

# HOLOMORPHIC FUNCTIONS OF POLYNOMIAL GROWTH ON BOUNDED DOMAINS

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In [4] R. Narasimhan proved the following theorem concerning holomorphic functions of polynomial growth: Suppose  $g$  is a holomorphic function defined on some open neighborhood of the closure of a bounded open subset  $\Omega$  of  $\mathbf{C}^n$ . If  $f$  is a holomorphic function of polynomial growth on  $\Omega$  such that  $f = gh$  for some holomorphic function  $h$  on  $\Omega$ , then  $h$  has polynomial growth on  $\Omega$ .

It is natural to raise the following question:

- (1) Suppose  $(\varphi_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$  is a matrix of holomorphic functions defined on some neighborhood of the closure of a bounded open subset  $\Omega$  of  $\mathbf{C}^n$ . Suppose  $(f_i)_{1 \leq i \leq r}$  is an  $r$ -tuple of holomorphic functions on  $\Omega$  having polynomial growth and for some  $s$ -tuple of holomorphic functions  $(g_j)_{1 \leq j \leq s}$  we have  $f_i = \sum_{j=1}^s \varphi_{ij} g_j$ ,  $1 \leq i \leq r$ , on  $\Omega$ . Can we always find an  $s$ -tuple of holomorphic functions  $(h_j)_{1 \leq j \leq s}$  on  $\Omega$  having polynomial growth such that  $f_i = \sum_{j=1}^s \varphi_{ij} h_j$ ,  $1 \leq i \leq r$ ?

In this paper we give an affirmative answer for (1) in the case when  $\Omega$  is Stein (Theorem 2 below). First we prove an infinitely differentiable analogue of our result by the partition of unity (Theorem 1 below) and then derive our result by the  $L^2$ -estimates for the  $\bar{\partial}$  operator.

**1. Notations.**  $n$  is a fixed natural number and  $m = 2n$ .  $\mathbf{N}$  = the set of all natural numbers.  $\mathbf{N}^*$  = the set of all nonnegative integers.  $\mathbf{R}_1 = \{c \in \mathbf{R} \mid c \geq 1\}$ .  $\mathbf{R}_+ = \{c \in \mathbf{R} \mid c > 0\}$ .

$x = (x_1, \dots, x_m)$  and  $z = (z_1, \dots, z_n)$  denote respectively points in  $\mathbf{R}^m$  and  $\mathbf{C}^n$ .  $x$  is identified with  $z$  by  $z_k = x_k + ix_{n+k}$ ,  $1 \leq k \leq n$ .  $dx = dx_1 \cdots dx_m$ .

$$|x| = \left( \sum_{k=1}^m |x_k|^2 \right)^{\frac{1}{2}}.$$

If  $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbf{N}^*)^m$ , then

$$|\alpha| = \sum_{k=1}^m \alpha_k, \quad \alpha! = \prod_{k=1}^m (\alpha_k)!$$

and

$$D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}.$$

If  $E \subset \mathbf{R}^m$ , then  $E^-$  = the closure of  $E$  in  $\mathbf{R}^m$ ,

$$d(x, E) = \inf_{y \in E} |x - y|, \quad d_E(x) = d(x, \mathbf{R}^m - E), \quad \text{and} \quad a(E) = \sup_{x, y \in E} |x - y|.$$

Received March 23, 1968.

For  $\epsilon \in \mathbf{R}_+$ ,

$$E(\epsilon) = \{x \in \mathbf{R}^m \mid d(x, E) < \epsilon\}, \quad \text{and} \quad E[\epsilon] = \{x \in \mathbf{R}^m \mid d_E(x) > \epsilon\}.$$

$\chi_E$  = the characteristic function of  $E$ .

All functions are complex-valued unless specified otherwise. For  $0 \leq k \leq n$ ,  $\mathcal{E}^{(0,k)}$  = the sheaf of germs of  $C^\infty(0, k)$ -forms on  $\mathbf{C}^n$ .  $\mathcal{E} = \mathcal{E}^{(0,0)}$ .  $\mathcal{O}$  = the sheaf of germs of holomorphic functions on  $\mathbf{C}^n$ . Suppose  $\Omega$  is an open subset of  $\mathbf{R}^m = \mathbf{C}^n$  and  $g = (g^{(1)}, \dots, g^{(r)}) \in \Gamma(\Omega, (\mathcal{E}^{(0,k)})^r)$  for some  $0 \leq k \leq n$  and  $r \in \mathbf{N}$ . Let

$$g^{(l)} = \sum_{1 \leq i_1 < \dots < i_k \leq n} g_{i_1 \dots i_k}^{(l)} d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_k}, \quad 1 \leq l \leq r.$$

