Adelic Poisson Summation

Siyan Daniel Li-Huerta

November 3, 2020
Recall $F$ is a number field. We had a continuous group homomorphism $\psi_Q : \mathbb{A}_Q / \mathbb{Q} \to S^1$, and then we set $\psi_F = \psi_Q \circ \text{tr}_{\mathbb{A}_F / \mathbb{A}_Q}$. Let’s now finish computing self-dual Haar measures for $\psi_F$.

Example

If $p \neq \infty$, then $\psi_{F,v}$ equals the composition

$$F_v \xrightarrow{\text{tr}_{F_v/\mathbb{Q}_p}} \mathbb{Q}_p \to \mathbb{Q}_p / \mathbb{Z}_p \subset \mathbb{Q} / \mathbb{Z} \subset \mathbb{R} / \mathbb{Z} \xrightarrow{\varphi} S^1.$$ 

Here, we take $f = 1_{\mathcal{O}_v}$. The Fourier transform of $f$ with respect to the Lebesgue measure on $F_v$ is

$$\hat{f}(a) = \int_{F_v} \text{d}x \ f(x) \psi_{F,v}(ax)^{-1} = \int_{\mathcal{O}_v} \text{d}x \ \psi_{F,v}(ax).$$

This integral doesn’t vanish if and only if $\psi_{F,v}(ax) = 1$ for all $x$ in $\mathcal{O}_v$, in which case it equals 1. In turn, this occurs if and only if $\text{tr}_{F_v/\mathbb{Q}_p}(a \mathcal{O}_v)$ lies in $\mathbb{Z}_p$, i.e. if and only if $a$ lies in $\mathfrak{d}_{F_v/\mathbb{Q}_p}^{-1}$. Therefore $\hat{f} = 1_{\mathfrak{d}_{F_v/\mathbb{Q}_p}^{-1}}$.

Write $\mathfrak{d}_{F_v/\mathbb{Q}_p} = \pi_v^d \mathcal{O}_v$, where $d \geq 0$. So $\mathfrak{d}_{F_v/\mathbb{Q}_p}^{-1} = \pi_v^{-d} \mathcal{O}_v$. 


Example (continued)

The Fourier transform of \( \hat{f} \) with respect to the Lebesgue measure on \( F_v \) is

\[
\hat{f}(x) = \int_{F_v} da \hat{f}(a) \psi_{F,v}(ax)^{-1} = \int_{d_{F_v/Q_p}^{-1}} da \psi_{F,v}(ax)
\]

\[
= \|\pi_v\|_v^{-d} \int_{O_v} da' \psi_{F,v}(\pi_v^{-d} a'x)
\]

\[
= \#(O_v/d_{F_v/Q_p}) \int_{O_v} da' \psi_{F,v}(\pi_v^{-d} a'x),
\]

where \( a = \pi_v^{-d} a' \). This integral doesn’t vanish if and only if

\[
\psi_{F,v}(\pi_v^{-d} a'x) = 1 \text{ for all } a' \text{ in } O_v,
\]

which occurs if and only if

\[
\text{tr}_{F_v/Q_p}(\pi_v^{-d} a'O_v) \subseteq \mathbb{Z}_p \iff \pi_v^{-d} a' \in d_{F_v/Q_p}^{-1} = \pi_v^{-d} O_v \iff a' \in O_v.
\]

Therefore \( \hat{\hat{f}} = \#(O_v/d_{F_v/Q_p})1_{O_v} \), so \( \#(O_v/d_{F_v/Q_p})^{-1/2} \) times the Lebesgue measure on \( F_v \) is self-dual.

We henceforth use the self-dual Haar measure on \( \mathbb{A}_F \).
Classical Poisson summation relates functions on $\mathbb{R}$ with their Fourier transforms via summing on the discrete subgroup $\mathbb{Z}$. Adelic Poisson summation does the same with $\mathbb{A}_F$ and $F$ instead.

