downarrow \\
(\text{Div}^d_Y)^I & \xleftarrow{\alpha} & Y^d×I
\end{array}
\]

that is Cartesian up to universal homeomorphism, where \( \mathfrak{S}_d^I \) acts naturally on the right-hand side. The induced \( \mathfrak{S}_d^I \)-action on the intersection cohomology groups with compact support of \( \text{Sht}_{H,d×I,d×\omega}|_{\text{F}_{d×I}/\Xi} \) intertwine with the \( \text{Weil}(F)^{d×I} \)-action by permutation, so together they yield a natural action of \( \mathfrak{S}_d^I \times \text{Weil}(F)^{d×I} = (\mathfrak{S}_d \ltimes \text{Weil}(F)^{d})^I \).

By applying proper base change to Diagrams (\( 1 \)) and (\( 3 \), the smoothness result of Xue [20 Theorem 4.2.3] identifies the intersection cohomology groups with compact support of \( \text{Sht}_{G,I,\omega}|_{\text{F}_{d×I}/\Xi} \) and the intersection cohomology groups with compact support of \( \text{Sht}_{H,d×I,d×\omega}|_{\text{F}_{d×I}/\Xi} \). Under this identification, the action of \( \text{Weil}(Q)^I \) agrees with the action of its image in \( (\mathfrak{S}_d \ltimes \text{Weil}(F)^{d})^I \), which completes the proof of Theorem A. From here, Theorem B follows by generalizing Hecke correspondences to \( \text{Sht}^{(d)}_{H,I,d×\omega} \) and showing that they are compatible with Diagrams (\( 1 \)) and (\( 3 \).

Finally, Theorem C follows from the fact that the \( \text{Weil}(F)^{d×I} \)-action on the intersection cohomology groups with compact support of \( \text{Sht}_{H,d×I,d×\omega}|_{\text{F}_{d×I}/\Xi} \) is constructed precisely using the \( F_{(h,i)} \), via Drinfeld’s lemma. Note that even when \( I \) is a singleton, \( d×I \) usually is not, so Drinfeld’s lemma and therefore multiple-leg phenomena play a crucial role in this paper.

**Remarks.** We conclude by discussing how to extend our work.

1. We prove the plectic conjecture on the level of cohomology groups rather than on the level of complexes because our action of a product of Weil groups relies on an \( \mathfrak{S}_d^I \)-equivariant application of Drinfeld’s lemma, which occurs on the level of cohomology groups. However, by using a derived \( \mathfrak{S}_d^I \)-equivariant version of Drinfeld’s lemma instead (e.g. developing an equivariant version of [13]), we expect our strategy to also yield the plectic conjecture on the level of complexes.

2. The smoothness result of Xue [20 Theorem 6.0.13] implies that the \( (\mathfrak{S}_d \ltimes \text{Weil}(F)^{d})^I \)-action given by Theorem A factors through \( (\mathfrak{S}_d \ltimes \text{Weil}(Y)^{d})^I \). As we use Xue’s results anyway, we phrase everything in the body of the paper in terms of Weil groups of curves. However, by working over a stack-theoretic quotient of the generic fiber by \( \mathfrak{S}_d^I \), we expect our strategy to also yield Theorem A and Theorem B without appealing to Xue’s results.

3. We require that \( F \) has the same constant field as \( Q \) in order for the results of Xue [20] to apply to the moduli of shtukas over \( Y \). Without this hypothesis, \( Y^d×I \) may be disconnected, so its local systems are no longer dictated by representations of a single group. However, we
expect Xue’s results to apply even without this hypothesis, provided that we use Weil groupoids instead. We similarly expect groupoid versions of Theorem A, Theorem B, and Theorem C to hold without this hypothesis.

(4) Our strategy, as well as Remarks (1) and (2), also applies to moduli spaces of local shtukas as in [10]. In particular, we expect a proof of the plectic conjecture for local Shimura varieties on the level of complexes, which should yield applications to (global) Shimura varieties via uniformization. We hope to report on this soon.

Tamiozzo considered a variant of Diagram (3) in his thesis, though he did not proceed further. After completing this paper, the author was informed that X. Zhu proposed a similar strategy for proving Theorem A and Theorem B.

Outline. In §1 we gather some facts on the moduli space of $G$-bundles, as well as certain relative variants thereof. In §2 we introduce symmetrized versions of the Hecke stack and the Beilinson–Drinfeld Grassmannian, and we also recall the Beauville–Laszlo theorem and the geometric Satake correspondence. In §3 we use the preceding material to define and study symmetrized versions of the moduli space of shtukas, which are the main characters of this paper. We also recall Xue’s smoothness result here. In §4 we discuss partial Frobenius morphisms, their relation to Weil groups, and how they arise in the moduli space of shtukas. In §5 we finally assemble everything and prove Theorem A, Theorem B, and Theorem C. We conclude by elaborating on a moduli-theoretic interpretation of Theorem C.

Notation. Unless otherwise specified, all fiber products and thus Cartesian powers are taken over $k$. We denote base changes with subscripts, possibly also with vertical restriction bars. For any algebraic stack $\mathcal{X}$ over $k$ with geometric point $\mathfrak{x}$, we write $\pi_1(\mathcal{X}, \mathfrak{x})$ for the associated étale fundamental group. We will usually omit basepoints and simply write $\pi_1(\mathcal{X})$. By a $G$-bundle, we always mean a principal homogeneous space for $G$.

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1. Moduli spaces of bundles

In this section, we gather some facts on the moduli space of $G$-bundles on $X$, as it plays a central role in our discussion. We begin by describing and fixing notation for our group schemes $G$ of interest over $X$, which serve as integral models for our structure group over $Q$. We then define the moduli space of $G$-bundles on $X$ with level structure, as well as certain relative variants which will be useful in §2. Finally, we conclude with some notation on Weil restrictions and how they affect Bun$_G$, which is crucial for the results of this paper.

1.1. We will use parahoric group schemes over $X$, since their corresponding Hecke stacks and Beilinson–Drinfeld affine Grassmannians in §2 enjoy nice properness properties. We now recall the definition of said group schemes. Let $k$ be a finite field of cardinality $q$, and let $X$ be a connected smooth proper curve over $k$. Write $Q$ for the function field of $X$, fix an algebraic closure $\overline{Q}$ of $Q$,
and write \( \Gamma_Q := \text{Gal}(\overline{Q}/Q) \) for the absolute Galois group of \( Q \) with respect to \( \overline{Q} \). For any closed point \( x \) of \( X \), write \( \mathcal{O}_x \) for the completion of the local ring \( \mathcal{O}_{X,x} \), and write \( Q_x \) for its fraction field.

**Definition.** We call a smooth affine group scheme \( G \) over \( X \) parahoric if it has geometrically connected fibers, its generic fiber \( G_Q \) is reductive, and for every closed point \( x \) of \( X \), the group scheme \( G_{\mathcal{O}_x} \) over \( \mathcal{O}_x \) is parahoric in the sense of \([7, 5.2.6]\).

Let \( G \) be a parahoric group scheme over \( X \). Then there exists a nonempty open subscheme \( U \) of \( X \) such that \( G_U \) is reductive over \( U \) \([9, \text{Exp. XIX 2.6}]\). Let \( \overline{Q} \) be a finite Galois extension of \( Q \) such that the \(*\)-action on a based root datum of \( G_Q \) factors through \( \text{Gal}(\overline{Q}/Q) \), and let \( \hat{Q} \) be a finite separable extension of \( Q \) such that \( G_{\hat{Q}} \) is split. Write \( f : \hat{X} \to X \) for the finite generically étale morphism corresponding to \( \hat{Q}/Q \), where \( \hat{X} \) is a connected smooth proper curve over \( k \). Write \( \hat{U} \) for the inverse image \( f^{-1}(U) \). After shrinking \( U \), we may assume that \( G_{\hat{U}} \) is split and \( f|_{\hat{U}} \) is étale.

Let \( T \) be a maximal subtorus of \( G_Q \), and let \( B \) be a Borel subgroup of \( G_{\hat{Q}} \) containing \( T_{\hat{Q}} \). After shrinking \( U \), we may assume that \( T \) extends to a split subtorus of \( G_{\hat{U}} \) over \( \hat{U} \) and \( B \) extends to a Borel subgroup of \( G_{\hat{U}} \) over \( \hat{U} \). Let \( \ell \) be a prime not dividing \( q \), and let \( E \) be a finite extension of \( \mathbb{Q}_\ell \). Write \((\hat{G}, \hat{T}, \hat{B})\) for the based dual group over \( E \) associated with the based root datum of \((G_{\hat{Q}}, T_{\hat{Q}}, B)\), and write \( L^G \) for the semidirect product \( \hat{G}(E) \rtimes \text{Gal}(\overline{Q}/Q) \). After enlarging \( E \), we may assume that \( E \) contains \( \sqrt{q} \).

1.2. **Remark.** Any connected reductive group \( G_Q \) over \( Q \) arises as the generic fiber of a parahoric group scheme as follows. By spreading out \( G_Q \) to a smooth affine group scheme over some nonempty open subscheme \( U \) of \( X \), applying \([9, \text{Exp. XIX 2.6}]\) and \([8, \text{Proposition 3.1.12}]\), and shrinking \( U \) if necessary, we obtain a reductive group scheme \( G_U \) over \( U \) with geometrically connected fibers whose generic fiber is isomorphic to \( G_Q \). For the finitely many \( x \) in \( X \setminus U \), there exists a parahoric group scheme \( G_{\mathcal{O}_x} \) over \( \mathcal{O}_x \) whose generic fiber is isomorphic to \( G_{Q_x} \) \([7, 5.1.9]\). Gluing the \( G_{\mathcal{O}_x} \) with \( G_U \) via fpqc descent yields a parahoric group scheme over \( X \) whose generic fiber is isomorphic to \( G_Q \).

1.3. Next, we form a parahoric group scheme with generic fiber \( G_{Q_u}^{\text{ad}} \). Write \( Z_U \) for the center of \( G_U \) over \( U \), and write \( G_U^{\text{ad}} \) for the quotient group \( G_U/Z_U \). Then \( G_U^{\text{ad}} \) is reductive over \( U \) \([9, \text{Exp. XII 4.7}]\), and compatibility with base change identifies the generic fiber of \( G_U \to G_{\overline{U}}^{\text{ad}} \) with the quotient morphism \( G_Q \to G_{\overline{Q}}^{\text{ad}} \). For the finitely many \( x \) in \( X \setminus U \), write \( G_{\mathcal{O}_x}^{\text{ad}} \) for the parahoric group scheme in the sense of \([7, 5.2.6]\) associated with the same subset of the reduced Bruhat–Tits building \( \mathcal{B}(G_{\overline{Q}}^{\text{ad}}) = \mathcal{B}(G_{Q_x}) \) as \( G_{\mathcal{O}_x} \) is. Then the quotient morphism \( G_{Q_x} \to G_{\mathcal{O}_x}^{\text{ad}} \) over \( \mathcal{O}_x \), so fpqc descent yields a parahoric group scheme \( G_{\mathcal{O}_x}^{\text{ad}} \) over \( X \) along with a morphism \( G \to G_{\mathcal{O}_x}^{\text{ad}} \).

A similar process yields a parahoric group scheme with generic fiber \( G_{Q_u}^{\text{ab}} \). Write \( G_U^{\text{ab}} \) for the abelianization of \( G_U \) over \( U \) as in \([9, \text{Exp. XXII 6.2.1}]\) which is a torus over \( U \). The formation of \( G_{\overline{U}}^{\text{ab}} \) is compatible with base change, so the generic fiber of \( G_U \to G_{\overline{U}}^{\text{ab}} \) is identified with the quotient morphism \( G_Q \to G_{\overline{Q}}^{\text{ab}} \). For the finitely many \( x \) in \( X \setminus U \), write \( G_{\mathcal{O}_x}^{\text{ab}} \) for the unique parahoric group scheme in the sense of \([7, 5.2.6]\) with generic fiber \( G_{Q_x}^{\text{ab}} \). Then the quotient morphism \( G_{Q_x} \to G_{\mathcal{O}_x}^{\text{ab}} \)

\(^4\text{In }[9], \text{this is called the coradical.}\)
over $\mathcal{O}_x$ uniquely extends to a morphism $G_{\mathcal{O}_x} \to G_{\mathcal{O}_x}^{ab}$ over $\mathcal{O}_x$, so fpqc descent yields a parahoric group scheme $G^{ab}$ over $X$ along with a morphism $G \to G^{ab}$.

By construction, the open subscheme $U$ of $X$, the finite Galois extension $\tilde{F}$ of $F$, and the finite separable extension $\tilde{F}$ of $F$ from 1.1 satisfy the same properties for $G^{ad}$ and $G^{ab}$ as they do for $G$. Similarly, the images of $T$ and $B$ under the quotient morphisms yield analogous objects for $G^{ad}$ and $G^{ab}$, and we use them to form $L G^{ad}$ and $L G^{ab}$. The central isogeny $G_Q \to G_Q^{ad} \times G_Q^{ab}$ induces a morphism $L G^{ad} \times L G^{ab} = (G^{ad} \times G^{ab}) \to L G$.

1.4. We now introduce a general, relative variant of the moduli space of $G$-bundles on $X$ with level structure. Let $T$ be a scheme over $k$, and let $D$ be a $T$-relative effective Cartier divisor of $X \times T$.

**Definition.** Write $\text{Bun}_{G,D}$ for the prestack over $T$ whose $S$-points parametrize data consisting of

i) a $G \times S$-bundle $\mathcal{G}$ on $X \times S$,
ii) an isomorphism $\psi: \mathcal{G}|_D \cong (G \times S)|_D$ of $(G \times S)|_D$-bundles.

When $T = k$ and $D = \emptyset$, we shorten this to $\text{Bun}_G$. For $T$-relative effective Cartier divisors $D_1$ and $D_2$ of $X \times T$ such that $D_1 \subseteq D_2$, pulling back $\psi$ yields a morphism $\text{Bun}_{G,D_1} \to \text{Bun}_{G,D_2}$.

Now $\text{Bun}_G$ is a smooth algebraic stack over $k$ [12, Proposition 1], and note that $\text{Bun}_{G,\emptyset} = \text{Bun}_G \times T$. In general, the Weil restriction $R_{D/T}((G \times T)|_D)$ has a left action on $\text{Bun}_{G,D}$ via composition with $\psi$, and we see that this exhibits the morphism $\text{Bun}_{G,D} \to \text{Bun}_G \times T$ as an $R_{D/T}((G \times T)|_D)$-bundle. Since $R_{D/T}((G \times T)|_D)$ is a smooth affine group scheme over $T$, we see that $\text{Bun}_{G,D}$ is a smooth algebraic stack over $T$.

1.5. **Example.** Let $r$ be a non-negative integer. Then the $S$-points of $\text{Bun}_{GL_r}$ parametrize rank-$r$ vector bundles $\mathcal{V}$ on $X \times S$, and the $S$-points of $\text{Bun}_{SL_r}$ parametrize rank-$r$ vector bundles on $\mathcal{V}$ equipped with an isomorphism $\mathcal{O}_{X \times S} \cong \text{det} \mathcal{V}$. Write $\text{Bun}_{GL_r}^0 \subseteq \text{Bun}_{GL_r}$ for the substack of vector bundles $\mathcal{V}$ such that, for all geometric points $\sigma$ of $S$, the fiber $\mathcal{V}_\sigma$ has degree 0. Then the image of the pushforward morphism $\text{Bun}_{SL_r} \to \text{Bun}_{GL_r}$ lies in $\text{Bun}_{GL_r}^0$. Because Euler characteristics and hence degrees are locally constant in $\sigma$, we see that $\text{Bun}_{GL_r}^0$ is an open substack of $\text{Bun}_{GL_r}$.

Let us describe the Harder–Narasimhan stratification for $\text{Bun}_{GL_r}$. Using the standard subtorus and Borel subgroup, we identify the set of dominant coweights of $GL_r$ with

\[ \{ (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r \mid \mu_1 \geq \cdots \geq \mu_r \text{ and } \mu_1 + \cdots + \mu_r = 0 \}. \]

Let $\mu$ be a dominant coweight of $GL_r$. Write $\text{Bun}_{GL_r}^{\leq \mu} \subseteq \text{Bun}_{GL_r}$ for the substack of vector bundles $\mathcal{V}$ such that, for all geometric points $\sigma$ of $S$, the normalized Harder–Narasimhan polygon of $\mathcal{V}_\sigma$ is bounded by the polygon associated with $\mu$. Then $\text{Bun}_{GL_r}^{\leq \mu}$ is an open substack of $\text{Bun}_{GL_r}$ [19, Lemma A.3]. Note that $\text{Bun}_{GL_r} = \bigcup_\mu \text{Bun}_{GL_r}^{\leq \mu}$, where $\mu$ runs over dominant coweights of $GL_r$.

Write $\text{Bun}_{GL_r}^{\leq \mu, 0}$ for the intersection $\text{Bun}_{GL_r}^{\leq \mu} \cap \text{Bun}_{GL_r}^0$, and for any finite closed subscheme $N$ of $X$, write $\text{Bun}_{GL_r, N}^{\leq \mu, 0}$ for the inverse image of $\text{Bun}_{GL_r, N}^{\mu, 0}$ in $\text{Bun}_{GL_r, N}$. When $\deg N$ is sufficiently large, $\text{Bun}_{GL_r, N}^{\leq \mu, 0}$ is a smooth quasi-projective scheme over $k$ [19, Lemma 3.1 a)].

1.6. Let $\mathcal{F}$ be a $G$-bundle on $X$, and write $E$ for the presheaf over $X$ whose $R$-points parametrize automorphisms of the $G_R$-bundle $\mathcal{F}_R$. As $\mathcal{F}$ is étale-locally isomorphic to the trivial $G$-bundle, we
see that $E$ is étale-locally isomorphic to $G$. Thus $E$ is also a smooth affine group scheme over $X$ with geometrically connected fibers. We can still form $\text{Bun}_E$ as in Definition 1.4, which remains a smooth algebraic stack over $k$ [12, Proposition 1].

Let $S$ be a scheme over $k$, and let $\mathcal{G}$ be a $G \times S$-bundle on $X \times S$. Note that the presheaf of isomorphisms $\text{Iso}_{G \times S}(\mathcal{G}, \mathcal{F} \times S)$ over $X \times S$ has a left action of $E \times S$ via composition, giving it the structure of an $E \times S$-bundle. Hence we obtain a morphism $\text{Bun}_G \to \text{Bun}_E$ by sending $\mathcal{G} \mapsto \text{Iso}_{G \times S}(\mathcal{G}, \mathcal{F} \times S)$. By viewing $\mathcal{F}$ as an $E$-bundle and exchanging the roles of $G$ and $E$, we get a similar morphism $\text{Bun}_E \to \text{Bun}_G$.

**Lemma.** This yields an isomorphism of stacks between $\text{Bun}_G$ and $\text{Bun}_E$.

**Proof.** For any $G \times S$-bundle $\mathcal{G}$ on $X \times S$, we have a morphism of $G \times S$-bundles

$$\mathcal{G} \to \text{Iso}_{E \times S}(\text{Iso}_{G \times S}(\mathcal{G}, \mathcal{F} \times S), \mathcal{F} \times S)$$

given by evaluation. We see that this is an isomorphism by pulling back to étale trivializations of $\mathcal{G}$ and $\mathcal{F}$. The other direction is proved similarly. \qed

1.7. By bootstrapping from Example 1.5 and Lemma 1.6, we define a version of the Harder–Narasimhan stratification for $\text{Bun}_G$ as follows. There exists a vector bundle $\mathcal{V}$ on $X$ equipped with an isomorphism $\mathcal{O}_X \xrightarrow{\sim} \text{det} \mathcal{V}$ such that [11 Proposition 2.2.(b), Theorem 2.6]

- we may identify $G^{ad}$ with a closed subgroup of $\text{SL}(\mathcal{V})$,
- the pushforward morphism $\rho^* : \text{Bun}_{G^{ad}} \to \text{Bun}_{\text{GL}(\mathcal{V})}$ is quasi-affine and of finite presentation.

Write $\rho$ for the rank of $\mathcal{V}$. Using Lemma 1.6 we compatibly identify $\text{Bun}_{\text{SL}(\mathcal{V})}$ with $\text{Bun}_{\text{GL}(\mathcal{V})}$ and $\text{Bun}_{\text{SL}(\mathcal{V})}$ with $\text{Bun}_{\text{GL}(\mathcal{V})}$. Since $\rho^*$ factors through $\text{Bun}_{\text{SL}(\mathcal{V})} \to \text{Bun}_{\text{GL}(\mathcal{V})}$, we see that $\rho^*$ factors through $\text{Bun}_{\text{SL}(\mathcal{V})}$ and hence $\text{Bun}_{\text{GL}(\mathcal{V})}^{\leq \mu}$. Write $\text{Bun}_G^{\leq \mu}$ for the inverse image of $\rho^*^{-1}(\text{Bun}_{\text{GL}(\mathcal{V})}^{\leq \mu})$ in $\text{Bun}_G$. Then Example 1.5 shows that $\text{Bun}_G^{\leq \mu}$ is an open substack of $\text{Bun}_G$, and we have $\text{Bun}_G = \bigcup_\mu \text{Bun}_G^{\leq \mu}$, where $\mu$ runs over dominant coweights of $\text{GL}_r$.

