More on Pontryagin Duals

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Let $G$ be an abelian topological group. Write $\varphi : \mathbb{R} \to S^1$ for the map $x \mapsto \exp(2\pi ix)$. Note $\varphi$ realizes $\mathbb{R}$ as the universal cover of $S^1$, sending the base-point $0$ to $1$.

**Example**

Let $G = S^1$. Let $\chi : S^1 \to S^1$ be in $\hat{G}$. Because $\mathbb{R}$ is simply connected, we can lift $\chi \circ \varphi : \mathbb{R} \to S^1$ uniquely to a continuous map $\tilde{\chi} : \mathbb{R} \to \mathbb{R}$ satisfying $\tilde{\chi}(0) = 0$. Thus we have a commutative diagram

$$
\begin{array}{ccc}
(\mathbb{R}, 0) & \xrightarrow{\tilde{\chi}} & (\mathbb{R}, 0) \\
\downarrow \varphi & & \downarrow \varphi \\
(S^1, 1) & \xrightarrow{\chi} & (S^1, 1).
\end{array}
$$

One can use the fact that $\chi \circ \varphi$ is a homomorphism to show $\tilde{\chi}$ is too, and $\tilde{\chi}$ preserves $\varphi^{-1}(0) = \mathbb{Z}$. Thus $\tilde{\chi}$ equals multiplication-by-$k$ for some $k$ in $\mathbb{Z}$, so $\chi$ equals $z \mapsto z^k$. This identifies $\hat{G}$ with $\mathbb{Z}$ as a group. The neighborhoods $W(G, 1, \varepsilon)$ show that $\hat{G}$ is discrete.
Let $t$ be in $(0,1]$, and write $N(t)$ for $\varphi((-\frac{t}{3}, \frac{t}{3}))$. As $t \to 0$, note the $N(t)$ form a basis of neighborhoods of $1$. Write $U^{(m)}$ for $\underbrace{U \cdots U}_{m \text{ times}}$, where $U \subseteq G$.

Lemma

Let $z$ be in $S^1$, and suppose $z, z^2, \ldots, z^m$ lie in $N(1)$. Then $z$ lies in $N(\frac{1}{m})$.

In particular, let $\chi : G \to S^1$ be a group homomorphism, and let $U$ be a subset of $G$ containing $1$. If $\chi(U^{(m)}) \subseteq N(1)$, then $\chi(U) \subseteq N(\frac{1}{m})$.

Proof.

We induct on $m$, where the $m = 1$ case is immediate. Assume now that $z, \ldots, z^{m+1}$ lie in $N(1)$. By induction, we know $z$ lies in $N(\frac{1}{m})$. Since $z^{m+1}$ lies in $N(1)$, there exists $y$ in $N(\frac{1}{m+1})$ such that $y^{m+1} = z^{m+1}$. This implies $z/y$ is an $(m+1)$-th root of unity, so $z = y \cdot (z/y)$ lies in $N(\frac{1}{m+1})\varphi(\frac{q}{m+1})$ for an integer $0 \leq q \leq m$.

I claim that $N(\frac{1}{m})$ and $N(\frac{1}{m+1})\varphi(\frac{q}{m+1})$ intersect if and only if $q = 0$. To see this, note that $N(\frac{1}{m})$ and $N(\frac{1}{m+1})\varphi(\frac{q}{m+1})$ are the homeomorphic images of $(-\frac{1}{3m}, \frac{1}{3m})$ and $(-\frac{3q-1}{3(m+1)}, \frac{3q+1}{3(m+1)})$, respectively.
Lemma

Let $z$ be in $S^1$, and suppose $z, z^2, \ldots, z^m$ lie in $N(1)$. Then $z$ lies in $N(\frac{1}{m})$. 

Proof (continued).

These images intersect if and only if

$$\frac{1}{3m} > \frac{3q-1}{3(m+1)} \iff m + 1 > 3qm - m \iff 2r + 1 > 3qr \iff q = 0.$$ 

Because $z$ lies in $N(\frac{1}{m})$, we have $q = 0$ and hence $z$ lies in $N(\frac{1}{m+1})$. □

Drawing a picture and using the law of cosines shows that

$$N(t) = \{z \in S^1 \mid |z - 1| < \sqrt{2 - 2 \cos(2\pi t / 3)}\}.$$ 

Therefore $W(K, 1, \sqrt{2 - 2 \cos(2\pi t / 3)})$ equals the set of $\chi$ in $\hat{G}$ such that $\chi(K) \subseteq N(t)$. As $t \to 0$, we see $\sqrt{2 - 2 \cos(2\pi t / 3)} \to 0$, so these form a basis of neighborhoods of 1.
Proposition

Let \( G \) be an abelian topological group.

1. Let \( \chi : G \to S^1 \) be a group homomorphism. Then \( \chi \) is continuous if and only if \( \chi^{-1}(N(1)) \) is open.

2. As \( K \) ranges over compact subsets of \( G \), the \( W(K, 1, \sqrt{3}) \) form a basis of neighborhoods of 1.

3. If \( G \) is discrete, then \( \hat{G} \) is compact.

4. If \( G \) is compact, then \( \hat{G} \) is discrete.

Proof.