For all $\nu$ in $M_F$, let $f_\nu$ be in $S(F^n_\nu)$, and suppose $f_\nu = 1_{\mathcal{O}_\nu}$ for cofinitely many $\nu$. Then we can form $f = \prod_{\nu \in M_F} f_\nu$. Since the $f_\nu$ are continuous and integrable, we see $f$ is as well.

**Definition**

A *Bruhat–Schwartz* function on $\mathbb{A}_F^n$ is a finite sum of functions of the above form.

Write $S(\mathbb{A}_F^n)$ for the set of Bruhat–Schwartz functions on $\mathbb{A}_F^n$. Note it is preserved under addition, multiplication, and scaling by $\mathbb{C}$.

**Remark**

Because $F_\nu/\mathbb{Q}_p$ is ramified only for finitely many $\nu$, we see the Fourier transform of $1_{\mathcal{O}_\nu}$ equals itself for cofinitely many $\nu$. As the Fourier transform on $F_\nu$ yields a $\mathbb{C}$-linear isomorphism $S(F_\nu) \sim S(F_\nu)$ for all $\nu$ in $M_F$, this implies the Fourier transform on $\mathbb{A}_F$ yields a $\mathbb{C}$-linear isomorphism $S(\mathbb{A}_F) \sim S(\mathbb{A}_F)$ too.
Proposition

Let $f$ be in $S(\mathbb{A}_F)$. Then $\mathcal{F}(x) = \sum_{\gamma \in F} f(x + \gamma)$ converges uniformly on compact subsets of $\mathbb{A}_F$ and defines a continuous function $\mathcal{F} : \mathbb{A}_F/F \to \mathbb{C}$.

Proof.

Let $S \supseteq M_{F,\infty}$ be finite. It suffices to consider convergence on $\prod_{v \in M_F} C_v$, where $C_v$ is a compact subset of $F_v$ such that $C_v = \mathcal{O}_v$ for all $v$ not in $S$ and $C_v = m_v^{a_v}$ for all $v$ in $S \setminus M_{F,\infty}$. By enlarging $S$, we can assume it contains all $v$ for which $f_v \neq 1_{\mathcal{O}_v}$. For $v$ in $S \setminus M_{F,\infty}$, by scaling $f_v$ and enlarging its support, we can assume $f_v = 1_{m_v^{b_v}}$.

Form the fractional ideal $I = \prod_{v \in S \setminus M_{F,\infty}} v^{\min\{a_v, b_v\}}$ of $\mathcal{O}_F$. For all $x$ in $\prod_{v \in M_F} C_v$, if $f(x + \gamma) \neq 0$, we see $x_v + \gamma$ lies in $m_v^{b_v}$ for all $v$ in $S \setminus M_{F,\infty}$ and $\mathcal{O}_v$ for all the other $v$. Thus $\gamma$ lies in $I$, so we get $|\mathcal{F}(x)| \leq \sum_{\gamma \in I} |\prod_{v \in M_F,\infty} f_v(x_v + \gamma)|$. Recall that $I$ is a lattice in $\prod_{v \in M_F,\infty} F_v$, and note $(x_v)_{v \in M_F,\infty}$ lies in the compact subset $\prod_{v \in M_F,\infty} C_v$ of $\prod_{v \in M_F,\infty} F_v$, so uniform convergence follows from the case of $\mathbb{R}^n$. This also implies $\mathcal{F}$ descends to a continuous function $\mathbb{A}_F/F \to \mathbb{C}$.  

\[\square\]
Lemma
Let $G$ be an abelian locally compact topological group, let $m$ be a Haar measure on $G$, let $H$ be a countable closed subgroup of $G$, and let $D$ be a Borel subset of $G$. If $D$ has compact closure, nonempty interior, and maps bijectively to $G/H$, then the pushforward of $m$ via $D \to G/H$ yields a Haar measure on $G/H$.

Proof.
Homework problem. \qed

We call this the quotient measure on $G/H$, and we call $D$ a fundamental domain for $G/H$.