**Remark.** We embed $G^{ad}$ instead of $G$ into $\text{GL}(\mathcal{V})$ to ensure that the resulting Harder–Narasimhan stratification is preserved under twisting. We will use this in 3.5. \[]

1.8. In this subsection, we relax our properness assumption on $X$ to separatedness. We now establish notation on the space of divisors of $X$. Let $d$ be a non-negative integer, and write $\text{Div}_{X}^d$ for the presheaf over $k$ whose $S$-points parametrize $S$-relative effective Cartier divisors of $X \times S$ with degree $d$. Also, write $X(d)$ for the scheme-theoretic quotient of $X^d$ by the permutation action of the symmetric group $S_d$. Since $X$ is a smooth curve over $k$, the morphism $\alpha : X^d \to \text{Div}_{X}^d$ that sends $(x_h)_{h=1}^d \mapsto \sum_{h=1}^d \Gamma_{x_h}$ induces an isomorphism $X(d) \xrightarrow{\sim} \text{Div}_{X}^d$, where $\Gamma_{x_h}$ denotes the graph of $x_h$ [2, Exp. XVII 6.3.9].

Write $\text{Div}_{X}^{d,0}$ for the subsheaf of $\text{Div}_{X}^d$ whose $S$-points parametrize $S$-relative effective Cartier divisors of $X \times S$ that are étale over $S$. We see that the preimage $\alpha^{-1}(\text{Div}_{X}^{d,0})$ consists of the $(x_h)_{h=1}^d$ such that $x_h \neq x_{h'}$ whenever $h \neq h'$, so $\text{Div}_{X}^{d,0}$ is an open subscheme of $\text{Div}_{X}^d$.

1.9. In 2, we will apply the relative variant of Definition 1.4 to the following groups and divisors. Let $I$ be a finite set. The summation morphism $(\text{Div}_{X}^d)^I \to \text{Div}_{X}^{d\# I}$ corresponds to a $(\text{Div}_{X}^d)^I$-relative effective Cartier divisor of $X \times (\text{Div}_{X}^d)^I$ with degree $d\# I$, which we denote by $\Gamma_{\sum_{i \in I} D_i}$. \[]
For any non-negative integer \( n \), write \( \Gamma_{\sum_{i \in I} nD_i} \) for the \((\text{Div}_X^d)^I\)-relative effective Cartier divisor \( n\Gamma_{\sum_{i \in I} D_i} \) of \( X \times (\text{Div}_X^d)^I \), and write \( G_{\Gamma_{\sum_{i \in I} nD_i}} \) for the Weil restriction

\[ R_{\Gamma_{\sum_{i \in I} nD_i}}(G \times_X \Gamma_{\sum_{i \in I} nD_i}). \]

Note that \( G_{\sum_{i \in I} nD_i} \) is a smooth affine group scheme over \((\text{Div}_X^d)^I\). For any \( n_1 \leq n_2 \), we can pull back the counit of the base change-Weil restriction adjunction

\[ G_{\sum_{i \in I} n_2 D_i} \times_{(\text{Div}_X^d)^I}(G \times_X \Gamma_{\sum_{i \in I} n_2 D_i}) \]

along \( \Gamma_{\sum_{i \in I} n_1 D_i} \to \Gamma_{\sum_{i \in I} n_2 D_i} \) to obtain a morphism

\[ G_{\sum_{i \in I} n_2 D_i} \times_{(\text{Div}_X^d)^I}(G \times_X \Gamma_{\sum_{i \in I} n_2 D_i}) \to G \times_X \Gamma_{\sum_{i \in I} n_1 D_i}, \]

which induces a morphism \( G_{\sum_{i \in I} n_2 D_i} \to G_{\sum_{i \in I} n_1 D_i} \) by adjunction. Write \( G_{\sum_{i \in I} \infty D_i} \) for the resulting inverse limit \( \varprojlim_n G_{\sum_{i \in I} nD_i} \), which is an affine group scheme over \((\text{Div}_X^d)^I\).

1.10. Finally, we conclude by introducing the setting of our Weil restrictions. Let \( m : Y \to X \) be a finite generically étale morphism, where \( Y \) is a connected smooth proper curve over \( k \). Write \( F \) for the function field of \( Y \), and let \( H \) be a parahoric group scheme over \( Y \). Applying the discussion in [13] to \( H \) over \( Y \) yields an open subscheme \( V \) of \( Y \), a finite Galois extension \( \hat{F} \) of \( F \), a finite separable extension \( \hat{F} \) of \( F \), a maximal subtorus \( A \) of \( H_F \), and a Borel subgroup \( C \) of \( H_F \). After shrinking \( V \), we may assume that \( m^{-1}(m(V)) = V \) and \( m|_V \) is étale. Write \( U \) for \( m(V) \).

Form the Weil restriction \( R_{Y/X} H \). Its generic fiber is the connected reductive group \( R_{F/Q}(H_F) \) over \( Q \), and for all closed points \( x \) of \( X \), we have

\[ (R_{Y/X} H)_\mathcal{O}_x = R_{(Y \times_X \mathcal{O}_x)/\mathcal{O}_x}(H_{Y \times_X \mathcal{O}_x}) = \prod_{y \in m^{-1}(x)} R_{\mathcal{O}_y/\mathcal{O}_x}(H_{\mathcal{O}_y}). \]

Now [13, Fact F.1] shows that this is parahoric in the sense of [7, 5.2.6]. Thus we may take our parahoric group scheme \( G \) to be \( R_{Y/X} H \) in this subsection.

The restriction \( G_U \) equals \( R_{V/U}(H_V) \), and because \( H_V \) is reductive over \( V \) and \( m|_V \) is finite étale, we see that \( G_U \) is reductive over \( U \). As the \(*\)-action of \( \Gamma_Q \) on a based root datum of \( G_Q \) is induced from the \(*\)-action of \( \Gamma_{\hat{F}} \) on a based root datum of \( H_{\hat{F}} \), after enlarging \( \hat{F} \) we may choose \( \hat{Q} \) to \( \hat{F} \). Then we may take \( \hat{Q} = \hat{F} \). Furthermore, we may choose \( T = R_{F/Q} A \). The natural commutative square

\[ \begin{array}{ccc}
G_{\hat{F}} & \xrightarrow{\sim} & \prod \hat{H}_{\hat{F}} \\
\hat{T}_{\hat{F}} & \xrightarrow{\sim} & \prod \hat{A}_{\hat{F}}
\end{array} \]

where \( \iota \) runs over \( \text{Hom}_Q(F, \hat{F}) \), indicates that we may take \( B = \prod \iota C \). Because \( \hat{V} \) is étale over \( U \), we see that \( T_{\hat{F}} \) and \( B_{\hat{F}} \) extend over \( \hat{V} \).
1.11. Maintain the notation of 1.10 and let $R$ be a scheme over $X$. Note that $R_{(Y \times X)/R}(H \times_X R) = G_R$. Write $\varepsilon : G_{Y \times X R} \to H \times_X R$ for the counit of the base change-Weil restriction adjunction, which is a morphism of group schemes over $Y \times_X R$. For any $H \times_X R$-bundle $\mathcal{H}$ on $Y \times_X R$, the Weil restriction $R_{(Y \times X)/R} \mathcal{H}$ is a $G_R$-bundle on $R$, as Weil restriction commutes with products. For any $G_R$-bundle $\mathcal{G}$ on $R$, the pullback $Y \times_X \mathcal{G}$ is a $G_{Y \times X}$-bundle on $Y \times_X R$, so we can form the pushforward $H \times_X R$-bundle $\varepsilon_*(Y \times_X \mathcal{G})$.

Since $m$ is a finite morphism of connected curves, [5, lemma 3.3] shows that this yields an equivalence of categories between $G_R$-bundles on $R$ and $H \times_X R$-bundles on $Y \times_X R$. Let $N$ be a finite closed subscheme of $X$, and write $M$ for $m^{-1}(N)$. By applying this to $R = X \times S$ and $R = N \times S$, we get an isomorphism $c : \text{Bun}_{G,N} \sim \text{Bun}_{H,M}$.

2. HECKE STACKS AND BEILINSON–DRINFELD AFFINE GRASSMANNIANS

In this section, we introduce symmetrized versions of Hecke stacks and Beilinson–Drinfeld affine Grassmannians. Instead of parameterizing $G$-bundles on $X$, points on $X$, and isomorphisms between these $G$-bundles away from said points, these symmetrized versions more generally parametrize divisors on $X$, along with the other data. This divisorial version naturally appears when taking preimages of points under $m : Y \to X$.

Our symmetrized Hecke stacks and Beilinson–Drinfeld affine Grassmannians continue to enjoy many of the same properties and structures as in the unsymmetrized special case. We begin by defining them, including convolution versions thereof, which will be invaluable in §4. Next, using the Beauville–Laszlo theorem, we study their relation to each other as well as their relative position stratifications. Finally, we recall the geometric Satake correspondence, which describes equivariant perverse sheaves on (usual, unsymmetrized) Beilinson–Drinfeld affine Grassmannians in terms of representations of the dual group.

2.1. First, we introduce a symmetrized, convolution version of the Hecke stack. Let $I_1, \ldots, I_k$ be an ordered partition of $I$, and let $N$ be a finite closed subscheme of $X$.

**Definition.** Write $\text{Hck}^{(d)(I_1, \ldots, I_k)}_{G,N,I}$ for the prestack over $k$ whose $S$-points parametrize data consisting of

i) for all $i$ in $I$, a point $D_i$ of $\text{Div}_{X,N}^d(S)$,

ii) for all $0 \leq j \leq k$, an object $(\mathcal{G}_j, \psi_j)$ of $\text{Bun}_{G,N}(S)$,

iii) for all $1 \leq j \leq k$, an isomorphism $\phi_j : \mathcal{G}_{j-1}|_{X \times S \setminus \sum_{i \in I_j} p_i} \to \mathcal{G}_j|_{X \times S \setminus \sum_{i \in I_j} p_i}$ such that $\psi_j \circ \phi_j|_{X \times S} = \psi_{j-1}$.

When $d = 1$, we omit it from our notation, and when $N = \emptyset$, we omit it from our notation. For finite closed subschemes $N_1$ and $N_2$ of $X$ such that $N_1 \subseteq N_2$, pulling back the $\psi_j$ yields a morphism $\text{Hck}^{(d)(I_1, \ldots, I_k)}_{G,N_2,I} \to \text{Hck}^{(d)(I_1, \ldots, I_k)}_{G,N_1,I}$.

For any $0 \leq j \leq k$, write $p_j : \text{Hck}^{(d)(I_1, \ldots, I_k)}_{G,N,I} \to \text{Bun}_{G,N}$ for the morphism sending the above data to $(\mathcal{G}_j, \psi_j)$. We also have a morphism $p : \text{Hck}^{(d)(I_1, \ldots, I_k)}_{G,N,I} \to (\text{Div}_{X,N}^d)^I$ that sends the above data to $(D_i)_{i \in I}$. And if $I_1, \ldots, I_k$ refines another ordered partition $I'_1, \ldots, I'_{k'}$ of $I$, we get a morphism $\pi_{(I_1, \ldots, I_k)}^{(I'_1, \ldots, I'_{k'})} : \text{Hck}^{(d)(I_1, \ldots, I_k)}_{G,N,I} \to \text{Hck}^{(d)(I'_1, \ldots, I'_{k'})}_{G,N,I}$ by preserving i), preserving $(\mathcal{G}_0, \psi_0)$, and for all $1 \leq j' \leq k'$, taking $\phi_{j'}$ to be the composition of $\phi_j$ over $1 \leq j \leq k$ with $I_j \subseteq I'_{j'}$. 

Because the $D_i$ are disjoint from $N \times S$, for any $0 \leq j \leq k$ we see that the square
\[
\begin{array}{ccc}
\text{Hck}_{G,N,I} \times \text{(Div}^d_{X,N})^I & \longrightarrow & \text{Hck}_{G,I}^d(I_1,\ldots,I_k) \\
(p_j,p) & & (p_j,p)
\end{array}
\]
is Cartesian. Therefore 1.4 shows that $\text{Hck}_{G,N,I}^d(I_1,\ldots,I_k)$ is a $R_{N/k}(G_N)$-bundle. As the morphism $(p_k,p) : \text{Hck}_{G,I}^d(I_1,\ldots,I_k) \to \text{Bun}_G \times \text{(Div}^d_X)^I$ is ind-projective \cite{1} Proposition 3.12\footnotemark[5] we see that $\text{Hck}_{G,I}^d(I_1,\ldots,I_k)$ and more generally $\text{Hck}_{G,N,I}^d(I_1,\ldots,I_k)$ is an ind-algebraic stack over $k$.

\[\text{2.2. Next, we similarly define the Beilinson–Drinfeld affine Grassmannian in our context.}\]

**Definition.** Write $\text{Gr}_{G,I}^d(I_1,\ldots,I_k)$ for the presheaf over $k$ whose $S$-points parametrize data consisting of

\begin{itemize}
  \item[i)] an object $((D_i)_{i\in I}, (G_j)_{j=0}^k, (\phi_j)_{j=1}^k)$ of $\text{Hck}_{G,I}^d(I_1,\ldots,I_k)(S)$,
  \item[ii)] an isomorphism $\theta : G_k \cong \mathbb{G} \times S$ of $G \times S$-bundles.
\end{itemize}

When $d = 1$, we omit it from our notation. We have a morphism $p : \text{Gr}_{G,I}^d(I_1,\ldots,I_k) \to \text{(Div}^d_X)^I$ as in \[2.1\] If $I_1,\ldots,I_k$ refines another ordered partition $I_1',\ldots,I_k'$ of $I$, we also get a morphism $\pi_{(I_1',\ldots,I_k')} : \text{Gr}_{G,I}^d(I_1,\ldots,I_k) \to \text{Gr}_{G,I}^d(I_1',\ldots,I_k')$ as in \[2.1\]

Since $\text{Gr}_{G,I}^d(I_1,\ldots,I_k)$ is defined via a Cartesian square
\[
\begin{array}{ccc}
\text{Gr}_{G,I}^d(I_1,\ldots,I_k) & \longrightarrow & \text{Hck}_{G,I}^d(I_1,\ldots,I_k) \\
\downarrow & & \downarrow p_k \\
\text{Spec } k & \longrightarrow & \text{Bun}_G,
\end{array}
\]
we see from \[2.1\] that $p : \text{Gr}_{G,I}^d(I_1,\ldots,I_k) \to \text{(Div}^d_X)^I$ is ind-projective.

\[\text{2.3. Our symmetrized objects are related to the unsymmetrized special case as follows. Write } [d] \text{ for the finite set } \{1,\ldots,d\}, \text{ and for any finite set } J, \text{ write } d \times J \text{ for } [d] \times J. \text{ We see that the squares}\]
\[
\begin{array}{ccc}
\text{Hck}_{G,N,I}^d(I_1,\ldots,I_k) & \longrightarrow & \text{Hck}_{G,N,I}^d(I_1,\ldots,I_k) \\
\downarrow p & & \downarrow p \\
(\text{Div}^d_{X,N})^I & \longrightarrow & (X \setminus N)^{d \times I}
\end{array}
\]
\[
\begin{array}{ccc}
\text{Gr}_{G,I}^d(I_1,\ldots,I_k) & \longrightarrow & \text{Gr}_{G,I}^d(I_1,\ldots,I_k) \\
\downarrow p & & \downarrow p \\
(\text{Div}^d_X)^I & \longrightarrow & X^{d \times I}
\end{array}
\]

are Cartesian, where the $\alpha$ send $(x_{h,i})_{h\in[d],i\in I}$ to $(\sum_{h=1}^d \Gamma_{x_{h,i}})_{i\in I}$ and preserve all other data. Since the bottom arrows are finite surjective, we see that the top arrows are finite surjective as well. In

\footnotetext[5]{In \cite{1}, only the $d = 1$ case is considered. However, the proof of the key step \cite{1} Proposition 3.7] is phrased entirely in terms of relative effective Cartier divisors, so it works for any $d$. Also, \cite{1} uses $p_0$ instead of $p_k$, but this makes no difference.
addition, if \( I_1, \ldots, I_k \) refines another ordered partition \( I'_1, \ldots, I'_{k'} \) of \( I \), we see that the squares

\[
\begin{array}{ccc}
\text{Hck}_{G,N,I}^{(d)|I_1,\ldots,I_k} & \xrightarrow{\alpha} & \text{Hck}_{G,N,d\times I}^{(d\times I_1,\ldots,d\times I_k)} \\
\downarrow & & \downarrow \\
\text{Gr}_{G,I}^{(d)|I_1,\ldots,I_k} & \xrightarrow{\alpha} & \text{Gr}_{G,d\times I}^{(d\times I_1,\ldots,d\times I_k)}
\end{array}
\]

are also Cartesian.

In all the above squares, note that \( \mathcal{G}_d^I \) has a right action on the right-hand sides via permuting the \((x_{h,i})_{h\in[d],i\in I}\). With respect to this action, the \( \alpha \) are invariant and the right arrows are equivariant.

2.4. We now recall the Beauville–Laszlo theorem. Let \( S \) be a scheme over \( k \), and let \( D \) be an \( S \)-relative effective Cartier divisor of \( X \times S \). For any non-negative integer \( n \), the \( S \)-relative effective Cartier divisor \( nD \) of \( X \times S \) is finite flat over \( S \), so its structure sheaf \( \mathcal{O}_{nD} \) yields a finite flat \( \mathcal{O}_S \)-algebra. For any \( n_1 \leq n_2 \), we obtain a morphism \( \mathcal{O}_{n_2D} \to \mathcal{O}_{n_1D} \). Write \( \mathcal{O}^\wedge_{D_i} \) for the resulting inverse limit \( \varprojlim_n \mathcal{O}_{nD} \), and write \( (X \times S)^\wedge_{D_i} \) for its relative spectrum \( \text{Spec}_S \mathcal{O}^\wedge_{D_i} \). The contravariance of \( \text{Spec}_S \) provides a closed immersion \( nD \to (X \times S)^\wedge_{D_i} \). By working locally and reducing to affines, we obtain a natural morphism \( i : (X \times S)^\wedge_{D_i} \to X \times S \) that preserves the closed subschemes \( nD \).

Write \( \text{Vect}(X \times S) \) for the category of vector bundles on \( X \times S \). Observe that we have an exact tensor functor

\[
\text{Vect}(X \times S) \to \left\{ \begin{array}{c} (\mathcal{V}_1, \mathcal{V}_2, \varphi) \\
\mathcal{V}_1 \text{ is a vector bundle on } X \times S \setminus D, \\
\mathcal{V}_2 \text{ is a vector bundle on } (X \times S)^\wedge_{D_i}, \text{ and} \\
\theta : \mathcal{V}_1 \mid_{(X \times S)^\wedge_{D_i} \setminus D} \sim \mathcal{V}_2 \mid_{(X \times S)^\wedge_{D_i} \setminus D} \text{ is an isomorphism} \end{array} \right\}
\]

given by \( \mathcal{V} \mapsto (\mathcal{V}\mid_{X \times S \setminus D}, \mathcal{V}\mid_{(X \times S)^\wedge_{D_i}}, \id) \).

**Theorem ([3] Theorem 2.12.1).** This yields an equivalence of categories.

More generally, the Tannakian description of \( G \)-bundles ([6] Theorem 4.8) implies that an analogous equivalence of categories holds if we replace “vector bundle” everywhere with “\( G \)-bundle.”

2.5. Using the Beauville–Laszlo theorem, we get the following reinterpretation of the Beilinson–Drinfeld affine Grassmannian. By pulling back, we see that an \( S \)-point of \( \text{Gr}_{G,I}^{(d)|I_1,\ldots,I_k} \) yields data consisting of

i) for all \( i \) in \( I \), a point \( D_i \) of \( \text{Div}_X^d(S) \),

ii) for all \( 0 \leq j \leq k \), a \( G\mid_{(X \times S)^\wedge_{\sum_{i\in I} D_i}} \)-bundle \( \mathcal{G}_j \) on \( (X \times S)^\wedge_{\sum_{i\in I} D_i} \),

iii) for all \( 1 \leq j \leq k \), an isomorphism \( \phi_j : \mathcal{G}_{j-1}\mid_{(X \times S)^\wedge_{\sum_{i\in I} D_i \setminus \sum_{i\in I} D_i}} \sim \mathcal{G}_j \mid_{(X \times S)^\wedge_{\sum_{i\in I} D_i \setminus \sum_{i\in I} D_i}} \),

iv) an isomorphism \( \theta : \mathcal{G}_k \sim \mathcal{G}_1 \mid_{(X \times S)^\wedge_{\sum_{i\in I} D_i \setminus \sum_{i\in I} D_i}} \) of \( G\mid_{(X \times S)^\wedge_{\sum_{i\in I} D_i \setminus \sum_{i\in I} D_i}} \)-bundles.

The Beauville–Laszlo theorem enables us to use iii) and iv) to glue ii) with the trivial bundle on \( X \times S \setminus \sum_{i\in I} D_i \). Hence conversely \( \text{Gr}_{G,I}^{(d)|I_1,\ldots,I_k}(S) \) parametrizes precisely the above data.