Then for  $x \in \mathbf{R}^m$  and  $\alpha \in (\mathbf{N}^*)^m$ ,

$$|D^\alpha g(x)| = \sup_{1 \leq i_1 < \dots < i_k \leq n, 1 \leq l \leq r} |D^\alpha g_{i_1 \dots i_k}^{(l)}(x)|.$$

For  $1 \leq i_1 < \dots < i_k \leq n$ ,  $(g_{i_1 \dots i_k}^{(1)}, \dots, g_{i_1 \dots i_k}^{(r)}) \in \Gamma(\Omega, \mathcal{E}^r)$  is called a coefficient of  $g$ .  $\bar{\partial}g$  denotes  $(\bar{\partial}g^{(1)}, \dots, \bar{\partial}g^{(r)})$ . If  $\psi$  is a real-valued continuous function on  $\Omega$  and  $h = (h^{(1)}, \dots, h^{(r)})$  is an  $r$ -tuple of  $(0, k)$ -forms on  $\Omega$  with locally integrable coefficients, then for  $1 \leq l \leq r$   $\|h^{(l)}\|_\psi$  is defined as in §4.1, [1] and  $\|h\|_\psi$  denotes  $\sup_{1 \leq l \leq r} \|h^{(l)}\|_\psi$ . If  $f \in \Gamma(\Omega, \mathcal{E})$ , then  $\text{Supp } f$  = the support of  $f$ .

**2. The infinitely differentiable case.** On  $\mathbf{R}^m$  define

$$\bar{\theta}(x) = \exp [(|x|^2 - 1)^{-1}] \quad \text{for} \quad |x| < 1 \quad \text{and} \quad \bar{\theta}(x) = 0 \quad \text{for} \quad |x| \geq 1.$$

Let

$$K = \int_{\mathbf{R}^m} \bar{\theta}(x) dx \quad \text{and} \quad \theta(x) = K^{-1} \bar{\theta}(x).$$

For  $\alpha \in (\mathbf{N}^*)^m$  let

$$K_\alpha = \sup_{x \in \mathbf{R}^m} |D^\alpha \theta(x)|.$$

For  $\epsilon \in \mathbf{R}_+$  let  $\theta_\epsilon(x) = \epsilon^{-m} \theta(x/\epsilon)$ . For any open subset  $\Omega$  of  $\mathbf{R}^m$  define

$$\sigma_{\Omega, \epsilon}(x) = \int_{\mathbf{R}^m} \theta_{\epsilon/4}(x - y) \chi_{\Omega(\epsilon/4)}(y) dy.$$

**LEMMA 1.**  $\sigma_{\Omega, \epsilon}$  is a  $C^\infty$  function on  $\mathbf{R}^m$  which is identically one on  $\Omega$  and is identically zero outside  $\Omega(\epsilon/2)$ . Moreover, for  $\alpha \in (\mathbf{N}^*)^m$  and  $x \in \mathbf{R}^m$   $|D^\alpha \sigma_{\Omega, \epsilon}(x)| \leq K'_\alpha \epsilon^{-|\alpha|}$ , where  $K'_\alpha = 4^{|\alpha|} K_\alpha \omega$  and  $\omega$  is the euclidean volume of the unit ball in  $\mathbf{R}^m$ .

*Proof.* By direct computations.

Q.E.D.

**LEMMA 2.** Suppose  $\Omega$  is an open subset of  $\mathbf{R}^m$  and  $a(\Omega) < 1$ . For  $k \in \mathbf{N}$  let  $G_k = \{x \in \Omega \mid 2^{-k-1} < d_\Omega(x) < 2^{-k}\}$ . Then there exist  $Q_\alpha \in \mathbf{R}_+$  for  $\alpha \in (\mathbf{N}^*)^m$

and non-negative real-valued  $\eta_k \in \Gamma(\Omega, \mathcal{E})$  for  $k \in \mathbf{N}$  such that

- (i)  $\text{Supp } \eta_k \subset G_k(2^{-k-2})$  for  $k \in \mathbf{N}$ ,
- (ii)  $\sum_{k \in \mathbf{N}} \eta_k \equiv 1$  on  $\Omega$ ,
- (iii)  $|D^\alpha \eta_k(x)| \leq Q_\alpha d_\Omega(x)^{-|\alpha|}$  for  $\alpha \in (\mathbf{N}^*)^m$ ,  $k \in \mathbf{N}$ , and  $x \in \Omega$ .