Examples

- Let $G = \mathbb{R}$, with $m$ being the Lebesgue measure, and $H = \mathbb{Z}$. We can take $D = [0, 1)$, which results in the usual measure on $\mathbb{R}/\mathbb{Z} = S^1$.

- Let $G = \mathbb{A}_\mathbb{Q}$ and $H = \mathbb{Q}$. It’s a homework problem to show we can take $D = \{(x_v)_v \in \mathbb{A}_\mathbb{Q} \mid \|x_v\|_v \leq 1 \text{ for } v \neq \infty \text{ and } 0 \leq x_\infty < 1\}$. 
Examples (continued)

- Let $G = \mathbb{A}_F$ and $H = F$. By choosing a $\mathbb{Q}$-basis of $F$, we can identify $F = \mathbb{Q}^n$ and hence $\mathbb{A}_F = \mathbb{A}_{\mathbb{Q}}^n$. Thus we can take $D$ to be the $n$-th power of the fundamental domain on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$.

Lemma

Write $m$ for the quotient measure on $\mathbb{A}_F/F$. Then the dual measure on $\widehat{\mathbb{A}_F}/F = F$ equals $m(\mathbb{A}_F/F)^{-1}$ times the counting measure.

Proof.

As $F$ is discrete and the dual measure is a Haar measure, we see it equals $c$ times the counting measure for some $c > 0$. Taking $f = 1$ in the Fourier inversion formula yields

$$1 = c \sum_{\gamma \in F} \hat{f}(\gamma) \psi_F(\gamma x)^{-1} = cm(\mathbb{A}_F/F),$$

since $\hat{f}$ equals $m(\mathbb{A}_F/F)$ times the indicator function on $0$. 
Theorem (adelic Poisson summation)

Let \( f \) be in \( S(\mathbb{A}_F) \). Then \( \sum_{\gamma \in F} f(\gamma) = \sum_{\gamma \in F} \hat{f}(\gamma) \).

Proof.

Let \( F(x) = \sum_{\gamma \in F} f(x + \gamma) \), considered as a function \( \mathbb{A}_F/F \to \mathbb{C} \). Note that \( F(0) \) equals the left-hand side above. Let \( D \subseteq \mathbb{A}_F \) be a fundamental domain for \( G/H \). First, I claim \( \hat{f}(c) = \hat{F}(c) \) for all \( c \) in \( F \), where we use the self-dual measure on \( \mathbb{A}_F \) and the quotient measure on \( \mathbb{A}_F/F \). To see this, note that

\[
\hat{F}(c) = \int_D d\mathcal{F}(x) \psi_F(cx)^{-1} = \int_D d\mathcal{F}(x) \sum_{\gamma \in F} f(x + \gamma) \psi_F(cx)^{-1}
\]

\[
= \int_D d\mathcal{F}(x) \sum_{\gamma \in F} f(x + \gamma) \psi_F(c(x + \gamma))^{-1} = \int_{\mathbb{A}_F} dy f(y) \psi_F(cy)^{-1} = \hat{f}(c),
\]

where \( y = x + \gamma \). Now \( \hat{f} \) lies in \( S(\mathbb{A}_F) \), so \( \sum_{\gamma \in F} |\hat{f}(\gamma)| = \sum_{\gamma \in F} |\hat{F}(\gamma)| \) converges. In other words, \( \hat{F} \) lies in \( L^1(F) \).
Theorem (adelic Poisson summation)

Let $f$ be in $S(\mathbb{A}_F)$. Then $\sum_{\gamma \in F} f(\gamma) = \sum_{\gamma \in F} \hat{f}(\gamma)$.

Proof (continued).