Write \( (\text{Div}_X^d)^I_{\leq} \subseteq (\text{Div}_X^d)^I \) for the subsheaf of \( (D_i)_{i\in I} \) such that the \( D_i \) are pairwise disjoint. As the preimage of \( (\text{Div}_X^d)^I \) in \( X^{d\times I} \) consists of the \((x_{h,i})_{h\in[d],i\in I}\) such that \( x_{h,i} \neq x_{h',i'} \) whenever \( i \neq i' \),
2.6. The above enables us to decompose the Beilinson–Drinfeld affine Grassmannian according to our ordered partition \( I_1, \ldots, I_k \) as follows. Recall the affine group scheme \( G_{\sum_{i \in I} \infty D_i} \) over \((\text{Div}^d X)\)' from 1.4. The description of \( \text{Gr}^{(d)}_{G, I}(I_1, \ldots, I_k) \) given in 2.5 shows that it has a left action of \( G_{\sum_{i \in I} \infty D_i} \) via composition with \( \theta \). This description further indicates that \( S \)-points of the stack-theoretic quotient \( \text{Gr}^{(d)}_{G, I}(I_1, \ldots, I_k) / G_{\sum_{i \in I} \infty D_i} \) parametrize data consisting of

i) for all \( i \) in \( I \), a point \( D_i \) of \( \text{Div}^d X(S) \),

ii) for all \( 0 \leq j \leq k \), a \( G_{\sum_{i \in I} \infty D_i} \)-bundle \( G_j \) on \((X \times S)\)\( \times_{\sum_{i \in I} D_i} \),

iii) for all \( 1 \leq j \leq k \), an isomorphism \( \phi_j : G_{\sum_{i \in I} \infty D_i} \rightarrow G_j \times_{\sum_{i \in I} D_i} \rightarrow G_j \times_{\sum_{i \in I} D_i} \).

In particular, we have a morphism

\[ \kappa : \text{Gr}^{(d)}_{G, I}(I_1, \ldots, I_k) / G_{\sum_{i \in I} \infty D_i} \rightarrow \left( \text{Gr}^{(d)}_{G, I_1} / G_{\sum_{i \in I_1} \infty D_i} \right) \times \cdots \times \left( \text{Gr}^{(d)}_{G, I_k} / G_{\sum_{i \in I_k} \infty D_i} \right) \]

that sends \( ((D_i)_{i \in I}, (G_j^k)_{j=0}^k, (\phi_j^k)_{j=1}^k) \) to \((((D_i)_{i \in I_1}, (G_j^1)_{j=0}^1, (\phi_0^1)), \ldots, ((D_i)_{i \in I_k}, (G_j^k)_{j=k-1}^k, (\phi_k)^1)) \).

2.7. We now explain how the Hecke stack combines the moduli space of \( G \)-bundles with the Beilinson–Drinfeld affine Grassmannian. Let \( n \) be a non-negative integer. Applying Definition 1.4 to \( T = (\text{Div}^d X) \) and \( D = \Gamma_{\sum_{i \in I} n D_i} \) yields a smooth algebraic stack \( \text{Bun}_G, \Gamma_{\sum_{i \in I} n D_i} \) over \((\text{Div}^d X)\)' as noted in 1.4. It is a \( G_{\sum_{i \in I} n D_i} \)-bundle over \( \text{Bun}_G \times (\text{Div}^d X) \), and the \( G_{\sum_{i \in I} n D_i} \)-action is even defined over \((\text{Div}^d X)\)'. Write \( \text{Bun}_G, \Gamma_{\sum_{i \in I} n D_i} \) for the inverse limit \( \lim_{\longrightarrow n} \text{Bun}_G, \Gamma_{\sum_{i \in I} n D_i} \), which consequently inherits a left action of \( G_{\sum_{i \in I} \infty D_i} \).

Consider the stack-theoretic quotient \( \left( \text{Gr}^{(d)}_{G, I_1} \times (\text{Div}^d X) \right) / \text{Bun}_G, \Gamma_{\sum_{i \in I} n D_i} \). Write \( \mathcal{A} \) for the prestack over \( k \) whose \( S \)-points parametrize data consisting of

i) an object \( ((D_i)_{i \in I}, (G_j^k)_{j=0}^k, (\phi_j^k)_{j=1}^k) \) of \( \text{Hck}^{(d)}_{G, I}(I_1, \ldots, I_k)(S) \),

ii) an isomorphism \( \theta : G_{\sum_{i \in I} \infty D_i} \times (\text{Div}^d X) / \text{Bun}_G, \Gamma_{\sum_{i \in I} n D_i} \) of \( \text{Bun}_G(S) \)-bundles.

Note that \( G_{\sum_{i \in I} \infty D_i} \) has a left action on \( \mathcal{A} \) via composition with \( \theta \). We see that this exhibits the natural morphism \( \mathcal{A} \rightarrow \text{Hck}^{(d)}_{G, I}(I_1, \ldots, I_k) \) as a \( G_{\sum_{i \in I} \infty D_i} \)-bundle. We also have a morphism

\[ \mathcal{A} \rightarrow \text{Gr}^{(d)}_{G, I}(I_1, \ldots, I_k) / \text{Bun}_G, \Gamma_{\sum_{i \in I} n D_i} \]

given by pulling back \( ((D_i)_{i \in I}, (G_j^k)_{j=0}^k, (\phi_j^k)_{j=1}^k) \), considering \( G_k \) in \( \text{Bun}_G(S) \), and taking \( \theta \) for the trivialization. The Beauville–Laszlo theorem implies that this is a \( G_{\sum_{i \in I} \infty D_i} \)-equivariant isomorphism.

Therefore quotienting by \( G_{\sum_{i \in I} \infty D_i} \) induces an isomorphism

\[ \text{Hck}^{(d)}_{G, I}(I_1, \ldots, I_k) \sim \left( \text{Gr}^{(d)}_{G, I}(I_1, \ldots, I_k) / \text{Bun}_G, \Gamma_{\sum_{i \in I} \infty D_i} \right) / G_{\sum_{i \in I} \infty D_i} \]

Under this identification, write \( \delta : \text{Hck}^{(d)}_{G, I}(I_1, \ldots, I_k) \rightarrow \text{Gr}^{(d)}_{G, I}(I_1, \ldots, I_k) / G_{\sum_{i \in I} \infty D_i} \) for projection onto the first factor.
2.8. We turn to the fibers of the Beilinson–Drinfeld affine Grassmannian. Let \( x \) be a closed point of \( X \), and write \( * \) for the singleton set. The description of \( \text{Gr}^{(s)}_{G,*} \) given in 2.5 shows that \( \text{Gr}^{(s)}_{G,*} \mid x \) is naturally isomorphic to the affine Grassmannian of \( G_{\mathbb{Q}} \) over \( \kappa(x) \) in the sense of [22] (1.2.1). Recall that this equals the fpqc sheaf quotient \( L(G_{\mathbb{Q}})/L^+(G_{\mathbb{Q}}) \), where \( L(G_{\mathbb{Q}}) \) denotes the loop group of \( G_{\mathbb{Q}} \) over \( \kappa(x) \), and \( L^+(G_{\mathbb{Q}}) \) denotes the positive loop group of \( G_{\mathbb{Q}} \) over \( \kappa(x) \) [22, Proposition 1.3.6]. We see from 2.2 that \( \text{Gr}^{(s)}_{G,*} \mid x \) is an ind-projective scheme over \( \kappa(x) \).

2.9. Now we describe the relative position stratification for unsymmetrized affine Grassmannians. Write \( X^+_x(T) \) for the set of dominant coweights of \( G \) with respect to \( T \) and \( B \), and let \( x \) be a closed point of \( U \). Because \( G_{\mathbb{Q}} \) is reductive, we see that \( G_{\mathcal{Q}_x} \) is quasi-split and splits over an unramified extension of \( \mathbb{Q}_x \).

Let \( \omega \) be in \( X^+_x(T) \), viewed as a dominant coweight of \( G_{\mathcal{Q}_x} \). Writing \( \kappa(\omega) \) for the residue field of the field of definition of \( \omega \), we see that \( \omega \) yields a closed affine Schubert variety \( \text{Gr}^\omega_{x,\omega} \subseteq \text{Gr}^{(s)}_{G,*} \mid x \times x \) Spec \( \kappa(x) \) as in [22, p. 25]. The union of the \( \text{Gal}(\kappa(x)/\kappa(x)) \)-translates of \( \text{Gr}^\omega_{x,\omega} \) descends to a closed subvariety \( \text{Gr}^\omega_{x,\omega} \subseteq \text{Gr}^{(s)}_{G,*} \mid x \). Recall that \( \text{Gr}^\omega \) and hence \( \text{Gr}^\omega_{x,\omega} \) is projective [22, Proposition 2.1.5 (1)].

Write \( \text{Gr}^{(s)}_{G,*} \mid x,\omega \subseteq \text{Gr}^{(s)}_{G,*} \mid U \) for the scheme-theoretic closure of \( \bigcup_x \text{Gr}^\omega_{x,\omega} \) in \( \text{Gr}^{(s)}_{G,*} \mid U \), where \( x \) runs over closed points of \( U \). More generally, for any \( \omega = (\omega_i)_{i \in I} \) in \( X^+_x(T)^I \), write \( \text{Gr}^{(I_1,\ldots,I_k)}_{G,I,\omega} \mid U^I \subseteq \text{Gr}^{(I_1,\ldots,I_k)}_{G,I} \mid U^I \) for the scheme-theoretic closure of

\[
\left( \prod_{i \in I} \text{Gr}^{(i)}_{G,I,\omega_i} \mid U \right) \mid U^I \subseteq \text{Gr}^{(I_1,\ldots,I_k)}_{G,I} \mid U^I
\]

in \( \text{Gr}^{(I_1,\ldots,I_k)}_{G,I} \mid U^I \), where we use 2.5 to view the left-hand side as a closed ind-subscheme of the right-hand side. From the projectivity of the \( \text{Gr}^\omega_{x,\omega} \) and the globalization procedure of [17, Remark 4.3], we see that \( \text{Gr}^{(I_1,\ldots,I_k)}_{G,I,\omega} \mid U^I \) is projective over \( U^I \), Note that \( \text{Gr}^{(I_1,\ldots,I_k)}_{G,I,\omega} \mid U^I \) depends only on the \( \Gamma^I_{\mathcal{Q}} \)-orbit of \( \omega \).

2.10. By bootstrapping from 2.9, we define the relative position stratification for symmetrized Beilinson–Drinfeld affine Grassmannians as follows. View elements of \( \mathcal{S}_d^I \) as bijections \( d \times I \xrightarrow{\sim} d \times I \) that preserve the \( I \)-factor. Let \( \Omega \) be a finite \( \mathcal{S}_d^I \)-stable and \( \Gamma_{\mathcal{Q}_d}^d \times I \)-stable subset of \( X^+_x(T)^d \times I \), and write \( \text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{G,d \times I,\Omega} \mid U^{d \times I} \) for the union

\[
\text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{G,d \times I,\Omega} \mid U^{d \times I} := \bigcup_{\omega \in \Omega} \text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{G,d \times I,\omega} \mid U^{d \times I} \subseteq \text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{G,d \times I} \mid U^{d \times I}.
\]

Note that \( \text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{G,d \times I,\Omega} \mid U^{d \times I} \) is projective over \( U^{d \times I} \). As \( \Omega \) is stable under \( \mathcal{S}_d^I \), we see that \( \text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{G,d \times I,\Omega} \mid U^{d \times I} \) is also stable under \( \mathcal{S}_d^I \). Therefore, writing \( \text{Gr}^{(d)(I_1,\ldots,I_k)}_{G,I,\Omega} \mid (\text{Div}_d^I)^I \) for the scheme-theoretic image of \( \text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{G,d \times I,\Omega} \mid U^{d \times I} \) under the morphism

\[
\alpha : \text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{G,d \times I} \mid U^{d \times I} \to \text{Gr}^{(d)(I_1,\ldots,I_k)}_{G,I} \mid (\text{Div}_d^I)^I
\]
obtained from \[2,3\] via restriction, we see that \(Gr_{G,I,\Omega}^{(d)(I_1,\ldots,I_k)}\) is schematic and proper over \(\text{Div}^d_U\).

Furthermore, the closed subset of \(Gr_{G,d\times I}^{(d\times I_1,\ldots,d\times I_k)}\dual_{U\times I}\) underlying \(\alpha^{-1}(Gr_{G,I,\Omega}^{(d)(I_1,\ldots,I_k)})\dual_{(\text{Div}^d_U)^I}\) is precisely \(Gr_{G,d\times I}^{(d\times I_1,\ldots,d\times I_k)}\dual_{U\times I}\). If \(I_1,\ldots,I_k\) refines another ordered partition \(I'_1,\ldots,I'_k\) of \(I\), we see that \(\pi^{(I_1,\ldots,I_k)}\) sends \(Gr_{G,I,\Omega}^{(d)(I_1,\ldots,I_k)}\dual_{(\text{Div}^d_U)^I}\) to \(Gr_{G,I,\Omega}^{(d)(I'_1,\ldots,I'_k)}\dual_{(\text{Div}^d_U)^I}\).

2.11. It will be useful to index relative position bounds with representations. Write \(X^+(\hat{T})\) for the set of dominant weights of \(\hat{G}\) with respect to \(\hat{T}\) and \(\hat{B}\). Recall that \(\text{Rep}_E(\hat{G}^I)\) is semisimple, and every irreducible object of \(\text{Rep}_E(\hat{G}^I)\) can be uniquely written as \(\boxtimes_{i\in I} W_i\), where the \(W_i\) are irreducible objects of \(\text{Rep}_E \hat{G}\). Now \(W_i\) is isomorphic to the Weyl module of a uniquely determined \(\omega_i\) in \(X^+(\hat{T}) = X^+_\bullet(T)\), so altogether we see that isomorphism classes of objects in \(\text{Rep}_E(\hat{G}^I)\) correspond to finite multisets of elements in \(X^+_\bullet(T)^I\).

For any \(W\) in \(\text{Rep}_E((L\hat{G})^{d\times I})\), write \(\Omega(W)\) for the finite subset of \(X^+_\bullet(T)^{d\times I}\) underlying the multiset corresponding to \(W\dual_{\hat{G}}\). Then the \(\text{Gal}(\hat{Q}/Q)\)-action of \(L\hat{G}\) shows that \(\Omega\) is \(\Gamma_{Q\hat{Q}}^{d\times I}\)-stable. If \(\Omega(W)\) is also \(\mathcal{E}_d^I\)-stable, write \(Gr_{G,I,W}^{(d)(I_1,\ldots,I_k)}\dual_{(\text{Div}^d_U)^I}\) for \(Gr_{G,I,\Omega(W)}^{(d)(I_1,\ldots,I_k)}\dual_{(\text{Div}^d_U)^I}\). Observe that \(\Omega(W)\) is always stable under \(\mathcal{E}_d^I\) in the \(d = 1\) setting.

2.12. Finally, we recall the geometric Satake correspondence. Let \(\zeta: I \rightarrow J\) be a map of finite sets, and suppose \(J_1,\ldots,J_k\) is an ordered partition of \(J\) such that \(I_j = \zeta^{-1}(J_j)\) for all \(1 \leq j \leq k\). Now \(\zeta\) induces morphisms \(\zeta^*: (L\hat{G})^J \rightarrow (L\hat{G})^I\) and \(\Delta_\zeta: U^J \rightarrow U^I\). We also write \(\Delta_\zeta\) for its base change \(Gr_{G,I}^{(I_1,\ldots,I_k)}\dual_{U^I} \times_{U^J} U^J \rightarrow Gr_{G,I}^{(I_1,\ldots,I_k)}\dual_{U^I}\). Observe that we may identify \(Gr_{G,I}^{(I_1,\ldots,I_k)}\dual_{U^I} \times_{U^J} U^J\) with \(Gr_{G,I}^{(I_1,\ldots,I_k)}\dual_{U^J}\).

Write \(\mathcal{P}_{G,I}^{(I_1,\ldots,I_k)}\) for the category of \(G\sum_{i\in I}\infty_i\)-equivariant perverse \(E\)-sheaves on \(Gr_{G,I}^{(I_1,\ldots,I_k)}\dual_{U^I}\) in the sense of \([11]\), Sect. A.2], with degree shifts normalized relative to \(U^I\).

**Theorem ([14] Theorem 12.16)\footnote{In [14], the field \(\hat{Q}\) is taken such that \(\text{Gal}(\hat{Q}/Q)\) equals the image of \(\Gamma_Q\) under the *-action. However, everything works for larger \(\hat{Q}\) as well.}.** We have a fully faithful functor \(\text{Rep}_E((L\hat{G})^I) \rightarrow \mathcal{P}_{G,I}^{(I_1,\ldots,I_k)}\), which we denote by \(W \mapsto \mathcal{I}_{I,W}^{(I_1,\ldots,I_k)}\). For all \(W\) in \(\text{Rep}_E((L\hat{G})^I)\), this functor satisfies the following properties:

a) The perverse sheaf \(\mathcal{I}_{I,W}^{(I_1,\ldots,I_k)}\) is supported on \(Gr_{G,I,W}^{(I_1,\ldots,I_k)}\dual_{U^I}\).

b) If \(I_1,\ldots,I_k\) refines another ordered partition \(I'_1,\ldots,I'_k\) of \(I\), we get a natural isomorphism

\[
(\mathcal{R}\pi^{(I_1,\ldots,I_k)})(\mathcal{I}_{I,W}^{(I_1,\ldots,I_k)}) \simeq \mathcal{I}_{I,W}^{(I'_1,\ldots,I'_k)}.
\]

c) If \(W = W_1 \boxtimes \cdots \boxtimes W_k\), where the \(W_j\) are objects in \(\text{Rep}_E((L\hat{G})^I)\), we have a natural isomorphism

\[
\mathcal{I}_{I,W}^{(I_1,\ldots,I_k)} \simeq \kappa^*\left(\mathcal{I}_{I,W_1}^{(I_1)} \boxtimes \cdots \boxtimes \mathcal{I}_{I,W_k}^{(I_k)}\right).
\]

d) We have a natural isomorphism

\[
\Delta_{\zeta}^*(\mathcal{I}_{I,W}^{(I_1,\ldots,I_k)}) \simeq \mathcal{I}_{J,W_{\zeta^*}}^{(J_1,\ldots,J_k)}.
\]
e) We naturally recover $W$ as the graded derived pushforward $\bigoplus_{p \in \mathbb{Z}} (R^p p_* \mathcal{F}_{I, W}^{(I_1, \ldots, I_k)})(\frac{p}{2})$, where $(\frac{p}{2})$ denotes the half-integral Tate twist given by our choice of $q^{1/2}$.

2.13. We conclude by explicitly describing the functor from Theorem 2.12 in certain cases. Let $\omega$ be in $X^+(T)$. Write $W_\omega$ for the Weyl module corresponding to $\omega$, and write $W_{\Gamma_Q, \omega}$ for the direct sum $\bigoplus_{\omega'} W_{\omega'}$, where $\omega'$ runs over the $\Gamma_Q$-orbit of $\omega$. Because the $\ast$-action of $\Gamma_Q$ preserves the based dual group $(\hat{G}, \hat{T}, \hat{B})$, we see that it naturally endows $W_{\Gamma_Q, \omega}$ with the structure of a finite-dimensional algebraic representation of $^L G$ over $E$. Note that $W_{\Gamma_Q, \omega}$ depends only on the $\Gamma_Q$-orbit of $\omega$.

The globalization procedure of [22, p. 81] and Theorem 2.12.e) show that $\mathcal{F}_{\ast, W_{\Gamma_Q}, \omega}$ equals the intersection complex of $\text{Gr}_{\ast, \omega}(U)$, with degree shifts normalized relative to $U$. More generally, for any $\omega$ in $X^+(T)^I$, write $W_{\Gamma_Q, \omega}$ for the exterior tensor product $\bigotimes_{i \in I} W_{\Gamma_Q, \omega_i}$. We see from Theorem 2.12.e) that $\mathcal{F}_{I, W_{\Gamma_Q}, \omega}$ equals the intersection complex of $\text{Gr}_{I, \omega}(U)$, with degree shifts normalized relative to $U^I$.

3. MODULI SPACES OF SHTUKAS

Essentially all of §1 and §2 holds for any perfect field $k$. By contrast, we have Frobenius morphisms when working over a finite field $k$, and in this section we use these Frobenius morphisms to introduce symmetrized versions of the moduli space of shtukas. These are the equi-characteristic analogues of Shimura varieties and their integral models. However, moduli spaces of shtukas admit richer variants than their number field counterparts: namely, the ability to have multiple legs, indexed by the finite set $I$. In the unsymmetrized special case, this phenomenon already plays a crucial role in applications to the Langlands program [14, 21], and it also plays a crucial role in this paper.

We start by defining our symmetrized moduli spaces of shtukas and explaining how they inherit various structures from §1 and §2. In the usual, unsymmetrized case, we describe how geometric Satake provides coefficient sheaves on the moduli spaces of shtukas, and we recall Xue’s result [20] that their relative cohomology over $(U \times N)^I$ is ind-smooth. After bootstrapping this to general, symmetrized setting, we conclude by describing Hecke correspondences for our symmetrized moduli spaces of shtukas.

3.1. We begin by establishing notation on the Harder–Narasimhan as well as relative position stratifications for Hecke stacks. For any dominant coweight $\mu$ of $\text{GL}_r$, recall the open substack $\text{Bun}^{\leq \mu}_G$ of $\text{Bun}_G$ from §1.7. Write $\text{Hck}_{G, \mu}^{(d)(I_1, \ldots, I_k), \leq \mu}$ for the preimage $p_{0,1}^{-1} (\text{Bun}^{\leq \mu}_G)$, and for any finite closed subscheme $N$ of $X$, write $\text{Hck}_{G,N,I}^{(d)(I_1, \ldots, I_k), \leq \mu}$ for its preimage in $\text{Hck}_{G,N,I}^{(d)(I_1, \ldots, I_k)}$. Then §1.7 shows that $\text{Hck}_{G,N,I}^{(d)(I_1, \ldots, I_k), \leq \mu}$ is an open substack of $\text{Hck}_{G,N,I}^{(d)(I_1, \ldots, I_k)}$.