*Proof.* For  $k \in \mathbf{N}$  let  $\tilde{\eta}_k = \sigma_{\alpha_k, 2^{-k-2}}$ . Then by Lemma 1,

$$|D^\alpha \tilde{\eta}_k(x)| \leq K'_\alpha 2^{(k+2)|\alpha|} \text{ for } \alpha \in (\mathbf{N}^*)^m \text{ and } x \in \mathbf{R}^m.$$

Since  $\text{Supp } \tilde{\eta}_k \subset G_k(2^{-k-2})$ ,  $\tilde{\eta}_k(x) = 0$  unless  $d_\Omega(x) < 2^{-k+1} + 2^{-k-2} < 2^{-k+2}$ . Hence

$$|D^\alpha \tilde{\eta}_k(x)| \leq K''_\alpha d_\Omega(x)^{-|\alpha|} \text{ for } \alpha \in (\mathbf{N}^*)^m \text{ and } x \in \Omega,$$

where  $K''_\alpha = 2^{4|\alpha|} K'_\alpha$ . Since for any five distinct  $k_i$  in  $\mathbf{N}$ ,  $1 \leq i \leq 5$ , we always have  $\bigcap_{i=1}^5 \text{Supp } \tilde{\eta}_{k_i} = \emptyset$ ,  $\eta = \sum_{k \in \mathbf{N}} \tilde{\eta}_k$  is well defined. Moreover,

$$|D^\alpha \eta(x)| \leq 4K''_\alpha d_\Omega(x)^{-|\alpha|} \text{ for } \alpha \in (\mathbf{N}^*)^m \text{ and } x \in \Omega.$$

It is clear that  $\eta(x) \geq 1$  for  $x \in \Omega$ . We are going to construct by induction on  $|\alpha|$ ,  $P_\alpha \in \mathbf{R}_+$  for  $\alpha \in (\mathbf{N}^*)^m$  such that

$$|D^\alpha \eta^{-1}(x)| \leq P_\alpha d_\Omega(x)^{-|\alpha|} \text{ for } \alpha \in (\mathbf{N}^*)^m \text{ and } x \in \Omega.$$

For  $|\alpha| = 0$ , set  $P_\alpha = 1$ . Suppose  $k \geq 1$ , and we have constructed  $P_\beta$  for  $|\beta| < k$ . Let  $|\alpha| = k$ . By applying the operator  $D^\alpha$  to  $\eta^{-1} \equiv 1$  on  $\Omega$ , we have

$$D^\alpha(\eta^{-1}) = -\eta^{-1} \sum_{\beta+\gamma=\alpha, \beta \neq \alpha} (\alpha!)(\beta!)^{-1}(\gamma!)^{-1}(D^\beta \eta^{-1})(D^\gamma \eta) \text{ on } \Omega.$$

It is clear that

$$P_\alpha = \sum_{\beta+\gamma=\alpha, \beta \neq \alpha} (\alpha!)(\beta!)^{-1}(\gamma!)^{-1}P_\beta \cdot 4K''_\alpha \text{ satisfies the requirement.}$$

The construction by induction is complete. Set  $\eta_k = \eta^{-1} \tilde{\eta}_k$  for  $k \in \mathbf{N}$  and set

$$Q_\alpha = \sum_{\beta+\gamma=\alpha} (\alpha!)(\beta!)^{-1}(\gamma!)^{-1}P_\beta \cdot 4K''_\alpha \text{ for } \alpha \in (\mathbf{N}^*)^m.$$

Then

$$D^\alpha \eta_k = \sum_{\beta+\gamma=\alpha} (\alpha!)(\beta!)^{-1}(\gamma!)^{-1}(D^\beta \eta^{-1})(D^\gamma \eta_k) \text{ on } \Omega \text{ for } k \in \mathbf{N}.$$

$$|D^\alpha \eta_k(x)| \leq Q_\alpha d_\Omega(x)^{-|\alpha|} \text{ for } k \in \mathbf{N}, \alpha \in (\mathbf{N}^*)^m, \text{ and } x \in \Omega.$$

It is clear that (i) and (ii) are satisfied. Q.E.D.

**THEOREM 1.** Suppose  $\Omega \subset \subset \tilde{\Omega}$  are bounded open subsets of  $\mathbf{R}^m$  and  $(\varphi_{ij})_{1 \leq i \leq s, 1 \leq j \leq r}$  is a matrix of holomorphic functions on  $\tilde{\Omega}$ . Let  $\Phi : \Gamma(\Omega, \mathcal{E}') \rightarrow \Gamma(\Omega, \mathcal{E})$  be induced by  $(\varphi_{ij})$ . Then for any  $p \in \mathbf{N}^*$  there exist  $q = q(p) \in \mathbf{N}^*$  and  $M = M(p) \in \mathbf{R}_1$  satisfying the following: if  $f \in \text{Im } \Phi$  and for some  $t \in \mathbf{N}^*$  and some  $A \in \mathbf{R}_+$  we have

$$|D^\beta f(x)| \leq A d_\Omega(x)^{-t} \text{ for } |\beta| \leq q \text{ and } x \in \Omega,$$

then there exists  $g \in \Gamma(\Omega, \mathcal{E}')$  such that  $\tilde{\Phi}(g) = f$  and

$$|D^\alpha g(x)| \leq M^{1+t} A d_\Omega(x)^{-t-p-\alpha} \quad \text{for } x \in \Omega \quad \text{and } |\alpha| \leq p.$$