Hence Fourier inversion applies to $\mathcal{F}$, so

$$\sum_{\gamma \in F} f(\gamma) = \mathcal{F}(0) = m(\mathbb{A}_F/F)^{-1} \sum_{\gamma \in F} \hat{f}(\gamma) \psi_F(0)^{-1} = m(\mathbb{A}_F/F)^{-1} \sum_{\gamma \in F} \hat{f}(\gamma).$$

Replacing $f$ with $\hat{f}$ in this formula and applying Fourier inversion to $f$ yields

$$\sum_{\gamma \in F} f(\gamma) = m(\mathbb{A}_F/F)^{-2} \sum_{\gamma \in F} f(-\gamma) = m(\mathbb{A}_F/F)^{-2} \sum_{\gamma \in F} f(\gamma).$$

Taking any $f$ with $\sum_{\gamma \in F} f(\gamma) \neq 0$ indicates $m(\mathbb{A}_F/F)^{-2} = 1$, so we see $m(\mathbb{A}_F/F) = 1$. Thus the original formula yields the desired result. \qed
Remark

This shows \( m(\mathbb{A}_F/F) = 1 \) when \( m \) is the quotient measure of the self-dual measure on \( \mathbb{A}_F \) with respect to \( \psi_F \). We won't use it, but it's often useful to take the measure on \( \mathbb{A}_F \) that takes the usual Lebesgue measure for \( \nu \nmid \infty \).

With the quotient measure of this, the volume of \( \mathbb{A}_F/F \) is

\[
\prod_{\nu \notin M_F, \infty} \#(\mathcal{O}_\nu/\mathfrak{o}_{F_v}/\mathbb{Q}_p)^{1/2} = \#(\mathcal{O}_F/\mathfrak{o}_F/\mathbb{Q})^{1/2} = |Nm_{F/\mathbb{Q}}(\mathfrak{o}_F/\mathbb{Q})|^{1/2} = |D_{F/\mathbb{Q}}|^{1/2}.
\]

We conclude by relating idelic and ray class group characters as follows.

Proposition

Let \( \chi : \mathbb{A}_F^\times/F^\times \rightarrow \mathbb{C}^\times \) be a continuous group homomorphism such that \( \chi^m = 1 \) for some \( m \). Then \( \ker \chi \) contains the image of \( K_{(I,S_0)} \) for some modulus \( (I,S_0) \) for \( F \).

Hence \( \chi \) induces a continuous group homomorphism from the ray class group \( \mathcal{C}_r(I,S_0)(F) = K_{(I,S_0)} \backslash \mathbb{A}_F^\times/F^\times \rightarrow \mathbb{C}^\times \).
Proposition

Let $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ be a continuous group homomorphism such that $\chi^m = 1$ for some $m$. Then $\ker \chi$ contains the image of $K(I, S_0)$ for some modulus $(I, S_0)$ for $F$.

Proof.

As $\chi^m = 1$, the image of $\chi$ lies in $\{ \zeta \in \mathbb{C} \mid \zeta^m = 1 \}$, which is discrete. Now $\mathbb{R}_{>0}$ and $\mathbb{C}^\times$ are connected, so their images under $\chi$ must be trivial by continuity. Thus if we take $S$ to be all the real embeddings, $\ker \chi$ contains $\prod_{v \in M_F, \infty} K(I, S_0), v$.

Let $U$ be a neighborhood of 1 in $\mathbb{C}^\times$ containing no nontrivial subgroups of $\mathbb{C}^\times$. As the preimage of $\chi^{-1}(U)$ in $\mathbb{A}_F^\times$ is a neighborhood of 1, we see it contains $\prod_{v \notin M_F, \infty} N_v$, where the $N_v$ are open subsets of $F_v^\times$ such that $N_v = O_v^\times$ for all $v$ not in some finite subset $S \supset M_F, \infty$, and $N_v = 1 + m^a_v$ for all $v$ in $S \setminus M_F, \infty$. Now the image of $\prod_{v \notin M_F, \infty} N_v$ in $\mathbb{C}^\times$ is a subgroup and thus trivial, so we can take $I = \prod_{v \in S \setminus M_F, \infty} v^a_v$. □