For any $W$ in $\text{Rep}_E((^L G)^{d \times I})$ with $\Omega(W)$ stable under $G^I$, write $\text{Hck}_{G,I,W}^{(d)(I_1, \ldots, I_k)} |_{(\text{Div}_{G^I})^I}$ for the preimage

$$\delta^{-1}(\text{Gr}_{G,I,W}^{(d)(I_1, \ldots, I_k)} |_{(\text{Div}_{G^I})^I} / G_{\sum_{i \in I} \infty D_i})$$
and write

\[ \text{Hck}^{(d)}_{G,N,I,W} \mid (\text{Div}_{U \setminus N})^I \subseteq \text{Hck}^{(d)}_{G,N,I} \mid (\text{Div}_{U \setminus N})^I \]

for the preimage of \( \text{Hck}^{(d)}_{G,N,I,W} \mid (\text{Div}_{U \setminus N})^I \). Note that \( \text{Hck}^{(d)}_{G,N,I,W} \mid (\text{Div}_{U \setminus N})^I \) is a closed substack of \( \text{Hck}^{(d)}_{G,N,I} \mid (\text{Div}_{U \setminus N})^I \).

Finally, write \( \text{Hck}^{(d)}_{G,N,I,W} \mid (\text{Div}_{U \setminus N})^I \) for the intersection

\[ \text{Hck}^{(d)}_{G,N,I} \mid (\text{Div}_{U \setminus N})^I \cap \text{Hck}^{(d)}_{G,N,I,W} \mid (\text{Div}_{U \setminus N})^I \subseteq \text{Hck}^{(d)}_{G,N,I} \mid (\text{Div}_{U \setminus N})^I. \]

3.2. We now define a symmetrized version of the moduli space of shtukas. For any prestack \( \mathcal{X} \) over \( k \), write Frobenius or Fro for its absolute \( q \)-Frobenius endomorphism.

**Definition.** Write \( \text{Sht}^{(d)}_{G,N,I} \) for the stack over \( k \) defined by the Cartesian square

\[
\begin{array}{ccc}
\text{Sht}^{(d)}_{G,N,I} & \xrightarrow{(p_0,p_k)} & \text{Hck}^{(d)}_{G,N,I} \\
\downarrow & & \downarrow \\
\text{Bun}_{G,N} & \xrightarrow{(\text{id}, \text{Frob})} & \text{Bun}_{G,N} \times \text{Bun}_{G,N}
\end{array}
\]

When \( d = 1 \), we omit it from our notation, and when \( N = \emptyset \), we omit it from our notation. For finite closed subschemes \( N_1 \) and \( N_2 \) of \( X \) such that \( N_1 \subseteq N_2 \), we get a morphism \( \text{Sht}^{(d)}_{G,N_2,I} \rightarrow \text{Sht}^{(d)}_{G,N_1,I} \) as in 2.1. We also have a morphism \( p : \text{Sht}^{(d)}_{G,N,I} \rightarrow (\text{Div}_{X \setminus N})^I \) as in 2.1. And if \( I_1, \ldots, I_k \) refines another ordered partition \( I_1', \ldots, I_k' \) of \( I \), we get a morphism \( \pi_{(I_1,\ldots,I_k)} : \text{Sht}^{(d)}_{G,N,I} \rightarrow \text{Sht}^{(d)}_{G,N,I'} \) as in 2.1.

If we replace \( \text{Hck}^{(d)}_{G,N,I} \) in the above square with

\[ \text{Hck}^{(d)}_{G,N,I} \mid (\text{Div}_{U \setminus N})^I, \text{ or } \text{Hck}^{(d)}_{G,N,I,W} \mid (\text{Div}_{U \setminus N})^I, \]

then we write

\[ \text{Sht}^{(d)}_{G,N,I} \mid (\text{Div}_{U \setminus N})^I, \text{ or } \text{Sht}^{(d)}_{G,N,I,W} \mid (\text{Div}_{U \setminus N})^I. \]

for the resulting fiber product, respectively.

We notate \( S \)-points of \( \text{Sht}^{(d)}_{G,N,I} \) using

\[ ((D_i)_{i \in I}, \{G_0, \psi_0\}) \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{k-1}} \{G_{k-1}, \psi_{k-1}\} \xrightarrow{\phi_k} \{G_0, \tau \psi_0\}), \]

where \( \tau \) denotes the pullback \((\text{id}_X \times \text{Frob}_S)^*\). We refer to this as a *shtuka* over \( S \), and we call \((D_i)_{i \in I}\) its *legs*.

3.3. Next, we consider level structure covers for moduli spaces of shtukas. Note that \( G(N) \) has a left action on \( \text{Sht}^{(d)}_{G,N,I} \) via composition with the \( \psi_j \). For finite closed subschemes \( N_1 \) and \( N_2 \) of \( X \) such that \( N_1 \subseteq N_2 \), the morphism \( \text{Sht}^{(d)}_{G,N_2,I} \rightarrow \text{Sht}^{(d)}_{G,N_1,I} \) is equivariant with respect to the homomorphism \( G(N_2) \rightarrow G(N_1) \).
Proposition. This exhibits the morphism $\text{Sh}_{G,N,I}^{(d)(I_1,\ldots,I_k)} \to \text{Sh}_{G,I}^{(d)(I_1,\ldots,I_k)} |_{\text{Div}_{X,N}^{d,I}}$ as a finite Galois morphism with Galois group $G(N)$. In general, this implies that the morphism

$$\text{Sh}_{G,N_2,I}^{(d)(I_1,\ldots,I_k)} \to \text{Sh}_{G,N_1,I}^{(d)(I_1,\ldots,I_k)} |_{\text{Div}_{X,N}^{d,I}}$$

is finite Galois with Galois group $\ker(G(N_2) \to G(N_1))$.

By pulling back, we see that analogous statements hold for

$$\text{Sh}_{G,N,I}^{(d)(I_1,\ldots,I_k), \leq \mu}, \text{Sh}_{G,N,I,W}^{(d)(I_1,\ldots,I_k)} |_{\text{Div}_{X,N}^{d,I}}, \text{and} \text{Sh}_{G,N,I,W}^{(d)(I_1,\ldots,I_k), \leq \mu} |_{\text{Div}_{X,N}^{d,I}}.$$

Proof. The equivariance of $\text{Sh}_{G,N_2,I}^{(d)(I_1,\ldots,I_k)} \to \text{Sh}_{G,N_1,I}^{(d)(I_1,\ldots,I_k)}$ shows that the first statement implies the second. As for the first statement, write $\mathcal{B}$ for the prestack over $k$ whose $S$-points parametrize a $G_N \times S$-bundle $\mathcal{G}$ on $N \times S$ along with an isomorphism $\phi : \mathcal{G} \to \mathcal{T} \mathcal{G}$ of $G_N \times S$-bundles. Since $N$ is finite over $k$, [1, lemma 3.3] shows that $G_N \times S$-bundles on $N \times S$ are equivalent to $\mathcal{R}_{N/k}(G_N)_S$-bundles on $S$. Because $G_N$ has geometrically connected fibers, we see that $\mathcal{R}_{N/k}(G_N)$ does as well, so applying [19, Lemma 3.3 b)] to the classifying stack $*/\mathcal{R}_{N/k}(G_N)$ shows that $\mathcal{B}$ is naturally isomorphic to the discrete stack $(*/\mathcal{R}_{N/k}(G_N))(k)$, which is $*/G(N)$ by Lang’s lemma.

Consider the morphism $\text{Sh}_{G,I}^{(d)(I_1,\ldots,I_k)} |_{\text{Div}_{X,N}^{d,I}} \to \mathcal{B}$ given by

$$((D_i)_{i \in I}, 0_0 \phi_1 \to G_1 \to \cdots \to G_k-1 \to G_0) \mapsto (G_0|_{N \times S}, \phi_k \circ \cdots \circ \phi_1|_{N \times S}).$$

Because the $D_i$ are disjoint from $N \times S$, we see that the square

$$\begin{array}{ccc}
\text{Sh}_{G,N,I}^{(d)(I_1,\ldots,I_k)} & \to & \text{Sh}_{G,I}^{(d)(I_1,\ldots,I_k)} |_{\text{Div}_{X,N}^{d,I}} \\
\downarrow & & \downarrow \\
\text{Spec } k & \to & \mathcal{B}
\end{array}$$

is Cartesian. As the bottom arrow is finite Galois with Galois group $G(N)$, the top arrow is as well. \hfill $\square$

3.4. Convolution morphisms for moduli spaces of shtukas inherit properties from the analogous morphisms for Beilinson–Drinfeld affine Grassmannians as follows. Write

$$\gamma : \text{Sh}_{G,N,I}^{(d)(I_1,\ldots,I_k)} \to \text{Hck}_{G,N,I}^{(d)(I_1,\ldots,I_k)}$$

for the projection morphism. If $I_1, \ldots, I_k$ refines another ordered partition $I'_1, \ldots, I'_k$ of $I$, we see that the square

$$\begin{array}{ccc}
\text{Sh}_{G,I,W}^{(d)(I_1,\ldots,I_k)} |_{\text{Div}_{X}^{d,I}} & \to & \text{Gr}_{G,I,W}^{(d)(I_1,\ldots,I_k)} |_{\text{Div}_{X}^{d,I}} / G_{\sum_{i \in I} \infty D_i} \\
\downarrow \pi^{(t_1,\ldots,t_k)}_{(I_1,\ldots,I_k)} & & \downarrow \pi^{(t_1,\ldots,t_k)}_{(I_1,\ldots,I_k)} \\
\text{Sh}_{G,I,W}^{(d)(I'_1,\ldots,I'_k)} |_{\text{Div}_{X}^{d,I}} & \to & \text{Gr}_{G,I,W}^{(d)(I'_1,\ldots,I'_k)} |_{\text{Div}_{X}^{d,I}} / G_{\sum_{i \in I} \infty D_i} \\
\end{array}$$
is Cartesian. Now 2.10 shows that the right arrow is schematic and proper, so the left arrow is as well. In general, we have a Cartesian square

\[
\begin{array}{ccc}
\text{Sht}_{G,N,I,W}^{(d)(I_1,\ldots,I_k)} \mid_{(\text{Div}^d_{U,N})^I} & \longrightarrow & \text{Sht}_{G,I,W}^{(d)(I_1,\ldots,I_k)} \mid_{(\text{Div}^d_{U,N})^I} \\
\pi^{(I_1,\ldots,I_k)} & & \pi^{(I_1,\ldots,I_k)} \\
\text{Sht}_{G,N,I,W}^{(d)(I_1',\ldots,I_k')} \mid_{(\text{Div}^d_{U,N})^I} & \longrightarrow & \text{Sht}_{G,I,W}^{(d)(I_1',\ldots,I_k')} \mid_{(\text{Div}^d_{U,N})^I}
\end{array}
\]

which implies that the left arrow here is schematic and proper as well. Finally, we see that the above restricts to a Cartesian square

\[
\begin{array}{ccc}
\text{Sht}_{G,N,I,W}^{(d)(I_1,\ldots,I_k)\leq \mu} \mid_{(\text{Div}^d_{U,N})^I} & \longrightarrow & \text{Sht}_{G,I,W}^{(d)(I_1,\ldots,I_k)\leq \mu} \mid_{(\text{Div}^d_{U,N})^I} \\
\pi^{(I_1,\ldots,I_k)} & & \pi^{(I_1,\ldots,I_k)} \\
\text{Sht}_{G,N,I,W}^{(d)(I_1',\ldots,I_k')\leq \mu} \mid_{(\text{Div}^d_{U,N})^I} & \longrightarrow & \text{Sht}_{G,I,W}^{(d)(I_1',\ldots,I_k')\leq \mu} \mid_{(\text{Div}^d_{U,N})^I}
\end{array}
\]

whose vertical arrows are schematic and proper.

3.5. We now describe the twisting action on the moduli space of shtukas. Write \( \mathbb{A}_Q \) for the adele ring of \( Q \), and write \( Z \) for the Zariski closure of the center of \( G_Q \) in \( G \). Then \( Z \) is a closed group subscheme of \( G \) that is flat over \( X \). Write \( \mathcal{O}_Q \) for the integral subring of \( \mathbb{A}_Q \), and write \( K_{Z,N} \) for the compact open subgroup \( \ker(Z(\mathcal{O}_Q) \to Z(N)) \) of \( Z(\mathcal{O}_Q) \). Fpqc descent yields an inclusion of groupoids

\[
Z(Q) \backslash Z(\mathbb{A}_Q)/K_{Z,N} \hookrightarrow \text{Bun}_{Z,N}(k)
\]

whose image consists precisely of the \( Z \)-bundles on \( X \) that are Zariski-locally trivial.

Recall that \( \text{Bun}_{Z,N}(k) \) acts on \( \text{Bun}_{G,N}(k) \) via twisting. As \( \text{Bun}_{Z,N}(k) \) is fixed by \( \text{Frob} \), we see that \( \text{Bun}_{Z,N}(k) \) acts on \( \text{Sht}_{G,N,I}^{(d)(I_1,\ldots,I_k)} \) via twisting. This action evidently preserves \( \text{Sht}_{G,N,I,W}^{(d)(I_1,\ldots,I_k)} \mid_{(\text{Div}^d_{U,N})^I} \), and because the image of \( Z \) in \( G^{ad} \) is trivial, we see that this action also preserves \( \text{Sht}_{G,N,I}^{(d)(I_1,\ldots,I_k)\leq \mu} \).

3.6. Let \( \Xi \) be a discrete subgroup of \( Z(Q) \backslash Z(\mathbb{A}_Q) \) such that \( Z(Q) \backslash Z(\mathbb{A}_Q)/\Xi \) is compact. After quotienting by \( \Xi \), we obtain the following finite type result for the Harder–Narasimhan stratification of moduli spaces of shtukas.

**Proposition.** The quotient stack \( \text{Sht}_{G,N,I,W}^{(d)(I_1,\ldots,I_k)\leq \mu} \mid_{(\text{Div}^d_{U,N})^I}/\Xi \) is Deligne–Mumford and of finite type over \( k \).

**Proof.** Since the \( \pi^{(I_1,\ldots,I_k)} \) from 3.4 are of finite type, it suffices to consider \( k = 1 \). I claim that it also suffices to consider \( N \) with \( \deg N \) sufficiently large. To see this, note that we can cover \( (\text{Div}^d_{U,N})^I \) with \( d\# I \) open subsets of the form \( (\text{Div}^d_{U,N\cup \{u\}})^I \), where \( u \) is a closed point of \( U \) and \( n \) is sufficiently large. Applying the statement to \( \text{Sht}_{G,N,I,W}^{(d)(I)\leq \mu} \mid_{(\text{Div}^d_{U,N\cup \{u\}})^I}/\Xi \), using Proposition 3.3 and covering \( \text{Sht}_{G,N,I,W}^{(d)(I)\leq \mu} \mid_{(\text{Div}^d_{U,N})^I}/\Xi \) with the finitely many resulting Deligne–Mumford open substacks of finite type over \( k \) yields the desired reduction.
First, we treat the case when $G_Q$ is adjoint. Then $Z_Q$ and hence $\Xi$ is trivial, and we see from 1.3 that $G = G^{\text{ad}}$. Recall the notation of 1.7 Since the pushforward morphism $\rho_* : \text{Bun}_G \to \text{Bun}_{\text{GL}_r}$ factors as the top row of the diagram

$$
\begin{array}{ccc}
\text{Bun}_{G,N} & \longrightarrow & \text{Bun}_{\text{GL}_r,N} \times \text{Bun}_{\text{GL}_r} \\
\downarrow & & \downarrow \\
\text{Bun}_G & \longrightarrow & \text{Bun}_{\text{GL}_r}
\end{array}
$$

where the square is Cartesian. Since its bottom arrow is of finite presentation, so is its top arrow. As the other top arrow $\text{Bun}_{G,N} \to \text{Bun}_{\text{GL}_r,N} \times \text{Bun}_{\text{GL}_r}$, $\text{Bun}_G$ is evidently a $R_{N/k}(\text{GL}(V)/G)$-bundle and hence of finite presentation, we see that $\rho_* : \text{Bun}_{G,N} \to \text{Bun}_{\text{GL}_r,N}$ is of finite presentation. Because the image of $\rho_*$ lies in $\text{Bun}_{G,N}^0$ and $\text{deg} \, N$ is sufficiently large, we see from Example 1.5 that the inverse image $\text{Bun}_{G,N}^{\leq \mu}$ of $\text{Bun}_{\text{GL}_r,N}^{\leq \mu}$ in $\text{Bun}_{G,N}$ is a scheme of finite type over $k$.

Now $(p_0, p) : \text{Hck}_{G,N,I,W}^{(d)(I), \leq \mu} |_{(\text{Div}_{U,N}^d)^I} \to \text{Bun}_{G,N}^{\leq \mu} \times (\text{Div}_{U,N}^d)^I$ is schematic and proper by 2.10 which makes $\text{Hck}_{G,N,I,W}^{(d)(I), \leq \mu} |_{(\text{Div}_{U,N}^d)^I}$ a scheme of finite type over $k$. Note that there exists a dominant coweight $\lambda(W)$ of $\text{GL}_r$ such that the image of $\text{Hck}_{G,N,I,W}^{(d)(I), \leq \mu} |_{(\text{Div}_{U,N}^d)^I}$ via $p_k$ lies in $\text{Bun}_{G,N}^{\leq \mu + \lambda(W)}$. Therefore $\text{Sht}_{G,N,I,W}^{(d)(I), \leq \mu} |_{(\text{Div}_{U,N}^d)^I}$ is defined by the Cartesian square

$$
\begin{array}{ccc}
\text{Sht}_{G,N,I,W}^{(d)(I), \leq \mu} |_{(\text{Div}_{U,N}^d)^I} & \longrightarrow & \text{Hck}_{G,N,I}^{(d)(I), \leq \mu} \\
\downarrow & & \downarrow \\
\text{Bun}_{G,N}^{\leq \mu} & \longrightarrow & \text{Bun}_{G,N}^{\leq \mu + \lambda(W)}
\end{array}
$$

which shows that $\text{Sht}_{G,N,I,W}^{(d)(I), \leq \mu} |_{(\text{Div}_{U,N}^d)^I}$ is also a scheme of finite type over $k$.

Next, we treat the case when $G_Q$ is a torus. Then $G^{\text{ad}}$ is trivial, so $\text{Sht}_{G,N,I}^{(d)(I), \leq \mu} = \text{Sht}_{G,N,I}^{(d)(I)}$. Because $G_U^0$ is a split torus, Lang’s lemma implies that the pullback of $\text{Sht}_{G,N,I,W}^{(d)(I), \leq \mu} |_{(\text{Div}_{U,N}^d)^I}$ to $(\text{Div}_{U,N}^{d,f-1(N)})^I$ is a union of $\text{Bun}_{G,N}(k)$-bundles over $(\text{Div}_{U,N}^{d} \times \text{Div}_{U,N}^{d,f-1(N)})^I$. Note that $(\text{Div}_{U,N}^{d} \times \text{Div}_{U,N}^{d,f-1(N)})^I$ is finite étale over $(\text{Div}_{U,N}^{d,0})^I$. As $\text{Bun}_{G,N}(k)$ is discrete and the image of $\Xi$ in $\text{Bun}_{G,N}(k)$ has finite index, we obtain an open substack of $\text{Sht}_{G,N,I,W}^{(d)(I)} |_{(\text{Div}_{U,N}^{d,0})^I}$ that is Deligne–Mumford, of finite type over $k$, and whose translates under $\Xi$ cover $\text{Sht}_{G,N,I,W}^{(d)(I)} |_{(\text{Div}_{U,N}^{d,0})^I}$. In particular, $\text{Sht}_{G,N,I,W}^{(d)(I)} |_{(\text{Div}_{U,N}^{d,0})^I}/\Xi$ is Deligne–Mumford and of finite type over $k$.

Finally, we treat the case of general $G$. The morphism $L(G^{\text{ad}} \times G^{\text{ab}}) \to LG$ from 1.3 allows us to view $W$ as a representation $W'$ of $(L(G^{\text{ad}} \times G^{\text{ab}}))^{d \times I}$. Then pushforward along $G \to G^{\text{ad}} \times G^{\text{ab}}$ yields a morphism

$$
\text{Sht}_{G,N,I,W}^{(d)(I)} |_{(\text{Div}_{U,N}^{d,0})^I} \to \text{Sht}_{G^{\text{ad}} \times G^{\text{ab}},N,I,W'}^{(d)(I)} |_{(\text{Div}_{U,N}^{d,0})^I},
$$

which is finite since the kernel of $G \to G^{\text{ad}} \times G^{\text{ab}}$ is finite flat over $X$. Note that this morphism is equivariant with respect to the action of $\Xi$, and by definition it preserves the $\mu$-truncated part.
As $L^G \times L^G = L(G \times G)$, there exist finite-dimensional algebraic representations $W^a$ and $W^b$ of $(L^G)^{\times I}$ and $(L^G)^{\times I}$ over $E$, respectively, such that $\Omega(W^a)$ and $\Omega(W^b)$ are stable under $\mathcal{S}_{\mathfrak{G}}^I$ and that $W'$ is a subrepresentation of $W^a \otimes W^b$. Now the natural morphism

$$\text{Sht}^a_{G^a \times G^b, N, I, W'} |_{(\text{Div}_{\mu, \alpha})^I} \to \text{Sht}^a_{G^a \times G^a, N, I, W^a \otimes W^b} |_{(\text{Div}_{\mu, \alpha})^I}$$

is a $\Xi$-equivariant closed immersion, and the right-hand side is naturally isomorphic to

$$\text{Sht}^a_{G^a, N, I, W} |_{(\text{Div}_{\mu, \alpha})^I} \times (\text{Div}_{\mu, \alpha})^I / \text{Sht}^a_{G^a, N, I, W} |_{(\text{Div}_{\mu, \alpha})^I} .$$

Restricting to the $\mu$-truncated part and applying our previous work to $G^a$ and $G^b$ shows that $\text{Sht}^a_{G, N, I, W} |_{(\text{Div}_{\mu, \alpha})^I} / \Xi$ is Deligne–Mumford and of finite type over $k$, as desired.