*Proof.* Fix  $p \in \mathbf{N}^*$ . By changing radially the scale of the coordinates system we can suppose w. l. o. g. that  $a(\Omega) < 1$ . Let  $\tilde{\Phi} : \Gamma(\tilde{\Omega}, \mathcal{E}') \rightarrow \Gamma(\tilde{\Omega}, \mathcal{E}')$  be induced by  $(\varphi_{ii})$ .  $\text{Im } \tilde{\Phi}$  is closed in the natural Fréchet space topology of  $\Gamma(\tilde{\Omega}, \mathcal{E}')$  [5, Theorem 5]. Therefore by the open mapping theorem for Fréchet spaces we can find  $g \in \mathbf{N}^*$  and  $L_1 \in \mathbf{R}_+$  satisfying:

(2) If  $f \in \text{Im } \tilde{\Phi}$  and for some  $A \in \mathbf{R}_+$   $|D^\beta f(x)| \leq A$  for  $|\beta| \leq q$  and  $x \in \tilde{\Omega}$ , then there exists  $g \in \Gamma(\tilde{\Omega}, \mathcal{E}')$  such that  $\tilde{\Phi}(g) = f$  and  $|D^\alpha g(x)| \leq L_1 A$  for  $|\alpha| \leq p$  and  $x \in \tilde{\Omega}^-$ .

Let

$$L_2 = \sup_{|\beta| \leq q} \sum_{\lambda + \mu = \beta} (\beta!) (\lambda!)^{-1} (\mu!)^{-1} K'_\mu.$$

From Lemma 2 we obtain  $Q_\alpha \in \mathbf{R}_+$  for  $\alpha \in (\mathbf{N}^*)^m$  and  $\eta_k \in \Gamma(\Omega, \mathcal{E})$  for  $k \in \mathbf{N}$  satisfying the conditions of Lemma 2. Choose  $M \in \mathbf{R}$  such that  $M \geq 2^5$  and

$$M \geq \sup_{|\alpha| \leq p} \sum_{\lambda + \mu = \alpha} 4(\alpha!) (\lambda!)^{-1} (\mu!)^{-1} Q_\lambda L_1 L_2 2^{4q}.$$

We claim that  $q$  and  $M$ , so constructed, satisfy the requirements.

Suppose  $f \in \text{Im } \tilde{\Phi}$  such that for some  $t \in \mathbf{N}^*$  and  $A \in \mathbf{R}_+$  we have

$$|D^\beta f(x)| \leq A d_\Omega(x)^{-t} \quad \text{for } |\beta| \leq q \quad \text{and } x \in \Omega.$$

For  $k \in \mathbf{N}$  let  $\tau_k = \sigma_{\Omega[2^{-k-2}], 2^{-k-2}}$ . By Lemma 1 we obtain

$$|D^\beta (f\tau_k)(x)| \leq L_2 A d_\Omega(x)^{-t+2^{(k+2)q}} \quad \text{for } |\beta| \leq q \quad \text{and } x \in \Omega.$$

Since  $f\tau_k \equiv 0$  outside  $\Omega[2^{-k-2}]$  ( $2^{-k-3}$ ),  $f\tau_k$  can be extended trivially to  $f_k \in \Gamma(\tilde{\Omega}, \mathcal{E}')$ .  $f_k \in \text{Im } \tilde{\Phi}$  and

$$|D^\beta f_k(x)| \leq L_2 A 2^{(k+3)t+(k+2)q} \quad \text{for } |\beta| \leq q \quad \text{and } x \in \tilde{\Omega}.$$

Hence by (2) there exists  $g_k \in \Gamma(\tilde{\Omega}, \mathcal{E}')$  such that  $\tilde{\Phi}(g_k) = f_k$  and

$$|D^\alpha g_k(x)| \leq L_1 L_2 A 2^{(k+3)t+(k+2)q} \quad \text{for } |\alpha| \leq p \quad \text{and } x \in \tilde{\Omega}^-.$$

Define  $g \in \Gamma(\Omega, \mathcal{E}')$  by  $g = \sum_{k \in \mathbf{N}} \eta_k g_k$ . Since

$$\text{Supp } \eta_k \subset G_k(2^{-k-2}) \subset \Omega[2^{-k-2}],$$

and  $\tau_k$  is identically one on  $\Omega[2^{-k-2}]$ ,  $\eta_k \tau_k = \eta_k$  for  $k \in \mathbf{N}$ . Hence on  $\Omega$

$$\tilde{\Phi}(g) = \sum_{k \in \mathbf{N}} \eta_k \tilde{\Phi}(g_k) = \sum_{k \in \mathbf{N}} \eta_k \tau_k f = \sum_{k \in \mathbf{N}} \eta_k f = f.$$