1. If \( \chi \) is continuous, then \( \chi^{-1}(N(1)) \) is open since \( N(1) \) is. Conversely, suppose \( \chi^{-1}(N(1)) \) is open. Let \( x \) be in \( G \), and consider the neighborhood \( N(t)\chi(x) \) of \( \chi(x) \). There exists an integer \( m \geq 1 \) such that \( \frac{1}{m} \leq t \), and \( \chi^{-1}(N(1)) \) contains a neighborhood \( V \) of 1 such that \( V^{(m)} \subseteq \chi^{-1}(N(1)) \). Therefore \( \chi(V)^{(m)} \subseteq N(1) \), so \( \chi(V) \subseteq N(\frac{1}{m}) \). Hence \( V \subseteq \chi^{-1}(N(\frac{1}{m})) \), so the image of \( V \chi \) under \( \chi \) lies in \( N(t)\chi(x) \).
Proposition

Let $G$ be an abelian topological group.

1. Let $\chi : G \to S^1$ be a group homomorphism. Then $\chi$ is continuous if and only if $\chi^{-1}(N(1))$ is open.

2. As $K$ ranges over compact subsets of $G$, the $W(K, 1, \sqrt{3})$ form a basis of neighborhoods of 1.

3. If $G$ is discrete, then $\hat{G}$ is compact.

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Proof (continued).

2. Consider a neighborhood $W(K', 1, \sqrt{2 - 2 \cos(2\pi/3m)})$ of 1. Let $K = K'^{(m)}$, which is compact. If $\chi$ lies in $W(K, 1, \sqrt{3})$, then $\chi(K'^{(m)}) \subseteq N(1)$ and thus $\chi(K') \subseteq N(\frac{1}{m})$. Thus we see $W(K, 1, \sqrt{3})$ is a neighborhood of 1 contained in $W(K', 1, \sqrt{2 - 2 \cos(2\pi/3m)})$.

3. Homework problem.

4. Let $\chi$ be in $\hat{G}$. Then $\chi(G)$ is a subgroup of $S^1$, but $N(1)$ contains no nontrivial subgroups. Thus $W(G, 1, \sqrt{3}) = \{1\}$. 
Proposition

If $G$ is locally compact, then $\hat{G}$ is too.

Proof.

Suppose $\chi$ in $\hat{G}$ is nontrivial. Then $\chi(g) \neq 1$ for some $g$ in $G$, so the open subset $W(\{g\}, \chi, \frac{1}{2}|\chi(g) - 1|)$ does not contain 1. Taking unions over all such $\chi$ shows that $\{1\}$ is closed in $\hat{G}$.

Next, we have a neighborhood $O$ of 1 whose closure is compact. I claim that $W(O, 1, \sqrt{2 - 2\cos(2\pi/12)})$ has compact closure. To see this, note it suffices to prove

$$W = \{\chi \in \hat{G} \mid \chi(O) \subseteq N(\frac{1}{4})\}$$

is compact. Write $G_0$ for the group $G$ with the discrete topology. Then $\hat{G}_0$ is compact, and we view $\hat{G}$ as a subgroup of $\text{Hom}(G, S^1) = \hat{G}_0$. 
Proposition

If $G$ is locally compact, then $\hat{G}$ is too.

Proof (continued).

Write $W_0 = \{\chi \in \hat{G}_0 \mid \chi(O) \subseteq N(\frac{1}{4})\}$. Then $W_0$ is an intersection of closed subsets of $\hat{G}_0$ and hence is closed in $\hat{G}_0$. Thus $W_0$ is compact. Next, we immediately have $W \subseteq W_0$, and because $O$ is a neighborhood of 1 and $N(\frac{1}{4}) \subseteq N(1)$, we have $W_0 \subseteq W$.

So we just have to show the topology on $W_0$ from $\hat{G}_0$ is finer than the topology on $W$ from $\hat{G}$. Let $\chi$ be in $W$, and consider

$$U = W \cap W(K, \chi, \sqrt{2 - 2 \cos(2\pi/6m)})$$. Now $O$ contains a neighborhood $V$ of 1 such that $V^{(2m)} \subseteq O$. As $K$ is compact, we have $K \subseteq FV$ for some finite subset $F$ of $G$.

Form $U_0 = W_0 \cap W_0(F, \chi, \sqrt{2 - 2 \cos(2\pi/6m)})$, and suppose $\rho$ lies in $U_0$. Since $\overline{N(1/4)}^{-1} = \overline{N(1/4)}$, we see $\xi = \chi^{-1}\rho$ sends $\overline{O}$ to $\overline{N(1/2)} \subseteq N(1)$. 
Proposition

If $G$ is locally compact, then $\hat{G}$ is too.

Proof (continued).

Therefore $\xi$ is continuous, and since $V^{(2m)} \subseteq O$, we get $\xi(V) \subseteq N(\frac{1}{2m})$. Because we translated by $\chi^{-1}$, we also see that $\xi$ lies in $W_0(F, 1, \sqrt{2 - 2\cos(2\pi/6m)})$, so $\xi(F) \subseteq N(\frac{1}{2m})$. Hence $\xi(K) \subseteq \xi(F)\xi(O) \subseteq N(\frac{1}{m})$, so $\xi$ lies in $W(K, 1, \sqrt{2 - 2\cos(2\pi/3m)})$. Multiplying by $\chi$ shows $\rho$ lies in $U$, so altogether $U_0$ is a neighborhood of $\chi$ in $W_0$ contained in $U$. Hence we obtain the desired fineness statement.