3.7. Remark. We expect Proposition 3.6 to hold for all of $\text{Sht}^a_{G, N, I, W, \Xi} |_{(\text{Div}_{\mu, \alpha})^I} / \Xi$. However, we will only need the open substack $\text{Sht}^a_{G, N, I, W, \Xi} |_{(\text{Div}_{\mu, \alpha})^I} / \Xi$, so Proposition 3.6 suffices. Alternatively, one can adapt the proof of [11, Theorem 3.15] to show that $\text{Sht}^a_{G, N, I, W, \Xi} |_{(\text{Div}_{\mu, \alpha})^I} / \Xi$ is Deligne–Mumford and locally of finite type over $k$. Proposition 3.6 enables us to use the six-functor formalism of [15] in treating cohomological correspondences and perverse sheaves, as in [14] §4.1, §4.2].

3.8. We now describe our coefficient sheaves in the usual, unsymmetrized case. Note that the composition

$$\text{Sht}^a_{G, N, I, W} \xrightarrow{\delta \cdot \gamma} \text{Gr}_{G, I}^a \mathcal{G}^{(d)(1, \ldots, I_k)} / G_{\sum_i I_i \infty D_i}$$

is invariant with respect to the action of $\text{Bun}_Z, N(k)$ and hence $\Xi$ on $\text{Sht}^a_{G, N, I, W}$. Thus it induces a morphism

$$\epsilon : \text{Sht}^a_{G, N, I, W} / \Xi \to \text{Gr}_{G, I}^a \mathcal{G}^{(d)(1, \ldots, I_k)} / G_{\sum_i I_i \infty D_i} .$$

In the $d = 1$ setting, write $\mathcal{F}_{N, I, W, \Xi}$ for the pullback $\epsilon^*(\mathcal{F}^{(d)(1, \ldots, I_k)}_{I, W})$, which Theorem 2.12(a) enables us to view as a complex of constructible $E$-sheaves on $\text{Sht}^{a}_{G, N, I, W} |_{(U \setminus N)^I} / \Xi$. Write $\mathcal{F}^{(d)(1, \ldots, I_k)}_{N, I, W, \Xi}$ for its restriction to $\text{Sht}^{a}_{G, N, I, W} |_{(U \setminus N)^I} / \Xi$. Applying proper base change to 3.4 and Theorem 2.12(b) shows that $R^p(\mathcal{F}^{(d)(1, \ldots, I_k)}_{N, I, W, \Xi})$ is independent of the ordered partition $I_1, \ldots, I_k$, so we denote this complex of constructible $E$-sheaves on $(U \setminus N)^I$ by $\mathcal{K}^{p, (d)(1, \ldots, I_k)}_{N, I, W, \Xi}$. For any integer $p$, write $\mathcal{K}^{p, (d)(1, \ldots, I_k)}_{N, I, W, \Xi}$ for its $p$-th cohomology, which is a constructible $E$-sheaf on $(U \setminus N)^I$, and finally write $\mathcal{K}^{p, (d)(1, \ldots, I_k)}_{N, I, W, \Xi}$ for the ind-constructible $E$-sheaf $\lim_{\mu \to \infty} \mathcal{K}^{p, (d)(1, \ldots, I_k)}_{N, I, W, \Xi}$ on $(U \setminus N)^I$.

3.9. To state Xue’s result, we recall the definition of ind-smoothness. Briefly, let $X$ be any normal connected noetherian scheme over $k$. Recall that we say an ind-constructible $E$-sheaf $\mathcal{M}$ on $X$ is ind-smooth if $\mathcal{M}$ is isomorphic to a directed colimit of smooth $E$-sheaves on $X$. This is equivalent to requiring that, for any geometric points $\overline{x}$ and $\overline{y}$ of $X$ and étale path $\overline{y} \sim \overline{x}$, the resulting specialization map $\mathcal{M}_\overline{x} \to \mathcal{M}_\overline{y}$ is an isomorphism [20, Lemma 1.1.5].

Theorem ([20, Theorem 6.0.12]). Assume that $X$ is geometrically connected over $k$. Then the ind-constructible $E$-sheaf $\mathcal{K}^{p, (d)(1, \ldots, I_k)}_{N, I, W, \Xi}$ on $(U \setminus N)^I$ is ind-smooth.
Even without the geometrically connected assumption on \( X \), we expect some form of Theorem 3.9 to hold.

3.10. Our symmetrized objects are related to the unsymmetrized special case as follows. Recall the Cartesian squares from \(^2.3\). By pulling back along \( \gamma \), we get analogous Cartesian squares

\[
\begin{array}{ccc}
\text{Sh}_{G,N,I}^{(d)(I_1,...,I_k)} & \xleftarrow{\alpha} & \text{Sh}_{G,N,d\times I}^{(d\times I_1,...,d\times I_k)} \\
\pi^{(I_1,...,I_k)} & & \pi^{(I_1,...,I_k)} \\
\text{Sh}_{G,N,I}^{(d)(I'_1,...,I'_{k'})} & \xleftarrow{\alpha} & \text{Sh}_{G,N,d\times I}^{(d\times I'_1,...,d\times I'_{k'})} \\
p & & p \\
(Div_{X,N}^d)^I & \xleftarrow{\alpha} & (X \setminus N)^{d\times I}
\end{array}
\]

for any ordered partition \( I_1, \ldots, I_k \) of \( I \) refined by \( I_1, \ldots, I_k \). The \( \alpha \) are evidently equivariant with respect to the action of \( \text{Bun}_{Z,N}(k) \) and hence \( \Xi \).

Let \( \Omega \) be a finite \( \mathcal{S}_d' \)-stable subset of \( X_+^\times(T)^{d\times I} \). We see from \(^2.10\) that the above diagram restricts to commutative squares

\[
\begin{array}{ccc}
\text{Sh}_{G,N,I,\Omega}^{(d)(I_1,...,I_k)} & \xleftarrow{\alpha} & \text{Sh}_{G,N,d\times I,\Omega}^{(d\times I_1,...,d\times I_k)} \\
\pi^{(I_1,...,I_k)} & & \pi^{(I_1,...,I_k)} \\
\text{Sh}_{G,N,I,\Omega}^{(d)(I'_1,...,I'_{k'})} & \xleftarrow{\alpha} & \text{Sh}_{G,N,d\times I,\Omega}^{(d\times I'_1,...,d\times I'_{k'})} \\
p & & p \\
(Div_{U,N}^d)^I & \xleftarrow{\alpha} & (U \setminus N)^{d\times I}
\end{array}
\]

that are Cartesian up to universal homeomorphism. Note that we may further restrict to the \( \mu \)-truncated part to obtain analogous commutative squares. As in \(^2.3\), we see that \( \mathcal{S}_d' \) has a right action on the right-hand sides for which the \( \alpha \) are invariant and the right arrows are equivariant.

3.11. By bootstrapping from Theorem 3.10, we obtain the following symmetrized version of the sheaves \( \mathcal{H}^p_{N,d\times I,W,\Xi} \). Let \( W \) be in \( \text{Rep}_E((\times^d G)^{d\times I}) \), and suppose that \( W \circ \sigma^* = W \) for all \( \sigma \in \mathcal{G}_d' \).

Then Theorem \(^2.12\) d) yields a natural isomorphism

\[
\Delta_{\sigma}^* (\mathcal{F}_{d\times I,W}^{(d\times I_1,...,d\times I_k)}) \cong \mathcal{F}_{d\times I,W^\sigma}^{(d\times I_1,...,d\times I_k)} = \mathcal{F}_{d\times I,W}^{(d\times I_1,...,d\times I_k)}.
\]

As the action of \( \sigma \) from \(^2.3\) is induced by \( \Delta_{\sigma} \) and compatible with the action from \(^3.10\), pulling this back along \( \epsilon \) gives an isomorphism \( \sigma^*(\mathcal{F}_{N,d\times I,W,\Xi}^{p}) \cong \mathcal{F}_{N,d\times I,W,\Xi}^{p} \). Taking \( R\pi_1 \) and applying proper base change gives an isomorphism \( \sigma^*(\mathcal{H}^p_{N,d\times I,W,\Xi}) \cong \mathcal{H}^p_{N,d\times I,W,\Xi} \). The naturality of Theorem \(^2.12\) d) indicates that this equips \( \mathcal{H}^p_{N,d\times I,W,\Xi} \) with the structure of a \( \mathcal{G}_d' \)-equivariant sheaf on \( (U \setminus N)^{d\times I} \).

Assume that \( X \) is geometrically connected over \( k \). Then Theorem 3.9 indicates that \( \mathcal{H}^p_{N,d\times I,W,\Xi} \) is isomorphic to a directed colimit \( \lim_{\rightarrow h} \mathcal{M}_h \) of smooth \( E \)-sheaves \( \mathcal{M}_h \) on \( (U \setminus N)^{d\times I} \). By replacing the \( \mathcal{M}_h \) with their eventual images, we may assume that the transition morphisms are injective.
Since $\mathcal{G}_d^I$ is finite, we may also replace the $\mathcal{M}_h$ with the $\mathcal{G}_d^I$-module they generate to ensure that the $\mathcal{M}_h$ are preserved by $\mathcal{G}_d^I$. The above isomorphism then restricts to isomorphisms $\sigma^*(\mathcal{M}_h) \simeq \mathcal{M}_h$, so the $\mathcal{M}_h$ descend to a directed system of smooth $E$-sheaves $\mathcal{M}_h^{(d)}$ on the stack-theoretic quotient $(U \smallsetminus N)^{d\times I}/\mathcal{G}_d^I = ((U \smallsetminus N)^d/\mathcal{G}_d)^I$ with injective transition morphisms. Write $\mathcal{E}_{N,I,W,\Xi}^{(d),p}$ for the resulting ind-smooth $E$-sheaf on $((U \smallsetminus N)^d/\mathcal{G}_d)^I$.

3.12. We recall some facts on the fundamental group of $((U \smallsetminus N)^d/\mathcal{G}_d)^I$. Maintain the assumptions of 3.11 and let $\bar{x}$ be a geometric point of $(U \smallsetminus N)^I$. Then $(U \smallsetminus N)^{d\times I}$ is a connected finite Galois cover of $((U \smallsetminus N)^d/\mathcal{G}_d)^I$ with Galois group $\mathcal{G}_d^I$, so we get a short exact sequence of topological groups

$$1 \to \pi_1((U \smallsetminus N)^{d\times I}, \Delta_{pr}(\bar{x})) \to \pi_1(((U \smallsetminus N)^d/\mathcal{G}_d)^I, \Delta_{pr}(\bar{x})) \to \mathcal{G}_d^I \to 1,$$

where $pr : d \times I \to I$ denotes the projection map. Because the image of $\Delta_{pr}(\bar{x})$ in $(((U \smallsetminus N)^d/\mathcal{G}_d)^I$ evidently has automorphism group $\mathcal{G}_d^I$ corresponding to the Galois action, said image is isomorphic to the stack-theoretic quotient $\bar{x}/\mathcal{G}_d^I$. Hence the above short exact sequence splits via $\mathcal{G}_d^I = \pi_1(\bar{x}/\mathcal{G}_d^I, \bar{x}) \to \pi_1(((U \smallsetminus N)/\mathcal{G}_d)^I, \Delta_{pr}(\bar{x}))$. In the resulting identification

$$\pi_1(((U \smallsetminus N)^d/\mathcal{G}_d)^I) = \mathcal{G}_d^I \times \pi_1(((U \smallsetminus N)^{d\times I}),$$

note that the action of $\mathcal{G}_d^I$ corresponds to the $\mathcal{G}_d^I$-action of $\mathcal{G}_d^I$-equivariant smooth $E$-sheaves on $(U \smallsetminus N)^{d\times I}$.

Observe that $\mathcal{G}_d^I$ preserves and acts freely on the open subscheme $((U \smallsetminus N)^d)^I$ of $(U \smallsetminus N)^{d\times I}$. Therefore the stack-theoretic quotient $((U \smallsetminus N)^d)^I/\mathcal{G}_d^I = ((U \smallsetminus N)^d/\mathcal{G}_d)^I$ coincides with the scheme-theoretic quotient, which equals $(\text{Div}_{U \smallsetminus N}^d)^I$. Note that $((U \smallsetminus N)^d/\mathcal{G}_d)^I \to (\text{Div}_{U \smallsetminus N}^d)^I$ becomes an isomorphism when restricted to this open subscheme.

Finally, note that this entire discussion also applies to $(U \smallsetminus N)^I_K$ in place of $U \smallsetminus N$.

3.13. Finally, we conclude by describing Hecke correspondences in our setup. We start by defining the adelic action at infinite level. Write $\eta_{(d),I}$ for the generic point of $(\text{Div}_{U}^d)^I$, and write $\text{Sh}_{G,\infty,I}^{(d),(I_1,\ldots,I_k)}$ for the inverse limit

$$\lim_{N} \text{Sh}_{G,N,I}^{(d),(I_1,\ldots,I_k)} |_{\eta_{(d),I}},$$

where $N$ runs through finite closed subschemes of $X$. Write $\text{Sh}_{G,\infty,I,W}^{(d),(I_1,\ldots,I_k)}$ for the analogous inverse limit. By Proposition 3.3, we see that $\text{Sh}_{G,\infty,I}^{(d),(I_1,\ldots,I_k)} \to \text{Sh}_{G,I}^{(d),(I_1,\ldots,I_k)} |_{\eta_{(d),I}}$ is a pro-Galois cover whose Galois group is $G(\mathbb{Q})$.

We extend the left $G(\mathbb{Q})$-action to a left $G(\mathbb{A}_{\mathbb{Q}})$-action as follows. Note that the $S$-points of $\text{Sh}_{G,\infty,I}^{(d),(I_1,\ldots,I_k)}$ parameterize data consisting of

i) for all $i$ in $I$, a point $D_i$ of $\eta_{(d),I}(S)$,

ii) for all $0 \leq j \leq k - 1$, a $G \times S$-bundle $\mathcal{G}_j$ on $X \times S$ and an isomorphism

$$\psi_j : \mathcal{G}_j |_{\Pi_x(X \times S)^{\wedge}_{x \times S}} \overset{\sim}{\to} G \times \Pi_x(X \times S)^{\wedge}_{x \times S}$$

of $G \times \Pi_x(X \times S)^{\wedge}_{x \times S}$-bundles, where $x$ runs over closed points of $X$,
Let $g = (g_x)_x$ be an element of $G(\mathbb{A}_Q)$. For every closed point $x$ of $X$, we get an isomorphism

$$g_x : G \times ((X \times S)^\wedge_{x \times S} \setminus x \times S) \xrightarrow{\sim} G \times ((X \times S)^\wedge_{x \times S} \setminus x \times S),$$

and for the cofinitely many $x$ such that $g_x$ lies in $G(\mathcal{O}_x)$, this extends to an isomorphism over $(X \times S)^\wedge_{x \times S}$. For the finitely many other $x$, the Beauville–Laszlo theorem enables us to use $g_x \circ \psi_j|((x \times S)^\wedge_{x \times S} \setminus x \times S)$ to glue $G_j|((x \times S)^\wedge_{x \times S})$ with the trivial bundle on $(X \times S)^\wedge_{x \times S}$.

We apply this to each of the finitely many other $x$. This yields a $G \times S$-bundle $g \cdot G_j$ on $X \times S$ along with an isomorphism $g \cdot \psi_j$ as in ii) such that, for every closed point $x$ of $X$, we have a commutative square

$$G \times ((X \times S)^\wedge_{x \times S} \setminus x \times S) \xrightarrow{g_x} G \times ((X \times S)^\wedge_{x \times S} \setminus x \times S),$$

This procedure is compatible with iii), and it yields a $G(\mathbb{A}_Q)$-action on $	ext{Sh}^{(d)(I_1, \ldots, I_k)}_{G, \infty, I}$. Since the above square extends to isomorphisms over $(X \times S)^\wedge_{x \times S}$ when $g_x$ lies in $G(\mathcal{O}_x)$, we see that this indeed extends the $G(\mathcal{O}_Q)$-action. We also see that this extends and hence commutes with the $Z(\mathbb{A}_Q)$-action from 3.5. Finally, compatibility with iii) indicates that the $G(\mathbb{A}_Q)$-action preserves $	ext{Sh}^{(d)(I_1, \ldots, I_k)}_{G, \infty, I, W}$.

3.14. Our adelic action satisfies the following compatibilities. If $I_1, \ldots, I_k$ refines another ordered partition $I'_1, \ldots, I'_k'$ of $I$, the morphisms $\pi^{(I_1, \ldots, I_k)}_{(I'_1, \ldots, I'_k')}$ pass to the inverse limit in 3.13 and yield a morphism

$$\pi^{(I_1, \ldots, I_k)}_{(I'_1, \ldots, I'_k')} : \text{Sh}^{(d)(I_1, \ldots, I_k)}_{G, \infty, I} \rightarrow \text{Sh}^{(d)(I'_1, \ldots, I'_k')}_{G, \infty, I}.$$

This commutes with the $G(\mathbb{A}_Q)$-action, and we see that it also restricts to a morphism

$$\pi^{(I_1, \ldots, I_k)}_{(I'_1, \ldots, I'_k')} : \text{Sh}^{(d)(I_1, \ldots, I_k)}_{G, \infty, I} \rightarrow \text{Sh}^{(d)(I'_1, \ldots, I'_k')}_{G, \infty, I, W}.$$

Similarly, the morphisms $\alpha$ pass to the inverse limit in 3.13 and yield a morphism

$$\alpha : \text{Sh}^{(d \times I_1, \ldots, d \times I_k)}_{G, \infty, I} \rightarrow \text{Sh}^{(d)(I_1, \ldots, I_k)}_{G, \infty, I}.$$

This commutes with the $G(\mathbb{A}_Q)$-action, and we see that it also restricts to a morphism

$$\alpha : \text{Sh}^{(d \times I_1, \ldots, d \times I_k)}_{G, \infty, I, W} \rightarrow \text{Sh}^{(d)(I_1, \ldots, I_k)}_{G, \infty, I, W}.$$

Note that the right $\mathcal{G}_d$-action on $\text{Sh}^{(d \times I_1, \ldots, d \times I_k)}_{G, \infty, I}$ and hence $\text{Sh}^{(d \times I_1, \ldots, d \times I_k)}_{G, \infty, I, W}$ commutes with the $G(\mathbb{A}_Q)$-action.

3.15. From here, we obtain an action of the Hecke algebra by correspondences as follows. Write $K_{G,N}$ for the compact open subgroup $\ker(G(\mathcal{O}_Q) \to G(N))$ of $G(\mathcal{O}_Q)$, and write $\mathfrak{H}_{G,N}$ for the ring of finitely-supported $E$-valued functions on $K_{G,N} \setminus G(\mathbb{A}_Q)/K_{G,N}$, where multiplication is given by convolution with respect to the Haar measure on $G(\mathbb{A}_Q)$ for which $K_{G,N}$ has measure 1. For any
closed point \( x \) of \( X \), write \( \mathcal{H}_{G,x} \) for the analogous ring of finitely-supported \( E \)-valued functions on \( G(\mathcal{O}_x) \setminus G(Q_x)/G(\mathcal{O}_x) \). Recall that \( \mathcal{H}_{G,N} \) contains the restricted tensor product \( \bigotimes'_u \mathcal{H}_{G,u} \), where \( u \) runs over closed points of \( X \setminus N \).

Let \( g \) be in \( G(\mathbb{A}_Q) \). The \( G(\mathbb{A}_Q) \)-action from \( \text{[3.13]} \) yields a finite étale correspondence

\[
\text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,\infty,\eta} \times (K_{G,N} \cap g^{-1} K_{G,N} g) \xrightarrow{g} \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,\infty,\eta} \cap K_{G,N}.
\]

using the fact that \( \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,\infty,\eta} / K_{G,N} = \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,N,\eta} \). By sending the indicator function of \( K_{G,N} g K_{G,N} \) to the above correspondence, we obtain a ring homomorphism from \( \mathcal{H}_{G,H} \) to the ring of \( E \)-valued finite étale correspondences on \( \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,N,\eta} \). We similarly obtain finite étale correspondences on \( \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,N,\eta} \), and because the \( G(\mathbb{A}_Q) \)-action commutes with the \( Z(\mathbb{A}_Q) \)-action, we also obtain analogous correspondences on \( \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,N,\eta} \).

We see from \( \text{[3.14]} \) that our correspondences are compatible with \( \pi_{(I_1,\ldots,I_k)}^{d(1,\ldots,I_k)} \) and \( \alpha \). In the \( d = 1 \) setting, proper base change shows that they induce an action of \( \mathcal{H}_{G,N} \) on \( \mathcal{H}_{G,N,\eta}^{d(1,\ldots,I_k)}[14, \S 4.1] \).