Fix  $x \in \Omega$  and  $|\alpha| \leq p$ . Let  $k$  be the greatest integer such that  $d_\Omega(x) < 2^{-k}$ .

$$D^\alpha g(x) = \sum_{i=k-1}^{k+2} D^\alpha (\eta_i g_i)(x).$$

$$|D^\alpha g(x)| \leq 4 \sum_{\lambda+\mu=\alpha} (\alpha!) (\lambda!)^{-1} (\mu!)^{-1} Q_\lambda d_\Omega(x)^{-|\lambda|} L_1 L_2 A 2^{(k+5)t + (k+4)\alpha}.$$

Hence for  $|\alpha| \leq p$  and  $x \in \Omega$

$$|D^\alpha g(x)| \leq M^{1+t} A d_\Omega(x)^{-t-p-\alpha} \quad \text{Q.E.D.}$$

**3. The holomorphic case.**

**LEMMA 3.** *Suppose  $\Omega$  is a bounded open subset of  $\mathbf{C}^n$  and  $f$  is a holomorphic function on  $\Omega$  such that for some  $A \in \mathbf{R}_+$  and  $t \in \mathbf{N}^*$  we have  $|f(z)| \leq A d_\Omega(z)^{-t}$  for  $z \in \Omega$ . Then  $|D^\alpha f(z)| \leq (|\alpha|!)^n A 2^{t+|\alpha|} n^{|\alpha|/2} d_\Omega(z)^{-t-|\alpha|}$  for  $\alpha \in (\mathbf{N}^*)^n$  and  $z \in \Omega$ .*

*Proof.* Fix  $z \in \Omega$  and  $\alpha \in (\mathbf{N}^*)^n$ . Let

$$\Gamma = \{(\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n \mid |z_i - \zeta_i| = (2\sqrt{n})^{-1} d_\Omega(z), \quad 1 \leq i \leq n\}.$$

Then

$$d_\Omega(\zeta) \geq \frac{1}{2} d_\Omega(z) \quad \text{for } \zeta \in \Gamma.$$

The required inequality follows from

$$\begin{aligned} D^\alpha f(z) &= i^{\alpha_{n+1} + \dots + \alpha_{2n}} \frac{\partial^{|\alpha|} f(z)}{\partial^{\alpha_1 + \alpha_{n+1}} z_1 \dots \partial^{\alpha_n + \alpha_{2n}} z_n} \\ &= i^{\alpha_{n+1} + \dots + \alpha_{2n}} (2\pi i)^{-n} [(\alpha_1 + \alpha_{n+1})!] \dots [(\alpha_n + \alpha_{2n})!] \\ &\quad \cdot \int_{\zeta \in \Gamma} \frac{f(\zeta) d\zeta_1 \dots d\zeta_n}{(\zeta_1 - z_1)^{\alpha_1 + \alpha_{n+1} + 1} \dots (\zeta_n - z_n)^{\alpha_n + \alpha_{2n} + 1}}. \quad \text{Q.E.D.} \end{aligned}$$

**LEMMA 4.** *Suppose  $\Omega$  is a bounded open subset of  $\mathbf{C}^n$ ,  $b \in \mathbf{R}_+$ , and  $\psi(z) = -2b \log d_\Omega(z)$  on  $\Omega$ . If  $f \in \Gamma(\Omega, \theta)$  and  $\|f\|_\psi \leq A$  for some  $A \in \mathbf{R}_+$ , then  $|f(z)| \leq \pi^{-n/2} 2^{b+n} n^{n/2} A d_\Omega(z)^{-b-n}$  for  $z \in \Omega$ .*

*Proof.* Fix  $z \in \Omega$ . Let  $D$  be the polydisc centered at  $z$  with all radii equal to  $(2\sqrt{n})^{-1} d_\Omega(z)$ . Let  $v$  be the euclidean volume of  $D$ . Then

$$v = \pi^n (2\sqrt{n})^{-2n} d_\Omega(z)^{2n}.$$

For  $x \in D$ ,  $d_\Omega(x) > \frac{1}{2} d_\Omega(z)$  and hence

$$e^{\psi(x)} = d_\Omega(x)^{-2b} < 2^{2b} d_\Omega(z)^{-2b}.$$

$$\int_{x \in D} e^{\psi(x)} dx \leq 2^{2b} d_\Omega(z)^{-2b} v.$$

$$\begin{aligned} |f(z)| &\leq v^{-1} \int_{x \in D} |f(x)| dx \leq v^{-1} \left( \int_{x \in D} |f(x)|^2 e^{-\psi(x)} dx \right)^{\frac{1}{2}} \left( \int_{x \in D} e^{\psi(x)} dx \right)^{\frac{1}{2}} \\ &\leq v^{-1} A 2^b d_\Omega(z)^{-b} v^{\frac{1}{2}} = \pi^{-n/2} 2^{b+n} n^{n/2} A d_\Omega(z)^{-b-n}. \quad \text{Q.E.D.} \end{aligned}$$

