MATH 99R PROBLEM SET 3

Due at 9am on Thursday, October 1.

(1) Let $F$ be a field, let $|\cdot|$ be a discretely valued norm on $F$, and let $\pi$ in $F$ be a uniformizer. Show that the map $\mathbb{Z} \times O^\times \to F^\times$ given by $(n, x) \mapsto \pi^n x$ is an isomorphism of topological groups.

(2) Let $F$ be a field, and let $|\cdot|$ be a nonarchimedean norm on $F$. Consider two closed balls $B_c(a, r) = \{ x \in F \ | \ |x - a| \leq r \}$ and $B_c(b, s) = \{ x \in F \ | \ |x - b| \leq s \}$, where $a$ and $b$ lie in $F$, and $r \geq s$ lie in $\mathbb{R}_{\geq 0}$. If $B_c(a, r)$ and $B_c(b, s)$ intersect, prove that $B_c(a, r)$ contains $B_c(b, s)$.

(3) Let $F$ be a field, and let $|\cdot|$ be a nonarchimedean norm on $F$. Recall that also we denote the the Gauss norm on the polynomial ring $F[t]$ by $|\cdot|$.

(a) Show that, for all $f$ in $F[t]$, we have $|f| = 0$ if and only if $f = 0$.

(b) Show that, for all $f$ and $g$ in $F[t]$, we have $|f + g| \leq \max\{|f|, |g|\}$.

(c) Prove that, for all $f$ and $g$ in $F[t]$, we have $|fg| = |f||g|$.

(d) Show that $|\cdot|$ extends uniquely to a nonarchimedean norm on the rational function field $F(t)$.

(4) Let $\kappa$ be a field, and write $v : \kappa((t)) \to \mathbb{Z} \cup \{\infty\}$ for the map sending $f \mapsto \ord_{t=0} f$.

(a) Show that $v$ is a valuation on $\kappa((t))$.

(b) Prove that $\kappa((t))$ is complete with respect to the norm induced by $v$.

(c) Prove that $\kappa((t))$ is locally compact if and only if $\kappa$ is finite.

(5) Let $F$ be a local field of characteristic $p > 0$, and write $q$ for the cardinality of its residue field. Prove that $F$ is isomorphic to $\mathbb{F}_q((t))$ as a topological field.

(Hint: use $\pi$-adic expansions, along with a special choice of representatives from Hensel’s lemma.)

(6) Let $F$ be field, let $|\cdot|$ be a discretely valued norm on $F$, and suppose $F$ is complete with respect to $|\cdot|$. Prove that $|\cdot|$ is the only discretely valued norm on $F$, up to isomorphism.

(Hint: use weak approximation and Hensel’s lemma to obtain a contradiction.)