3.16. \textbf{Remark}. Let \( N(g) \) be a finite set of closed points \( x \) of \( X \) containing those for which \( g \) does not lie in \( G(\mathcal{O}_x) \). Note that the construction in \( \text{[3.13]} \) similarly yields a left \( (\prod_{x \in N(g)} G(\mathcal{O}_x)) \times (\prod_{x \in N(g)} G(Q_x)) \)-action on

\[
\lim_{N} \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,N,\eta} \sqcup (\text{Div}^{d}_{X \setminus N(g)}[\div])',
\]

where \( N \) runs through finite closed subschemes of \( X \) supported on \( N(g) \), that extends the left \( G(\mathbb{O}_Q) \)-action. Therefore the construction in \( \text{[3.15]} \) can naturally be extended to a finite étale correspondence on \( \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,N,\eta} \times (\text{Div}^{d}_{X \setminus (N \cup N(g))}[\div]) ', which restricts to one on \( \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,N,\eta} \times (\text{Div}^{d}_{U \setminus (N \cup N(g))}[\div])' \). Quotienting by \( \Xi \) yields a finite étale correspondence on \( \text{Sh}^{(d)(I_1,\ldots,I_k)}_{G,N,\eta} / (\text{Div}^{d}_{U \setminus (N \cup N(g))}[\div])' \).

4. \textbf{Weil groups and partial Frobenii}

In positive characteristic algebraic geometry, the formation of fundamental groups rarely commutes with taking products—even over an algebraically closed field. One can remedy this by asking for additional structure: namely, partial Frobenius morphisms. In this section, we discuss this general phenomenon, as well as how it arises in the setting of moduli spaces of shtukas.

First, we define partial Frobenii and package their action into the \textit{Frobenius–Weil group}, which generalizes the relation between Weil sheaves and Weil groups. Next, we recall Drinfeld’s lemma and explain the link between partial Frobenii and Frobenius elements in the Weil group. We then describe how partial Frobenii arise in the setting of symmetrized moduli spaces of shtukas. In the usual, unsymmetrized case, their relative cohomology over \( (U \setminus N)^{!} \) is only ind-smooth, but nonetheless work of Xue \([20]\) shows that they still satisfy the conclusion of Drinfeld’s lemma. The resulting action of a product of Weil groups is crucial for the results of this paper.
4.1. We begin by defining the Frobenius–Weil group. Let $I$ be a finite set, and let $\mathcal{X}$ be a prestack over $k$. For any subset $I_1$ of $I$, write $\text{Frob}_{I_1} : \mathcal{X}^I \rightarrow \mathcal{X}^I$ for the product $\left( \prod_{i \in I_1} \text{Frob}_X \right) \times \left( \prod_{i \in I \setminus I_1} \text{id}_X \right)$. Note that $\text{Frob}_{I_1}$ and $\text{Frob}_{I_2}$ commute for any subsets $I_1$ and $I_2$ of $I$, and the composition of the $\text{Frob}_{(i)}$ for all $i$ in $I$ equals the absolute $q$-Frobenius endomorphism of $\mathcal{X}$.

Now let $\mathcal{X}$ be a geometrically connected algebraic stack over $k$. Then $\text{Frob}_{(i),K}$ induces a continuous homomorphism $\pi_1(\mathcal{X}^I_K) \rightarrow \pi_1(\mathcal{X}^I_K)$. As absolute $q$-Frobenius induces the identity on $\pi_1(\mathcal{X}^I_K)$, this yields an isomorphism of topological groups and hence a continuous action of the discrete group $\mathbb{Z}^I$ on $\pi_1(\mathcal{X}^I_K)$.

**Definition.** Write $F\text{Weil}_I(\mathcal{X})$ for the semidirect product $\pi_1(\mathcal{X}^I_K) \rtimes \mathbb{Z}^I$. When $I$ is a singleton, write $\text{Weil}(\mathcal{X})$ instead.

4.2. Next, we describe some functoriality properties of Frobenius–Weil groups. Let $\zeta : I \rightarrow J$ be a map of finite sets, and let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of geometrically connected algebraic stacks over $k$. This induces a morphism $\mathcal{X}^J_K \rightarrow \mathcal{Y}^J_K$ and thus a continuous homomorphism $\pi_1(\mathcal{X}^J_K) \rightarrow \pi_1(\mathcal{Y}^J_K)$. Because absolute $q$-Frobenius is functorial, we see that this extends to a natural continuous homomorphism $F\text{Weil}_J(\mathcal{X}) \rightarrow F\text{Weil}_I(\mathcal{Y})$.

By applying this to $\zeta : \{i\} \rightarrow I$ and $f = \text{id}_{\mathcal{X}}$, we obtain a continuous homomorphism

$$F\text{Weil}_I(\mathcal{X}) \rightarrow \text{Weil}(\mathcal{X})$$

for all $i$ in $I$. In particular, we get a continuous homomorphism

$$F\text{Weil}_I(\mathcal{X}) \rightarrow \text{Weil}(\mathcal{X})^I,$$

which is surjective since $\pi_1(\mathcal{X}^I_K) \rightarrow \pi_1(\mathcal{X}^I_K)^I$ is surjective.

4.3. We now recall the monodromy interpretation of Frobenius–Weil groups as well as Drinfeld’s lemma. Let $\mathcal{X}$ be a geometrically connected algebraic stack over $k$. By restricting to $\pi_1(\mathcal{X}^I_K)$ and separately considering the action of $\mathbb{Z}^I$, we see that finite-dimensional continuous representations of $F\text{Weil}_I(\mathcal{X})$ over $E$ are equivalent to smooth $E$-sheaves $\mathcal{M}$ on $\mathcal{X}^I_K$ equipped with isomorphisms $\overline{F}_{(i)} : \text{Frob}_{(i),K}^* \mathcal{M} \simeq \mathcal{M}$ such that

$$\overline{F}_{(i')} \circ \text{Frob}_{(i')}^* \circ \overline{F}_{(i)} = \overline{F}_{(i)} \circ \text{Frob}_{(i),K}^* \circ \overline{F}_{(i')}$$

for all $i$ and $i'$ in $I$. These $\overline{F}_{(i)}$ are called partial Frobenius morphisms. A similar equivalence holds for ind-smooth $E$-sheaves on $\mathcal{X}^I_K$ equipped with partial Frobenii, and precomposing with the continuous homomorphisms from $4.2$ corresponds to pullback.

Suppose that $\mathcal{X}$ is a noetherian scheme. Then, as in the proof of [18, Theorem 3.2], we see that [14, Lemme 8.11] and [18, Lemma 3.3] imply that any finite-dimensional continuous representation of $F\text{Weil}_I(\mathcal{X})$ over $E$ factors through $\text{Weil}(\mathcal{X})^I$ via the continuous homomorphism $F\text{Weil}_I(\mathcal{X}) \rightarrow \text{Weil}(\mathcal{X})^I$ from $4.2$.

**Remark.** Without the geometrically connected assumption on $\mathcal{X}$, we expect some form of 4.3 to hold with a groupoid version of Definition 4.1. For example, see the proof of [18, Theorem 3.2] when $\mathcal{X}$ is a scheme over $k$. 

4.4. Partial Frobenius morphisms and Frobenius elements in the Weil group are related as follows. Maintain the assumptions of 4.3. Suppose that \( \mathcal{X} \) is also geometrically integral, write \( \eta_I \) for the generic point of \( \mathcal{X}^I \), and choose a geometric point \( \eta_I \) lying over \( \eta_I \). Note that \( \mathcal{X}^I \) remains integral, and \( \eta_I \) lifts to a geometric generic point of \( \mathcal{X}^I_k \). Let \( k' \) be a finite extension of \( k \) with degree \( r \), and let \( \underline{x} = (x_i)_{i \in I} \) be a point of \( \mathcal{X}^I(k') \). Choose a geometric point \( \underline{x} \) lying over \( \underline{x} \) as well as an étale path \( \eta_I \sim \underline{x} \). We see that \( \underline{x} \) also lifts to a geometric point of \( \mathcal{X}^I_k \).

For any \( i \) in \( I \), write \( \psi_i : \mathcal{X}^I \to \mathcal{X}^I \) for projection onto the \( i \)-th factor. This induces an étale path \( \psi_i(\eta_I) \sim \underline{x}_i \) and hence a continuous homomorphism \( \text{Weil}(x_i) \to \text{Weil}(\mathcal{X}) \), where we form \( \text{Weil}(x_i) \) using absolute \( q^r \)-Frobenius instead of absolute \( q \)-Frobenius as in Definition 4.1. Write \( \gamma_{x_i} \) for the image of the generator of \( \mathbb{Z} = \text{Weil}(x_i) \) in \( \text{Weil}(\mathcal{X}) \).

Let \( \mathcal{M} \) be a smooth \( E \)-sheaf on \( \mathcal{X}^I_k \) equipped with partial Frobenii \( \mathcal{F}_{\{i\}} \). Then we get a specialization isomorphism \( \mathcal{M} \mid \underline{x} \sim \mathcal{M} \mid \eta_I \), and 4.3 endows \( \mathcal{M} \mid \eta_I \) with an action of \( \text{Weil}(\mathcal{X})^I \). On the other hand, since \( \text{Frob}_{\{i\}}^* \) fixes \( \underline{x} \), we see that

\[
\mathcal{F}_{\{i\}} \circ \text{Frob}_{\{i\}}^*(\mathcal{F}_{\{i\}}) \circ \cdots \circ \text{Frob}_{\{i\}}^{r-1,*}(\mathcal{F}_{\{i\}})
\]

restricts to an action on \( \mathcal{M} \mid \underline{x} \).

**Proposition.** Under the isomorphism \( \mathcal{M} \mid \underline{x} \sim \mathcal{M} \mid \eta_I \), this corresponds to the action of \( \gamma_{x_i} \) in the \( i \)-th entry of \( \text{Weil}(\mathcal{X})^I \).

**Proof.** By passing to finite-dimensional continuous representations of \( \text{Weil}(\mathcal{X})^I \), we obtain a surjection \( \bigotimes_{i \in I} \mathcal{E}_i \to \mathcal{M} \) that is compatible with partial Frobenii, where the \( \mathcal{E}_i \) are Weil \( E \)-sheaves on \( \mathcal{X} \). Therefore it suffices to prove the analogous claim for \( \bigotimes_{i \in I} \mathcal{E}_i \). The identification \( \bigotimes_{i \in I} \mathcal{E}_i \mid \underline{x} = \bigotimes_{i \in I} (\mathcal{E}_i \mid \eta_I) \) is compatible with partial Frobenii, and the identification \( \bigotimes_{i \in I} (\mathcal{E}_i \mid \eta_I) \) is compatible with the \( \text{Weil}(\mathcal{X})^I \)-action, so it suffices to prove the analogous claim for \( \mathcal{E}_i \) and \( I = \{i\} \). But \( \mathbb{Z} = \text{Weil}(x_i) \to \text{Weil}(\mathcal{X}) \) sends the generator of \( \mathbb{Z} \) to \( r \) times the generator of \( \mathbb{Z} \subseteq \text{Weil}(\mathcal{X}) \), and this image acts via

\[
\mathcal{F}_{\{i\}} \circ \text{Frob}_{\{i\}}^*(\mathcal{F}_{\{i\}}) \circ \cdots \circ \text{Frob}_{\{i\}}^{r-1,*}(\mathcal{F}_{\{i\}})
\]

by definition. \( \square \)

4.5. At this point, we specialize to the setting of moduli spaces of shtukas. Consider the morphism \( \text{Fr}_{1,\mathcal{X},I}^{(d)(I_1,\ldots,I_k)} : \text{Sh}_{G,N,I}^{(d)(I_2,\ldots,I_k)} \to \text{Sh}_{G,N,I}^{(d)(I_2,\ldots,I_k)} \) that sends

\[
((D_i)_{i \in I},(G_0,\psi_0)) \mapsto (G_1,\psi_1) \leftarrow \cdots \leftarrow (G_{k-1},\psi_{k-1}) \leftarrow (\tau G_0,\tau \psi_0)
\]

to

\[
((D_i)_{i \in I},(\tau D_i)_{i \in I},(G_1,\psi_1)) \leftarrow (G_2,\psi_2) \leftarrow \cdots \leftarrow (\tau G_0,\tau \psi_0) \leftarrow (\tau G_1,\tau \psi_1).
\]

Note that \( \text{Fr}_{1,\mathcal{X},I}^{(d)(I_1,\ldots,I_k)} \) commutes with the action of \( \text{Bun}_{Z,N}(k) \) and hence \( \Xi \) on both sides. Furthermore, for any finite \( \mathcal{G}_I \)-stable and \( \Gamma_{Q,I}^{dx} \)-stable subset \( \Omega \) of \( X_+^s(T)^{d \times I} \), we see that \( \text{Fr}_{1,\mathcal{X},I}^{(d)(I_1,\ldots,I_k)} \) sends \( \text{Sh}_{G,N,I,\Omega}^{(d)(I_2,\ldots,I_k)} \mid (\text{Div}_{U,N}^d) \) to \( \text{Sh}_{G,N,I,\Omega}^{(d)(I_2,\ldots,I_k)} \mid (\text{Div}_{U,N}^d) \).
Observe that the square

\[
\begin{array}{ccc}
\text{Sh}_{G,N,I}^{(d)(I_1,...,I_k)} & \xrightarrow{\text{Fr}_{I_1,N,I}^{(d)(I_1,...,I_k)}} & \text{Sh}_{G,N,I}^{(d)(I_2,...,I_{k-1},I_1)} \\
P & & P \\
(\text{Div}_{X,N}^d)^I & \xrightarrow{\text{Frob}_{I_1}} & (\text{Div}_{X,N}^d)^I
\end{array}
\]

commutes. Furthermore, the composition

\[
\text{Sh}_{G,N,I}^{(d)(I_1,...,I_k)} \xrightarrow{\text{Fr}_{I_1,N,I}^{(d)(I_1,...,I_k)}} \text{Sh}_{G,N,I}^{(d)(I_2,...,I_{k-1},I_1)} \xrightarrow{\text{Fr}_{I_2,N,I}^{(d)(I_2,...,I_{k-1},I_1)}} \cdots \xrightarrow{\text{Fr}_{I_k,N,I}^{(d)(I_1,...,I_{k-1})}} \text{Sh}_{G,N,I}^{(d)(I_1,...,I_k)}
\]

equals \(\text{Frob} : \text{Sh}_{G,N,I}^{(d)(I_1,...,I_k)} \to \text{Sh}_{G,N,I}^{(d)(I_1,...,I_k)}\). Since \(\text{Frob}\) is a universal homeomorphism, this shows that \(\text{Fr}_{I_1,N,I}^{(d)(I_1,...,I_k)}\) is a universal homeomorphism too.

### 4.6. Our \(\text{Fr}_{I_1,N,I}^{(d)(I_1,...,I_k)}\) commute with the Hecke action.

To see this, note that the \(\text{Fr}_{I_1,N,I}^{(d)(I_1,...,I_k)}\) pass to the inverse limit in \(\text{3.13}\) and yield a morphism \(\text{Fr}_{I_1,N,I}^{(d)(I_1,...,I_k)} : \text{Sh}_{G,N,I}^{(d)(I_1,...,I_k)} \to \text{Sh}_{G,\infty,I}^{(d)(I_1,...,I_k)}\). This commutes with the \(G(\mathcal{A}_Q)\)-action, and we see that it also restricts to a morphism

\[
\text{Fr}_{I_1,N,I}^{(d)(I_1,...,I_k)} : \text{Sh}_{G,\infty,I,W}^{(d)(I_1,...,I_k)} \to \text{Sh}_{G,\infty,I,W}^{(d)(I_2,...,I_{k-1},I_1)}.
\]

Therefore the finite étale correspondences from \(\text{3.13}\) and Remark \(\text{3.16}\) commute with \(\text{Fr}_{I_1,N,I}^{(d)(I_1,...,I_k)}\).

### 4.7. In the usual, unsymmetrized case, we now convert our \(\text{Fr}_{d \times I_1,N,d \times I}^{(d \times I_1,...,d \times I_k)}\) into partial Frobenius morphisms in the sense of \(\text{4.3}\). Let \(W\) be in \(\text{Rep}_F((G^d)^{d \times I})\), and suppose that \(W \circ \sigma^* = W\) for all \(\sigma \in \mathcal{G}_d^I\). Write \(a_1\) for \(\text{Fr}_{d \times I_1,N,d \times I}^{(d \times I_1,...,d \times I_k)}\). Then \(\text{4.5}\) and the proof of \([14]\) Proposition \(\text{3.3}\) yield a canonical isomorphism

\[
a_1^{-1} \mathcal{F}_{N,d \times I,W,(\Phi)} \cong \mathcal{F}_{N,d \times I,W,(\Phi)}.
\]

In addition, there exists a dominant coweight \(\lambda(W)\) of \(GL_r\) such that \([14]\) p. 868

\[
(\text{Fr}_{d \times I_1,N,d \times I}^{(d \times I_1,...,d \times I_k)})^{-1}(\text{Sh}_{G,N,d \times I,W}^{(d \times I_1,...,d \times I_k) \leq \mu}|(U \setminus N)^{d \times I}) \subseteq \text{Sh}_{G,N,d \times I,W}^{(d \times I_1,...,d \times I_k) \leq \mu + \lambda(W)|(U \setminus N)^{d \times I}}.
\]

This allows us to form the commutative diagram

\[
\begin{array}{ccc}
\text{Sh}_{G,N,d \times I,W}^{(d \times I_2,...,d \times I_k,d \times I_1) \leq \mu}|(U \setminus N)^{d \times I} & \xrightarrow{\text{Fr}_{d \times I_1}} & \text{Sh}_{G,N,d \times I,W}^{(d \times I_1,...,d \times I_k) \leq \mu + \lambda(W)||(U \setminus N)^{d \times I} \\
P & & P \\
(U \setminus N)^{d \times I} & \xrightarrow{\text{Frob}_{d \times I}} & (U \setminus N)^{d \times I}
\end{array}
\]

7Now \([14]\) Proposition \(\text{3.3}\) only treats the case of split \(G\). However, it extends to the general case, which is already implicitly used in \([14]\) §12.
After quotienting the top row by $\Xi$ and using the isomorphism $a^*_{\ell}(d_{xI_2,\ldots,d_{xI_6,d_{xI_1}}}) \sim F_{N,d_{xI},W,\Xi}$, this forms a cohomological correspondence and thus induces a morphism
\[
F_{d_{xI}_1} : \text{Frob}^*_{d_{xI}_1} \mathcal{H}^{\leq \mu}_{N,d_{xI},W,\Xi} \to \mathcal{H}^{\leq \mu+\lambda(W)}_{N,d_{xI},W,\Xi}
\]
of constructible sheaves on $(U \setminus N)^{d_{xI}}$. Taking $p$-th cohomology as well as the colimit over dominant coweights $\mu$ of $\text{GL}_r$ yields a morphism
\[
F_{d_{xI}_1} : \text{Frob}^*_{d_{xI}_1} \mathcal{H}_{N,d_{xI},W,\Xi}^p \to \mathcal{H}_{N,d_{xI},W,\Xi}^p
\]
of ind-constructible $E$-sheaves on $(U \setminus N)^{d_{xI}}$. By Theorem 2.12(b), we see that $F_{d_{xI}_1}$ is independent of the ordered partition $I_1, \ldots, I_k$. In particular, reordering the $I_1, \ldots, I_k$ allows us to see that
\[
F_{d_{xI}_1} \circ \text{Frob}^*_{d_{xI}_1}(F_{d_{xI}_1}) = F_{d_{xI}_1} \circ \text{Frob}^*_{d_{xI}_1}(F_{d_{xI}_1}).
\]
In the $d = 1$ setting, Theorem 4.6 implies that we have
\[
h \circ F_{d_{xI}_1}|_{\eta_1} = F_{d_{xI}_1}|_{\eta_1} \circ \text{Frob}^*_{d_{xI}_1}(h)
\]
for all $h$ in $\mathfrak{h}_{G,N}$.

Next, we descend our partial Frobenii to the sheaves $\mathcal{H}^{(d)}_{N,1,W,\Xi}$. Maintain the assumptions of 4.7 and observe that the cohomological correspondence from 4.7 is equivariant with respect to the $\mathfrak{H}^I_{d}$-action. Thus the pullback $\text{Frob}^*_{d_{xI}_1} \mathcal{H}^{(d)}_{N,d_{xI},W,\Xi}$ inherits an $\mathfrak{H}^I_{d}$-equivariant structure analogous to the one on $\mathcal{H}^{(d)}_{N,d_{xI},W,\Xi}$ from 3.11 and $F_{d_{xI}_1}$ preserves these structures. We have a commutative diagram
\[
\begin{array}{ccc}
((U \setminus N)^d/\mathfrak{G}_d)^I & \longrightarrow & (U \setminus N)^{d_{xI}} \\
\downarrow \text{Frob}_{d_{xI}_1} & & \downarrow \text{Frob}_{d_{xI}_1} \\
((U \setminus N)^d/\mathfrak{G}_d)^I & \longrightarrow & (U \setminus N)^{d_{xI}}
\end{array}
\]
so assuming that $X$ is geometrically connected over $k$, we can identify the descent of
\[
\text{Frob}^*_{d_{xI}_1} \mathcal{H}^{(d)}_{N,d_{xI},W,\Xi}
\]
to $((U \setminus N)^d/\mathfrak{G}_d)^I$ with $\text{Frob}^*_{d_{xI}_1} \mathcal{H}^{(d)}_{N,1,W,\Xi}$. Therefore $F_{d_{xI}_1}$ descends to a morphism
\[
F_{(d)I} : \text{Frob}^*_{(d)I} \mathcal{H}^{(d)}_{N,1,W,\Xi} \sim \mathcal{H}^{(d)}_{N,1,W,\Xi}
\]
of ind-smooth $E$-sheaves on $((U \setminus N)^d/\mathfrak{G}_d)^I$. Now 4.5 indicates that the composition
\[
F_{(d)I} \circ \text{Frob}^*_{I_k}(F_{(d)I_{k-1}}) \circ \text{Frob}^*_{I_{k-1} \cup I_k}(F_{(d)I_{k-2}}) \circ \cdots \circ \text{Frob}^*_{I_{2\cup \cdots \cup I_k}}(F_{(d)I_1})
\]
equals the canonical isomorphism $\text{Frob}^* \mathcal{H}^{(d)}_{N,1,W,\Xi} \sim \mathcal{H}^{(d)}_{N,1,W,\Xi}$. As 4.7 shows that the $F_{d_{xI}_1}$ and hence $F_{(d)I_j}$ commute, this implies that $F_{(d)I_k}$ is also an isomorphism.