**THEOREM 2.** *Suppose  $\Omega$  is a bounded Stein open subset of  $\mathbf{C}^n$  and  $(\varphi_{ij})_{1 \leq i \leq r_0, 1 \leq j \leq r_1}$  is a matrix of holomorphic functions defined in some open neigh-*

borhood  $\tilde{\Omega}$  of  $\Omega^-$ . Let  $\Phi : \Gamma(\Omega, \mathcal{O}^{r_1}) \rightarrow \Gamma(\Omega, \mathcal{O}^{r_0})$  be induced by  $(\varphi_{ij})$ . Then there exist  $M \in \mathbf{R}_1$  and  $p \in \mathbf{N}^*$  satisfying the following: if  $f \in \text{Im } \Phi$  and for some  $t \in \mathbf{N}^*$  and some  $A \in \mathbf{R}_+$   $|f(z)| \leq A d_{\tilde{\Omega}}(z)^{-t}$  for  $z \in \Omega$ , then there exists  $g \in \Gamma(\Omega, \mathcal{O}^{r_1})$  such that  $\Phi(g) = f$  and  $|g(z)| \leq M^{1+t} A d_{\tilde{\Omega}}(z)^{-t-p}$ .

*Proof.* By changing radially the scale of the coordinates system of  $\mathbf{C}^n$  we can assume that  $a(\Omega) < 1$ . By considering  $\tilde{\Omega}$  as an open subset of its envelope of holomorphy and by shrinking  $\tilde{\Omega}$  if necessary, we can assume that there is an exact sequence of sheaf-homomorphisms on  $\tilde{\Omega}$ :

$$\mathcal{O}^{r_{n+1}} \xrightarrow{\varphi^{(n)}} \mathcal{O}^{r_n} \xrightarrow{\varphi^{(n-1)}} \dots \xrightarrow{\varphi^{(2)}} \mathcal{O}^{r_2} \xrightarrow{\varphi^{(1)}} \mathcal{O}^{r_1} \xrightarrow{\varphi^{(0)}} \mathcal{O}^{r_0}$$

such that  $\varphi^{(0)}$  is induced by  $(\varphi_{ij})$ . For  $0 \leq k \leq n$  let  $\varphi^{(k)}$  be represented by the matrix  $(\varphi_{ij}^{(k)})_{1 \leq i \leq r_k, 1 \leq j \leq r_{k+1}}$  and let

$$\Phi^{(k)} : \Gamma(\Omega, \mathcal{E}^{r_{k+1}}) \rightarrow \Gamma(\Omega, \mathcal{E}^{r_k}) \quad \text{be induced by } (\varphi_{ij}^{(k)}).$$

Since at a point in a complex number space the ring of  $C^\infty$  function-germs as a module over the ring of holomorphic function-germs is flat [3, Theorems 1 and 2 bis] and moreover  $\mathcal{E}$  is fine we have

$$(3) \quad \text{Ker } \Phi^{(k)} = \text{Im } \Phi^{(k+1)}, \quad 0 \leq k \leq n-1.$$

Set  $q_{n+1} = -1$ . By applying Theorem 1, we obtain by backward induction on  $k$ ,  $q_k \in \mathbf{N}^*$  and  $M_k \in \mathbf{R}_1$  for  $0 \leq k \leq n$  such that (4)<sub>k</sub> holds for  $0 \leq k \leq n$ :

$$(4)_k \quad \text{if } f^{(k)} \in \text{Im } \Phi^{(k)} \text{ and for some } t \in \mathbf{N}^* \text{ and } A \in \mathbf{R}_+ \text{ } |D^\beta f^{(k)}(z)| \leq A d_{\tilde{\Omega}}(z)^{-t} \text{ for } z \in \Omega \text{ and } |\beta| \leq q_k, \text{ then there exists } g^{(k)} \in \Gamma(\Omega, \mathcal{E}^{r_{k+1}}) \text{ such that } \Phi^{(k)}(g^{(k)}) = f^{(k)} \text{ and } |D^\alpha g^{(k)}(z)| \leq M_k^{1+t} A d_{\tilde{\Omega}}(z)^{-t-\alpha_k-\alpha_{k+1}-1} \text{ for } z \in \Omega \text{ and } |\alpha| \leq q_{k+1} + 1.$$

Let

$$N_1 = \sup_{0 \leq k \leq n} M_k, \quad q = \sup_{0 \leq k \leq n+1} (q_k + 1),$$

and

$$L = \sup_{|\alpha| \leq q_0} (|\alpha|!)^n 2^{|\alpha|} n^{|\alpha|/2}.$$

Let  $v$  be the volume of  $\Omega$ ,

$$N_2 = N_1^{(n+1)(n+1+q)} n^n v^{\frac{1}{2}} L, \quad \text{and} \quad \eta = 2N_1^{n+1}.$$

Let

$$N_3 = \sup \{ |\varphi_{ij}^{(k)}(z)| \mid z \in \Omega^-, \quad 1 \leq i \leq r_k, \quad 1 \leq j \leq r_{k+1}, \quad 0 \leq k \leq n \},$$

$$\mu = \sup \{ (1 + |z|^2) \mid z \in \Omega^- \}, \quad \text{and} \quad r = \sup_{0 \leq k \leq n} r_k.$$

Let

$$p = 2(n+1)q + n \quad \text{and} \quad M = \max(2\eta, \pi^{-n/2} 2^{(2n+1)q+n} n^{n/2} (1 + \mu r N_3)^n N_2).$$

We claim that  $p$  and  $M$  satisfy the requirements.

Fix  $f \in \text{Im } \Phi$  such that for some  $t \in \mathbf{N}^*$  and some  $A \in \mathbf{R}_+$   $|f(z)| \leq A d_\Omega(z)^{-t}$  for  $z \in \Omega$ . By Lemma 3

$$|D^\alpha f(z)| \leq LA 2^t d_\Omega(z)^{-t-|\alpha|} \quad \text{for } |\alpha| \leq q_0 \quad \text{and } z \in \Omega.$$

By (4)<sub>0</sub> we obtain  $f^{(1)} \in \Gamma(\Omega, \mathcal{E}^{r_1})$  such that

$$|D^\alpha f^{(1)}(z)| \leq N_1^{1+t+q} LA 2^t d_\Omega(z)^{-t-3q} \quad \text{for } |\alpha| \leq q_1 + 1 \quad \text{and } z \in \Omega$$

and

$$(5) \quad \Phi^{(0)}(f^{(1)}) = f.$$

We are going to construct by induction on  $k$

$$f^{(k)} = (f_1^{(k)}, \dots, f_{r_k}^{(k)}) \in \Gamma(\Omega, (\mathcal{E}^{(0,k-1)})^{r_k}), \quad 1 \leq k \leq n + 1,$$

such that for  $1 \leq k \leq n + 1$  (6)<sub>k</sub> and (7)<sub>k</sub> hold and for  $1 \leq k \leq n$  (8)<sub>k</sub> holds:

$$(6)_k \quad |D^\alpha f^{(k)}(z)| \leq N_1^{k(1+t)+k^2q} n^{k-1} LA 2^t d_\Omega(z)^{-t-(2k+1)q} \quad \text{for } |\alpha| \leq q_k + 1 \quad \text{and } z \in \Omega,$$

$$(7)_k \quad \sum_{i=1}^{r_k} \varphi_{ii}^{(k-1)} \bar{\partial} f_i^{(k)} = 0, \quad 1 \leq i \leq r_{k-1},$$

$$(8)_k \quad \bar{\partial} f_i^{(k)} = \sum_{i=1}^{r_{k+1}} \varphi_{ii}^{(k)} f_i^{(k+1)}, \quad 1 \leq i \leq r_k.$$

By applying  $\bar{\partial}$  to both sides of (5) we obtain (7)<sub>1</sub>. Hence  $f^{(1)}$  satisfies both (6)<sub>1</sub> and (7)<sub>1</sub>. Suppose for some  $1 \leq k \leq n$ ,  $f^{(1)}, \dots, f^{(k)}$  have been constructed. (7)<sub>k</sub> and (3) imply that the coefficients of  $\bar{\partial} f^{(k)}$  are in  $\text{Im } \Phi^{(k)}$ . (6)<sub>k</sub> implies that

$$|D^\alpha \bar{\partial} f^{(k)}(z)| \leq N_1^{k(1+t)+k^2q} n^k LA 2^t d_\Omega(z)^{-t-2(k+1)q} \quad \text{for } |\alpha| \leq q_k \quad \text{and } z \in \Omega.$$

By applying (4)<sub>k</sub> to the coefficients of  $\bar{\partial} f^{(k)}$ , we obtain  $f^{(k+1)} \in \Gamma(\Omega, (\mathcal{E}^{(0,k)})^{r_{k+1}})$  satisfying (6)<sub>k+1</sub> and (8)<sub>k</sub>. Applying  $\bar{\partial}$  to (8)<sub>k</sub> we obtain (7)<sub>k+1</sub>. The construction by induction is complete.