Finally, we conclude by recalling work of Xue. Assume that $X$ is geometrically connected over $k$. By taking $\overline{F}_{(i)} = \overline{F}_{(i)}^I_{\Xi}$ from 4.8 and invoking Theorem 3.9, we can use 4.3 to obtain a representation $\mathcal{H}^{p}_{N,1,W,\Xi}$ of $F\text{Weil}_{I}(U \setminus N)$ over $E$ whose restriction to $\pi_1((U \setminus N)^I_{\Xi})$ is a
union of finite-dimensional continuous representations. We see from 4.7 that the partial Frobenius and $\mathcal{S}_{G,N}$-actions commute.

**Theorem** ([20 Proposition 6.0.10, Theorem 6.0.13]). The action of $\text{FWeil}_I(U \setminus N)$ on $\mathcal{H}_{N,I,W,\Xi}^p|_{\Pi}$ factors through an action of $\text{Weil}(U \setminus N)^I$ via the continuous homomorphism

$$\text{FWeil}_I(U \setminus N) \to \text{Weil}(U \setminus N)^I$$

from 4.2.

4.10. **Remark.** In [20], Theorem 4.9 is obtained by first proving a similar statement for a variant of $\text{FWeil}$ that uses $\eta_I$ in place of $(U \setminus N)^I$, applying the resulting triviality of the action of

$$\ker(\text{FWeil}(\eta_I) \to \text{Weil}(\eta)^I)$$

to prove Theorem 3.9, and finally deducing Theorem 4.9 from this.

4.11. We use work of Xue to extend the relationship between partial Frobenii and Frobenius elements in the Weil group to our setting. Maintain the assumptions of 4.9. Let $k'$ be a finite extension of $k$ with degree $r$, and let $\xi$ be a point of $(U \setminus N)^I(k')$. Choose a geometric point $\overline{\xi}$ lying over $\xi$ as well as an étale path $\eta_{I\xi} \sim \overline{\xi}$. For any $i$ in $I$, write $\gamma_{x_i}$ for the resulting element of $\text{Weil}(U \setminus N)$ as in 4.4.

By Theorem 3.9, we get a specialization isomorphism $\mathcal{H}_{N,I,W,\Xi}^p|_\Xi \sim \mathcal{H}_{N,I,W,\Xi}^p|_{\Pi}$. As in 4.4, we obtain an action of

$$F_{\{i\}} \circ \text{Frob}_{(i)}^{*}(F_{\{i\}}) \circ \cdots \circ \text{Frob}_{(i)}^{r-1,*}(F_{\{i\}})$$

on $\mathcal{H}_{N,I,W,\Xi}^p|_\Xi$, where we can use $F_{\{i\}}$ instead of $F_{\{i\}}$ because our $E$-sheaves and their morphisms are already defined over $k$.

**Proposition.** Under the isomorphism $\mathcal{H}_{N,I,W,\Xi}^p|_\Xi \sim \mathcal{H}_{N,I,W,\Xi}^p|_{\Pi}$, this corresponds to the action of $\gamma_{x_i}$ in the $i$-th entry of $\text{Weil}(U \setminus N)^I$.

**Proof.** Now [20] Lemma 6.0.9] and Theorem 3.9 indicate that $\mathcal{H}_{N,I,W,\Xi}^p$ is a union of ind-smooth $E$-subsheaves $\mathcal{M}$ over $(U \setminus N)^I$, where $\mathcal{M}|_{\Pi}$ is preserved by $\bigotimes_{i \in I} \mathcal{S}_{G,u_i}$ for some closed points $(u_i)_{i \in I}$ of $U \setminus N$, the $\mathcal{M}|_{\Pi}$ is finitely generated over $\bigotimes_{i \in I} \mathcal{S}_{G,u_i}$, and $\mathcal{M}$ is preserved isomorphically by the $F_{\{i\}}$. So it suffices to prove the analogous claim for $\mathcal{M}$.

Let $m$ be a maximal ideal of $\bigotimes_{i \in I} \mathcal{S}_{G,u_i}$, and let $n$ be a non-negative integer. Since the $\mathcal{S}_{G,u_i}$ are finitely generated $E$-algebras, we see that $(\mathcal{M}|_{\Pi})/m^n$ corresponds to a smooth $E$-sheaf on $(U \setminus N)^I$ equipped with partial Frobenii. Then Proposition 4.4 shows that the analogous claim holds for this smooth $E$-sheaf.

Because $\bigotimes_{i \in I} \mathcal{S}_{G,u_i}$ is noetherian, we have an injection $\mathcal{M}|_{\Pi} \to \bigoplus_{m} (\mathcal{M}|_{\Pi})^\wedge_m$, where $m$ runs over maximal ideals of $\bigotimes_{i \in I} \mathcal{S}_{G,u_i}$, and $(-)^\wedge_m$ denotes $m$-adic completion. The partial Frobenii and $\bigotimes_{i \in I} \mathcal{S}_{G,u_i}$-actions commute, so this injection is $\text{Weil}(U \setminus N)^I$-equivariant. Similarly, we obtain a injective map $\mathcal{M}|_{\Xi} \to \bigoplus_{m} (\mathcal{M}|_{\Xi})^\wedge_m$ that preserves the action of

$$F_{\{i\}} \circ \text{Frob}_{(i)}^{*}(F_{\{i\}}) \circ \cdots \circ \text{Frob}_{(i)}^{r-1,*}(F_{\{i\}})$$

Therefore the analogous claim for the sheaves corresponding to the $(\mathcal{M}|_{\Pi})/m^n$ implies the claim for $\mathcal{M}$, as desired. □
5. The plectic conjecture

In this section, we prove our main theorems on the cohomology of (usual, unsymmetrized) moduli spaces of shtukas. For this, we begin by recalling the setup of Weil restrictions from \cite{[1]} which provides one incarnation of the conjectured plectic diagram from \cite{[16]} (1.3)). This yields a connection between unsymmetrized shtukas for $G$ and symmetrized shtukas for $H$. We then use the link between symmetrized and unsymmetrized shtukas for $H$ to prove Theorem A. We then use the Hecke compatibility of this relation to prove Theorem B, and we conclude by explicating our constructions to prove Theorem C.

5.1. We start by describing the Hecke stack incarnation of the \textit{plectic diagram}. Recall the notation of \cite{[1]} and write $d$ for the degree of $m$. Since $Y$ and hence $\Div_Y^d$ is proper, the morphism $m^{-1}: X \to \Div_Y^d$ sending $x \mapsto m^{-1}(\Gamma_x)$ is proper as well. As $m^{-1}$ is also a monomorphism, we see that it is a closed immersion. Because $m$ is étale over $U$, we see that $m^{-1}$ sends $U$ to $\Div_Y^{d,0}$.

For any $0 \leq j_0 \leq k$, recall the morphisms $p_{j_0}$ and $p$ from Definition \ref{2.1}

\begin{proposition}
We have a natural Cartesian square
\[
\begin{array}{ccc}
\text{Hck}_{G,N,I}^{(I_1,...,I_k)} & \longrightarrow & \text{Hck}_{H,M,I}^{(d)(I_1,...,I_k)} \\
\downarrow{(p_{j_0},p)} & & \downarrow{(p_{j_0},p)} \\
\text{Bun}_{G,N} \times (X \setminus N)^I & \xrightarrow{\times(m^{-1})^I} & \text{Bun}_{H,M} \times (\Div_Y^{d,I})^I.
\end{array}
\]
\end{proposition}

\begin{proof}
An $S$-point of $\text{Hck}_{G,N,I}^{(I_1,...,I_k)}$ consists of

i) for all $i$ in $I$, a point $x_i$ of $(X \setminus N)(S)$,

ii) for all $0 \leq j \leq k$, an object $(G_j, \psi_j)$ of $\text{Bun}_{G,N}(S)$,

iii) for all $1 \leq j \leq k$, an isomorphism $\phi_j: G_{j-1} \big|_{X \times S \setminus \sum_{i \in I_j} \Gamma_{x_i}} \cong G_{j} \big|_{X \times S \setminus \sum_{i \in I_j} \Gamma_{x_i}}$ with $\psi_{j} \circ \phi_{j} \big|_{N \times S} = \psi_{j-1}$.

By using the isomorphism $c$ and additionally applying \cite{[1]} to $R = X \times S \setminus \sum_{i \in I_j} \Gamma_{x_i}$, we see that ii) and iii) are equivalent to objects $(H_j, \psi'_j)$ of $\text{Bun}_{H,M}(S)$ and isomorphisms $\phi'_j: H_{j-1} \big|_{Y \times S \setminus \sum_{i \in I_j} m^{-1}(\Gamma_{x_i})} \cong H_{j} \big|_{Y \times S \setminus \sum_{i \in I_j} m^{-1}(\Gamma_{x_i})}$ with $\psi'_{j} \circ \phi'_{j} \big|_{M \times S} = \psi'_{j-1}$. Combined with i), this is precisely the data parametrized by the fiber product of $\text{Hck}_{H,M,I}^{(d)(I_1,...,I_k)}$ and $\text{Bun}_{G,N} \times (X \setminus N)^I$ over $\text{Bun}_{H,M} \times (\Div_Y^{d,I})^I$.
\end{proof}

5.2. Next, we proceed to the Beilinson–Drinfeld affine Grassmannian version of the plectic diagram. Take $N = \varnothing$ and $j_0 = k$ in Proposition \ref{5.1}. After pulling back along the $k$-point of $\text{Bun}_G \cong \text{Bun}_H$ corresponding to the trivial bundle, this yields a Cartesian square

\[
\begin{array}{ccc}
\text{Gr}_{G,I}^{(I_1,...,I_k)} & \longrightarrow & \text{Gr}_{H,I}^{(d)(I_1,...,I_k)} \\
\downarrow{p} & & \downarrow{p} \\
X^I & \xrightarrow{(m^{-1})^I} & (\Div_Y^d)^I.
\end{array}
\]
5.3. We now describe how the relative position stratification fits into the plectic diagram. Because $T = R_{F/Q} A$, we see that $X_\bullet(T)$ is isomorphic as a $\Gamma_Q$-module to
\[
\{ \varphi : \Gamma_Q \to X_\bullet(A) \mid \varphi(xg) = g^{-1}\varphi(x) \text{ for all } g \in \Gamma_F \text{ and } x \in \Gamma_Q \},
\]
whose $\Gamma_Q$-action is given by inverse left multiplication. Under this identification, $X_\bullet(T)^{\Gamma_Q}$ corresponds to the subset of functions $\varphi$ taking constant values in $X_\bullet(A)^{\Gamma_F}$. As $B = \prod_i C$, we see that $X_\bullet^+(T)$ corresponds to the subset of functions $\varphi$ that take values in $X_\bullet^+(A)$. After choosing representatives for $\Gamma_Q/\Gamma_F$ in $\Gamma_Q$ and enumerating them, we may identify $X_\bullet(T)$ with $X_\bullet(A)^d$ and $X_\bullet^+(T)$ with $X_\bullet^+(A)^d$.

Let $\Omega$ be a finite $\mathcal{G}^d_I$-stable and $\Gamma_F^{d \times I}$-stable subset of $X_\bullet^+(A)^{d \times I}$. In particular, $\Omega$ is also $\Gamma_Q^I$-stable when viewed as a subset of $X_\bullet^+(A)^I$. Hence we can form $\text{Gr}^{(d)(I_1,\ldots,I_k)}_{H,I,\Omega}\big|_{(\text{Div}^d_V)^I}$ and $\text{Gr}^{(I_1,\ldots,I_k)}_{G,I,\Omega}\big|_{U^I}$ as in 2.10. Note that the Cartesian square in 5.2 restricts to a Cartesian square
\[
\begin{array}{ccc}
\text{Gr}^{(I_1,\ldots,I_k)}_{G,I,\Omega}\big|_{U^I} & \longrightarrow & \text{Gr}^{(d)(I_1,\ldots,I_k)}_{H,I,\Omega}\big|_{(\text{Div}^d_V)^I} \\
\downarrow & & \downarrow \\
U^I & \longrightarrow & (\text{Div}^d_V)^I.
\end{array}
\]

**Remark.** Note that stability under $\Gamma_F^{d \times I}$ is a condition that is independent of our choice of representatives for $\Gamma_Q/\Gamma_F$ in $\Gamma_Q$. Similarly, stability under $\mathcal{G}^d_I$ is a condition that is independent of our enumeration of said representatives.

5.4. In our setup, intersection complexes on relative position strata correspond to the following representations. First, suppose that $\Omega$ equals $\prod_{i \in I} \Omega_i$, where the $\Omega_i$ are finite $\mathcal{G}_d$-stable and $\Gamma_F^d$-stable subsets of $X_\bullet^+(A)^d$. In particular, $\Omega_i$ is a finite disjoint union of $\Gamma_F^d$-orbits $O$. Then we can form $W_O$ as in 2.13 and we write $W_{\Omega_i,H}$ for the object $\bigoplus O W_O$ of $\text{Rep}_E((LH)^d)$. Note that the $\mathcal{G}_d$-stability of $\Omega_i$ implies that $W_{\Omega_i,H} \circ \sigma^* = W_{\Omega_i,H}$ for all $\sigma$ in $\mathcal{G}_d$. Finally, write $W_{\Omega,H}$ for the exterior tensor product $\Box_{i \in I} W_{\Omega_i,H}$, and recall that 2.13 indicates that $\mathcal{S}^{(d \times I_1,\ldots,d \times I_k)}_{d \times I,\Omega,H}$ equals the intersection complex of $\text{Gr}^{(d \times I_1,\ldots,d \times I_k)}_{H,d \times I,\Omega}\big|_{V^{d \times I}}$, with degree shifts normalized relative to $V^{d \times I}$.

By viewing $\Omega_i$ as a subset of $X_\bullet^+(T)$ instead, we see that it is $\Gamma_Q$-stable. Form $W_{\Omega_i,G}$ and $W_{\Omega,G}$ as above. We analogously see that $\mathcal{S}^{(I_1,\ldots,I_k)}_{I,\Omega,G}$ equals the intersection complex of $\text{Gr}^{(I_1,\ldots,I_k)}_{G,I,\Omega}\big|_{U^I}$, with degree shifts normalized relative to $U^I$.

More generally, let $\Omega$ be any finite $\mathcal{G}_d^d$-stable and $\Gamma_F^{d \times I}$-stable subset of $X_\bullet^+(A)^{d \times I}$. Then $\Omega$ is a finite union of subsets of the form considered above, and we write $W_{\Omega,H}$ and $W_{\Omega,G}$ for the corresponding direct sum of algebraic representations over $E$. We see that the above relation with intersection complexes continues to hold.
5.5. Finally, we arrive at the plectic diagram for moduli spaces of shtukas. By pulling back \[5.1\] along $\gamma$, we get an analogous Cartesian square

$$
\begin{array}{ccc}
\text{Sht}^{(I_1, \ldots, I_k)}_{G,N,I} & \longrightarrow & \text{Sht}^{(d)(I_1, \ldots, I_k)}_{H,M,I} \\
\downarrow p & & \downarrow p \\
(U \setminus N)^I & \longrightarrow & (\text{Div}^d_{V \setminus M})^I.
\end{array}
$$

Because $(m^{-1})^I$ is a locally closed immersion, we see that further restricting to \[5.4\] via $\gamma \circ \delta$ yields a Cartesian square

$$
\begin{array}{ccc}
\text{Sht}^{(I_1, \ldots, I_k)}_{G,N,I,W_{0,G}} | (U \setminus N)^I & \longrightarrow & \text{Sht}^{(d)(I_1, \ldots, I_k)}_{H,M,I,W_{0,H}} | (\text{Div}^d_{V \setminus M})^I \\
\downarrow p & & \downarrow p \\
(U \setminus N)^I & \longrightarrow & (\text{Div}^d_{V \setminus M})^I.
\end{array}
$$

Finally, note that the square

$$
\begin{array}{ccc}
\text{Sht}^{(I_1, \ldots, I_k)}_{G,N,I} & \longrightarrow & \text{Sht}^{(d)(I_1, \ldots, I_k)}_{H,M,I} \\
\downarrow \text{Fr} (I_1, \ldots, I_k) & & \downarrow \text{Fr} (d)(I_1, \ldots, I_k) \\
\text{Sht}^{(J_1, \ldots, J_k,J_1)}_{G,N,I} & \longrightarrow & \text{Sht}^{(d)(J_1, \ldots, J_k,J_1)}_{H,M,I}
\end{array}
$$

commutes.

5.6. Let us identify the coefficient sheaves in our cases of interest. Recall the morphism $\epsilon$ from \[3.8\] In the $d = 1$ setting, it induces an isomorphism from $\text{Sht}^{(I_1, \ldots, I_k)}_{G,N,I,W_{0,G}} | (U \setminus N)^I / \Xi$ to $\text{Gr}^{(I_1, \ldots, I_k)}_{G,I} | (U \setminus N)^I$ étale-locally by the proof of \[14\] Proposition 2.11. Therefore $\mathcal{F}^{(I_1, \ldots, I_k)}_{N,I,W_{0,G},\Xi} = \epsilon^* (\mathcal{F}^{(I_1, \ldots, I_k)}_{I,W_{0,G}})$ equals the intersection complex of $\text{Sht}^{(I_1, \ldots, I_k)}_{G,N,I,W_{0,G}} | (U \setminus N)^I / \Xi$, with degree shifts normalized relative to $(U \setminus N)^I$. As the center of $G$ equals the Weil restriction of the center of $H$, we can use $\Xi$ for both $G$ and $H$, and applying the above discussion to $H$ shows that $\mathcal{F}^{(d \times I_1, \ldots, d \times I_k)}_{M,d \times I,W_{0,H},\Xi}$ equals the intersection complex of $\text{Sht}^{(d \times I_1, \ldots, d \times I_k)}_{H,M,d \times I,W_{0,H}} | (V \setminus M)^{d \times I} / \Xi$, with degree shifts normalized relative to $(V \setminus M)^{d \times I}$.

Write $\mathcal{F}^{(d)(I_1, \ldots, I_k)}_{M,I,W_{0,H},\Xi}$ for the intersection complex of $\text{Sht}^{(d)(I_1, \ldots, I_k)}_{H,M,I,W_{0,H}} | (\text{Div}^d_{V \setminus M})^I / \Xi$, with degree shifts normalized relative to $(\text{Div}^d_{V \setminus M})^I$. In the commutative squares from \[5.5\] note that the horizontal arrows are locally closed immersions, up to universal homeomorphism. Therefore the pullback of $\mathcal{F}^{(d)(I_1, \ldots, I_k)}_{M,I,W_{0,H},\Xi}$ to $\text{Sht}^{(I_1, \ldots, I_k)}_{G,N,I,W_{0,G}} | U^I / \Xi$ equals $\mathcal{F}^{(I_1, \ldots, I_k)}_{N,I,W_{0,G},\Xi}$. Similarly, the commutative squares from \[3.10\] are finite generically étale, so the pullback of $\mathcal{F}^{(d)(I_1, \ldots, I_k)}_{M,I,W_{0,H},\Xi}$ to $\text{Sht}^{(d \times I_1, \ldots, d \times I_k)}_{H,M,d \times I,W_{0,H}} | (V \setminus M)^{d \times I} / \Xi$ equals $\mathcal{F}^{(d \times I_1, \ldots, d \times I_k)}_{M,d \times I,W_{0,H},\Xi}$.