Let  $\psi(z) = -2(t + (2n + 1)q) \log d_\Omega(z)$ . Then for  $1 \leq k \leq n + 1$  we have

$$(9)_k \quad \|f^{(k)}\|_\psi \leq N_2 A \eta^t.$$

By Lemma 4, [2] there exists an  $r_{n+1}$ -tuple of  $(0, n - 1)$ -forms  $g^{(n+1)} = (g_1^{(n+1)}, \dots, g_{r_{n+1}}^{(n+1)})$  on  $\Omega$  with locally square integrable coefficients such that

$$(10) \quad \bar{\partial} g^{(n+1)} = f^{(n+1)}$$

and

$$(11) \quad \|g^{(n+1)}\|_\psi \leq \mu \|f^{(n+1)}\|_\psi \leq \mu N_2 A \eta^t.$$

We are going to construct by backward induction on  $k$  an  $r_k$ -tuple of  $(0, k - 2)$ -forms  $g^{(k)} = (g_1^{(k)}, \dots, g_{r_k}^{(k)})$  on  $\Omega$  with locally square integrable coefficients for  $2 \leq k \leq n + 1$  such that (12)<sub>k</sub> holds for  $2 \leq k \leq n$  and (13)<sub>k</sub> and (14)<sub>k</sub> hold for  $2 \leq k \leq n + 1$ :

$$(12)_k \quad \bar{\partial} g_i^{(k)} = f_i^{(k)} - \sum_{i=1}^{r_{k+1}} \varphi_{ij}^{(k)} g_j^{(k+1)}, \quad 1 \leq i \leq r_k,$$

$$(13)_k \quad \bar{\partial} \left( f_i^{(k-1)} - \sum_{j=1}^{r_k} \varphi_{ij}^{(k-1)} g_j^{(k)} \right) = 0, \quad 1 \leq i \leq r_{k-1},$$

$$(14)_k \quad \|g^{(k)}\|_{\psi} \leq \mu(1 + \mu r N_3)^{n-k+1} N_2 A \eta^t.$$

(14)<sub>n+1</sub> follows from (11). (13)<sub>n+1</sub> follows from (8)<sub>n</sub> and (10). Hence  $g^{(n+1)}$  satisfies the requirements. Suppose now for some  $2 < k \leq n+1$ ,  $g^{(k)}, \dots, g^{(n+1)}$  have been constructed. From (9)<sub>k-1</sub>, (14)<sub>k</sub> and the definitions of  $r$  and  $N_3$ , we obtain

$$\begin{aligned} \left\| f_i^{(k-1)} - \sum_{j=1}^{r_k} \varphi_{ij}^{(k-1)} g_j^{(k)} \right\|_{\psi} &\leq N_2 A \eta^t + r N_3 \mu (1 + \mu r N_3)^{n-k+1} N_2 A \eta^t \\ &= (1 + \mu r N_3)^{n-k+2} N_2 A \eta^t, \quad 1 \leq i \leq r_{k-1}. \end{aligned}$$

By (13)<sub>k-1</sub> and Lemma 4, [2], we obtain an  $r_{k-1}$ -tuple of  $(0, k-2)$ -forms  $g^{(k-1)} = (g_1^{(k-1)}, \dots, g_{r_{k-1}}^{(k-1)})$  on  $\Omega$  with locally square integrable coefficients such that (12)<sub>k-1</sub> and (14)<sub>k-1</sub> are satisfied. (8)<sub>k-2</sub> and (12)<sub>k-1</sub> imply that

$$\bar{\partial} f_i^{(k-2)} = \sum_{j=1}^{r_{k-1}} \varphi_{ij}^{(k-2)} \left( \bar{\partial} g_j^{(k-1)} + \sum_{l=1}^{r_k} \varphi_{il}^{(k-1)} g_l^{(k)} \right) = \sum_{j=1}^{r_{k-1}} \varphi_{ij}^{(k-2)} \bar{\partial} g_j^{(k-1)}.$$

This gives us (13)<sub>k-1</sub>. The construction by induction is complete. Set

$$g_i = f_i^{(1)} - \sum_{j=1}^{r_2} \varphi_{ij}^{(1)} g_j^{(2)} \quad \text{and} \quad g = (g_1, \dots, g_{r_1}).$$

(13)<sub>2</sub> implies that  $g \in \Gamma(\Omega, \mathcal{O}^{r_1})$ .  $\Phi(g) = f$ . (9)<sub>1</sub> and (14)<sub>2</sub> yield  $\|g\|_{\psi} \leq (1 + \mu r N_3)^n N_2 A \eta^t$ .  $|g(z)| \leq M^{1+t} A d_0(z)^{-t-p}$  for  $z \in \Omega$  by Lemma 4. Q.E.D.

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