5.7. Now that the plectic diagram relates unsymmetrized shtukas for $G$ to symmetrized shtukas for $H$, we leverage the connection between symmetrized and unsymmetrized shtukas for $H$ as

\[\text{Now \[14\] Proposition 2.11] only treats the case of split $G$. However, it extends to the general case, which is already implicitly used in \[14\] §12.}\]
follows. We henceforth assume that \( Y \) is geometrically connected over \( k \). Restricting the commutative square from \( 3.10 \) to \( (\text{Div}^d_{V \setminus M})^I \) yields a Cartesian square

\[
\begin{array}{ccc}
\text{Sh}_H^{(d)(I_1 \ldots, I_k)} & \xleftarrow{\alpha} & \text{Sh}_H^{(d \times I_1 \ldots, d \times I_k)} \\
\downarrow p & & \downarrow p \\
(\text{Div}^d_{V \setminus M})^I & \xleftarrow{\alpha} & ((V \setminus M)^d)^I.
\end{array}
\]

After quotienting by \( \Xi \) and applying proper base change, the identifications in \( 3.12 \) and \( 5.6 \) show that

\[
Rp^p_\Xi(\mathcal{F}_{M,I,W,H}^{(d)(I_1 \ldots, I_k)} | (\text{Div}^d_{V \setminus M})^I) = \mathcal{H}_{M,I,W,H}^{(d)p} \Xi | (\text{Div}^d_{V \setminus M})^I.
\]

The evident commutative diagram

\[
\begin{array}{ccc}
\text{Sh}_I^{(d)(I_1 \ldots, I_k)} & \xleftarrow{\alpha} & \text{Sh}_I^{(d \times I_1 \ldots, d \times I_k)} \\
\downarrow F_{I_1,M,I}^{(d)(I_1 \ldots, I_k)} & & \downarrow F_{I_1,M,I}^{(d \times I_1 \ldots, d \times I_k)} \\
\text{Sh}_I^{(d)(I_2 \ldots, I_k,I_1)} & \xleftarrow{\alpha} & \text{Sh}_I^{(d \times I_2 \ldots, d \times I_k,d \times I_1)}
\end{array}
\]

enables us to identify the morphism \( F_{(d)I_1} \) from \( 4.8 \) with the morphism induced by \( F_{I_1,M,I}^{(d)(I_1 \ldots, I_k)} \).

5.8. We now explain how the \textit{plectic homomorphism} arises naturally in our setup. Since \( V \setminus M \) is étale over \( U \setminus N \), our \( m^{-1} \) induces a morphism \( U \setminus N \to (V \setminus M)^d/\mathcal{G}_d \) by \( 3.12 \). Applying \( 4.2 \) to this morphism with \( \zeta = \text{id}_I \) yields a continuous homomorphism

\[
\text{FWeil}_I(U \setminus N) \to \text{FWeil}_I((V \setminus M)^d/\mathcal{G}_d).
\]

Next, recall that \( 3.12 \) identifies

\[
\pi_1((V \setminus M)^d/\mathcal{G}_d)^I) = \mathcal{G}_d^I \times \pi_1((V \setminus M)^{d \times I})^I,
\]

where \( \mathcal{G}_d^I \) acts via permutation on \( (V \setminus M)^{d \times I} \). As this commutes with \( \text{Frob}_{d \times (i)} \) for any \( i \) in \( I \), we see that

\[
\text{FWeil}_I((V \setminus M)^d/\mathcal{G}_d) = (\mathcal{G}_d^I \times \pi_1((V \setminus M)^{d \times I}) \times \mathbb{Z}^I = \mathcal{G}_d^I \times (\pi_1((V \setminus M)^{d \times I}) \times \mathbb{Z}^I),
\]

where \( \mathcal{G}_d^I \) acts trivially on \( \mathbb{Z}^I \). By considering the action of \( \text{Frob}_{(h,i)} \) for all \( (h,i) \) in \( I \), we get a continuous homomorphism

\[
\mathcal{G}_d^I \times (\pi_1((V \setminus M)^{d \times I}) \times \mathbb{Z}^I) \to \mathcal{G}_d^I \times (\pi_1((V \setminus M)^{d \times I}) \times \mathbb{Z}^{d \times I}) = \mathcal{G}_d^I \times \text{FWeil}_{d \times I}(V \setminus M)
\]

induced by \( \mathbb{Z}^I \to \mathbb{Z}^{d \times I} \), where \( \mathcal{G}_d^I \) acts via permutation on \( \mathbb{Z}^{d \times I} \). Finally, the naturality of \( 4.2 \) applied to \( \zeta = \sigma \) in \( \mathcal{G}_d^I \) indicates that the continuous homomorphism

\[
\text{FWeil}_{d \times I}(V \setminus M) \to \text{Weil}(V \setminus M)^{d \times I}
\]

is equivariant with respect to the \( \mathcal{G}_d^I \)-action. Therefore we obtain a continuous homomorphism

\[
\mathcal{G}_d^I \times \text{FWeil}_{d \times I}(V \setminus M) \to \mathcal{G}_d^I \times \text{Weil}(V \setminus M)^{d \times I} = (\mathcal{G}_d \times \text{Weil}(V \setminus M)^d)^I.
\]
5.9. We prove the following generalization of Theorem A. Applying 5.8 when \( I \) is a singleton yields a continuous homomorphism \( \text{Weil}(U \setminus N) \to \mathcal{G}_d \ltimes \text{Weil}(V \setminus M)^d \). Write \( \beta_{d \times I} \) for the generic point of \((V \setminus M)^{d \times I}\), and let \( \beta_{d \times I} \) be a geometric point lying over it.

**Theorem.** The action of \( \text{Weil}(U \setminus N)^I \) on \( \mathcal{H}_{N,I,W_{0,G}}^{p,\Omega,G,\Xi}|_\pi \) naturally extends to an action of \((\mathcal{G}_d \ltimes \text{Weil}(V \setminus M)^d)^I\) via the continuous homomorphism \( \text{Weil}(U \setminus N)^I \to (\mathcal{G}_d \ltimes \text{Weil}(V \setminus M)^d)^I \).

**Proof.** As \( \mathcal{H}_{N,I,W_{0,G}}^{p,\Omega,G,\Xi} \) equals \( \mathcal{R}_{\text{Weil}}^p(\mathfrak{S}_{N,I,W_{0,G}}^\Xi) \), proper base change and 5.6 show that \( \mathcal{H}_{N,I,W_{0,G}}^{p,\Omega,G,\Xi} \) equals the pullback of \( \mathcal{R}_{\text{Weil}}^p(\mathfrak{S}_{M,I,W_{0,H}}^\Xi) \) to \((U \setminus N)^I\). Now 5.7 identifies this with the pullback of \( \mathcal{H}^{(d),p}_{M,I,W_{0,H}} \) to \((U \setminus N)^I\), and by 4.3 we can use the ind-smoothness of \( \mathcal{H}^{(d),p}_{M,I,W_{0,H}} \) and the \( F_{d(i)} \) from 4.8 to obtain an action of \( \text{FWeil}_I((V \setminus M)^d/\mathcal{G}_d) \) on \( \mathcal{H}^{(d),p}_{M,I,W_{0,H}} \). By using an étale path \( \alpha(\beta_{d \times I}) \rightsquigarrow (m^{-1})^{(\eta_I)} \), we see from 5.5 and ind-smoothness that the action of \( \text{FWeil}_I((V \setminus M)^d/\mathcal{G}_d) \) arises from the action of \( \text{FWeil}_I((V \setminus M)^d/\mathcal{G}_d) \) on \( \mathcal{H}^{(d),p}_{M,I,W_{0,H}} \) via composing with

\[
\text{FWeil}_I((V \setminus M)^d/\mathcal{G}_d) \to \mathcal{G}_d \ltimes \text{FWeil}_{d \times I}(V \setminus M).
\]

Finally, the naturality of 4.2 for \( \zeta = \sigma \) in \( \mathcal{G}_d \) implies that the square

\[
\begin{array}{ccc}
\text{FWeil}_I(U \setminus N) & \longrightarrow & \mathcal{G}_d \ltimes \text{FWeil}_{d \times I}(V \setminus M) \\
\downarrow & & \downarrow \\
\text{Weil}(U \setminus N)^I & \longrightarrow & (\mathcal{G}_d \ltimes \text{Weil}(V \setminus M)^d)^I
\end{array}
\]

commutes. Since the vertical arrows are surjective, applying Theorem 4.9 to \( \mathcal{H}_{N,I,W_{0,G}}^{p,\Omega,G,\Xi}|_\pi \) and \( \mathcal{H}^{(d),p}_{M,d \times I,W_{0,H}} \) concludes the proof.

5.10. Using the Hecke compatibility of our constructions, we now prove the following generalization of Theorem B.

**Theorem.** The \((\mathcal{G}_d \ltimes \text{Weil}(V \setminus M)^d)^I\)-action on \( \mathcal{H}_{N,I,W_{0,G}}^{p,\Omega,G,\Xi}|_\pi \) from Theorem 5.9 commutes with the \( \mathcal{S}_{G,N}\)-action from 5.15.
Proof. Under the identification $G(\mathbb{A}_Q) = H(\mathbb{A}_F)$, we see that the compact open subgroup $K_{G,N}$ corresponds to $K_{H,M}$. This identifies $\mathcal{S}_G$ with $\mathcal{S}_H$. For any $g$ in $G(\mathbb{A}_Q)$, write $N(g)$ for the finite set of closed points of $X$ where $g$ does not lie in $G(\mathcal{O}_x)$, and write $M(g)$ for $m^{-1}(N(g))$. Remark 3.16 gives an associated finite étale correspondence on $\Sh_{G,N,I,W_\Omega,G}((U \setminus (N \cup N(g))) \setminus \Xi)$ over $(U \setminus (N \cup N(g)))^I$. By viewing $g$ as an element of $H(\mathbb{A}_F)$ instead, we obtain analogous correspondences on $\Sh_{H,M,I,W_\Omega,H}((\Div_{V,\Xi}((U \setminus (N \cup N(g))))^I \setminus \Xi)$ and $\Sh_{H,M,d \times I,W_\Omega,H}((V \setminus (M \cup M(g))))^d \setminus \Xi$. Because these correspondences are finite étale, their pullbacks preserve intersection complexes, so we obtain a cohomological correspondence on $\mathcal{F}^{(d)}_{M,I,W_\Omega,H,\Xi}$ after quotienting by $\Xi$.

Restricting 5.5 to $(\Div_{V,\Xi}((U \setminus (N \cup N(g))))^I$ yields a morphism

$$\Sh_{G,N,I,W_\Omega,G}((U \setminus (N \cup N(g)))^I \rightarrow \Sh_{H,M,I,W_\Omega,H}((\Div_{V,\Xi}((U \setminus (N \cup N(g))))^I).$$

Note that our correspondence on $\Sh_{G,N,I,W_\Omega,G}((U \setminus (N \cup N(g)))^I$ is precisely the pullback of our correspondence on $\Sh_{H,M,I,W_\Omega,H}((\Div_{V,\Xi}((U \setminus (N \cup N(g))))^I$ along the above morphism. Similarly, we see that our correspondence on

$$\Sh_{H,M,d \times I,W_\Omega,H}((V \setminus (M \cup M(g))))^d \setminus \Xi$$

equals the pullback of our correspondence on $\Sh_{H,M,I,W_\Omega,H}((\Div_{V,\Xi}((U \setminus (N \cup N(g))))^I$ along $\alpha$, up to universal homeomorphism.

The proof of Theorem 5.9 constructs the $(\mathcal{G}_d \cong \text{Weil}(V \setminus M)^d)^I$-action on $\mathcal{H}^p_{N,I,W_\Omega,G,\Xi}|_\mathcal{M}$ by identifying the latter with $\mathcal{H}^p_{M,d \times I,W_\Omega,H,\Xi}|_\mathcal{M}$ by $\alpha$. The above shows that, under this identification, the $\mathcal{S}_{G,N}$-action on $\mathcal{H}^p_{N,I,W_\Omega,G,\Xi}|_\mathcal{M}$ coincides with the $\mathcal{S}_{H,M}$-action on $\mathcal{H}^p_{M,d \times I,W_\Omega,H,\Xi}|_\mathcal{M}$. From here, the desired commutativity follows from 5.14 and 4.7.

5.11. Before turning to Theorem C, we need some notation on the splitting behavior of points along $Y \rightarrow X$. Let $k'$ be a finite extension of $k$ with degree $r$, and let $x = (x_i)_{i \in I}$ be a point of $(U \setminus N)^I(k')$ such that every $x_i$ splits completely in $V \setminus M$, i.e. the inverse image $m^{-1}(x_i)$ is a disjoint union of $d$ points $(y_{h,i})_{h=1}^d$ of $(V \setminus M)(k')$.

Because $(m^{-1})^I$ is a monomorphism, we get a Cartesian square

$$
\begin{array}{ccc}
x & \xrightarrow{(m^{-1})^I} & (m^{-1})^I(x) \\
\downarrow & & \downarrow \\
(U \setminus N)^I & \xrightarrow{(m^{-1})^I} & (\Div_{V,\Xi}((U \setminus N)^I(x))
\end{array}
$$

whose top arrow is an isomorphism. Therefore restricting 5.5 to $(m^{-1})^I(x)$ yields an isomorphism

$$
\Sh_{G,N,I,W_\Omega,G}(x) \xrightarrow{(m^{-1})^I \circ (m^{-1})^I} \Sh_{H,M,I,W_\Omega,H}(x).
$$

Note that further restriction yields an isomorphism $\Sh_{G,N,I,W_\Omega,G}(x) \xrightarrow{(m^{-1})^I} \Sh_{H,M,I,W_\Omega,H}(x)$.

As every $x_i$ splits completely in $V \setminus M$, we see that the preimage of $(m^{-1})^I(x)$ under $\alpha$ equals

$$
\{(y_{\sigma(h,i)})_{j \in [d],i \in I} \in (V \setminus M)^{d \times I}(k') \mid \sigma \in \mathcal{G}_d^I\},
$$
where each $(y_{\theta(h,i)})_{h \in [d], i \in I}$ maps isomorphically to $(m^{-1})^I(\xi)$ under $\alpha$. Write $y$ for the point $(y_{h,i})_{h \in [d], i \in I}$ of $(V \setminus M)^{d \times I}(k')$ induced by our enumeration of the $m^{-1}(x_i)$. Then pulling back along $(m^{-1})^I(\xi) \stackrel{\sim}{\leftarrow} y$ yields an isomorphism
\[
\text{Sh}^{(d)(I_1,\ldots,I_k)}_{H,M,I} \mid (m^{-1})^I(\xi) \stackrel{\sim}{\leftarrow} \text{Sh}^{(d \times I_1,\ldots,d \times I_k)}_{H,M,d \times I} \mid y,
\]
and further restriction yields a universal homeomorphism
\[
\text{Sh}^{(d)(I_1,\ldots,I_k)}_{H,M,I,W_{\Omega,H}} \mid (m^{-1})^I(\xi) \leftarrow \text{Sh}^{(d \times I_1,\ldots,d \times I_k)}_{H,M,d \times I,W_{\Omega,H}} \mid y.
\]

5.12. The above yields the following identifications. Our zig-zag of universal homeomorphisms
\[
\text{Sh}^{(I_1,\ldots,I_k)}_{G,N,I,W_{\Omega,G}} \mid x \sim \text{Sh}^{(d)(I_1,\ldots,I_k)}_{H,M,I,W_{\Omega,H}} \mid (m^{-1})^I(\xi) \leftarrow \text{Sh}^{(d \times I_1,\ldots,d \times I_k)}_{H,M,d \times I,W_{\Omega,H}} \mid y
\]
from 5.11 allows us to identify the étale cohomology of the left and right terms. As these morphisms are equivariant with respect to the action of $\Xi$, the same holds after quotienting by $\Xi$.

Choose a geometric point $\overline{x}$ lying over $x$. Now $\mathcal{F}^{(I_1,\ldots,I_k)}_{N,I,W_{\Omega,G}}$ restricts to the intersection complex on $\text{Sh}^{(I_1,\ldots,I_k)}_{G,N,I,W_{\Omega,G}} \mid \overline{x}/\Xi$, so proper base change indicates that $\mathcal{H}^p_{N,I,W_{\Omega,G},\Xi/\Xi}$ is naturally isomorphic to the $p$-th intersection cohomology with compact support of $\text{Sh}^{(I_1,\ldots,I_k)}_{G,N,I,W_{\Omega,G}} \mid \overline{x}/\Xi$ with coefficients in $E$. Analogous statements hold for $\text{Sh}^{(d)(I_1,\ldots,I_k)}_{H,M,I,W_{\Omega,H}} \mid (m^{-1})^I(\xi)/\Xi$ and $\text{Sh}^{(d \times I_1,\ldots,d \times I_k)}_{H,M,d \times I,W_{\Omega,H}} \mid y/\Xi$, in a manner compatible with the identifications induced by the above zig-zag.

5.13. Finally, we prove the following generalization of Theorem C. After choosing an étale path $\overline{\eta} \sim \overline{x}$, we get a specialization isomorphism $\mathcal{H}^p_{N,I,W_{\Omega,G},\Xi/\Xi} \sim \mathcal{H}^p_{N,I,W_{\Omega,G},\Xi/\Xi}$ by Theorem 3.9.

Applying $(m^{-1})^I$ to $\overline{\eta} \sim \overline{x}$ induces an étale path $(m^{-1})^I(\overline{\eta}) \sim (m^{-1})^I(\overline{x})$, and composing this with our étale path $\alpha(\overline{\beta_{d \times I}}) \sim (m^{-1})^I(\overline{\eta})$ yields an étale path $\alpha(\overline{\beta_{d \times I}}) \sim (m^{-1})^I(\overline{\xi})$. The isomorphism $(m^{-1})^I(\xi) \sim y$ enables us to view $\overline{x}$ as a geometric point $\overline{y}$ lying over $y$, so this amounts to an étale path $\alpha(\overline{\beta_{d \times I}}) \sim \alpha(\overline{y})$. Because $\alpha$ is étale at $\overline{\beta_{d \times I}}$ and $\overline{y}$, this corresponds to an étale path $\overline{\beta_{d \times I}} \sim \overline{y}$, which gives us a specialization isomorphism
\[
\mathcal{H}^p_{M,d \times I,W_{\Omega,H},\Xi/\Xi} \sim \mathcal{H}^p_{M,d \times I,W_{\Omega,H},\Xi/\Xi}
\]
by Theorem 3.9.

Let $(h,i)$ be in $d \times I$. From our étale path $\overline{\beta_{d \times I}} \sim \overline{y}$, we obtain an element $\gamma_{y_{h,i}}$ of $\text{Weil}(V \setminus M)$ as in 4.4. Since $\text{Frob}^+(h,i)$ fixes $\overline{y}$, we see that
\[
F_{\{h,i\}} \circ \text{Frob}^+(h,i) \circ \cdots \circ \text{Frob}^+(h,i) \circ (F_{\{h,i\}}) \circ \cdots \circ (F_{\{h,i\}})
\]
restricts to an action on $\mathcal{H}^p_{M,d \times I,W_{\Omega,H},\Xi/\Xi}$.

**Theorem.** Under the isomorphisms $\mathcal{H}^p_{N,I,W_{\Omega,G},\Xi/\Xi} \sim \mathcal{H}^p_{M,d \times I,W_{\Omega,H},\Xi/\Xi}$, this corresponds to the action of $\gamma_{y_{h,i}}$ in the $(h,i)$-th entry of $\text{Weil}(V \setminus M)^{d \times I}$.

**Proof.** The proof of Theorem 5.9 constructs the $\text{Weil}(V \setminus M)^{d \times I}$-action on $\mathcal{H}^p_{N,I,W_{\Omega,G},\Xi/\Xi}$ via the isomorphism $\mathcal{H}^p_{N,I,W_{\Omega,G},\Xi/\Xi} \sim \mathcal{H}^p_{M,d \times I,W_{\Omega,H},\Xi/\Xi}$. Thus it suffices to see that this corresponds to the action of $\gamma_{y_{h,i}}$ in the $(h,i)$-th entry of $\text{Weil}(V \setminus M)^{d \times I}$ on $\mathcal{H}^p_{M,d \times I,W_{\Omega,H},\Xi/\Xi}$, which follows immediately from Proposition 4.11.
5.14. When \( r = 1 \), the action from Theorem 5.13 has the following description in terms of the special fiber of the moduli space of shtukas. First, recall from 3.8 that we can use any ordered partition of \( d \times I \) to compute \( \mathcal{H}_{M,d \times I,W_{\Omega,H}}^P \). Let \( P \) be any ordered partition of \( d \times I \) \( \setminus \{ (h,i) \} \). Then \( F_{\{ (h,i) \}} \) is induced by the morphism \( F_{\{ (h,i) \}}^{\{ (h,i) \}} : \text{Sh}_{H,M,d \times I,W_{\Omega,H}}^P |_{\mathbb{Z}} \to \text{Sh}_{H,M,d \times I,W_{\Omega,H}}^P |_{\mathbb{Z}} \).

5.15. The case of general \( r \) is more complicated for the following reason. One uses Theorem 2.12 (b) to identify \( R\pi_! \left( \mathcal{F}^{\{ (h,i) \}}_{M,d \times I,W_{\Omega,H}} \right) \) with \( R\pi_! \left( \mathcal{F}^{(P,\{ (h,i) \})}_{M,d \times I,W_{\Omega,H}} \right) \), which is what enables us to iteratively compose \( F_{\{ (h,i) \}} \). This identification does not seem to arise from an explicit cohomological correspondence, so for general \( r \) and \( \Xi \) this impedes us from similarly describing the action of \( F_{\{ (h,i) \}} \circ \text{Frob}_{\{ (h,i) \}}^* \circ \cdots \circ \text{Frob}_{\{ (h,i) \}}^r \circ \gamma \) on \( \mathcal{H}_{M,d \times I,W_{\Omega,H}}^P |_{\mathbb{Z}} \). However, when all the \( x_i \) are disjoint, the \( y_{h,i} \) will also be disjoint, so we can use 2.5 to explicitly identify \( \text{Sh}_{H,M,d \times I,W_{\Omega,H}}^P |_{\mathbb{Z}} \) with \( \text{Sh}_{H,M,d \times I,W_{\Omega,H}}^P |_{\mathbb{Z}} \) via pulling back along \( \delta \circ \gamma \). Therefore we obtain a description of the action from Theorem 5.13 in terms of the special fiber of the moduli space of shtukas in this case.

REFERENCES


