

## Invariance of plurigenera

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In this paper we give a proof of the following long conjectured result on the invariance of the plurigenera.

*Main Theorem.* Let  $\pi : X \rightarrow \Delta$  be a smooth projective family of compact complex manifolds parametrized by the open unit 1-disk  $\Delta$ . Assume that the fibers  $X_t = \pi^{-1}(t)$ ,  $t \in \Delta$ , are of general type. Then for every positive integer  $m$  the plurigenus  $\dim_{\mathbb{C}} \Gamma(X_t, mK_{X_t})$  is independent of  $t \in \Delta$ , where  $K_{X_t}$  is the canonical line bundle of  $X_t$ .

*Notations and Terminology.* The canonical line bundle of a complex manifold  $Y$  is denoted by  $K_Y$ . A compact manifold  $Y$  is said to be of general type if there is a point  $P \in Y$  with the property that one can find elements  $s_0, s_1, \dots, s_n \in \Gamma(Y, mK_Y)$  such that  $s_0$  is nonzero at  $P$  and  $\frac{s_1}{s_0}, \dots, \frac{s_n}{s_0}$  form a local coordinate system of  $Y$  at  $P$ , where  $\dim Y = n$ .

By the family  $\pi : X \rightarrow \Delta$  being projective we mean that there exists a positive holomorphic line bundle on the total space  $X$  of the family. Let  $n$  be the complex dimension of each  $X_t$  for  $t \in \Delta$ . Let  $K_{X,\pi}$  be the line bundle on  $X$  whose restriction to  $X_t$  is  $K_{X_t}$  for each  $t \in \Delta$ . Since the normal bundle of  $X_t$  in  $X$  is trivial, the two line bundles  $K_X$  and  $K_{X,\pi}$  are naturally isomorphic. Under this natural isomorphism a local section  $s$  of  $K_{X,\pi}$  corresponds to the local section  $s \wedge \pi^*(dt)$  of  $K_X$ . Unless there is some risk of confusion, in this paper we will, without any further explicit mention, always identify  $K_{X,\pi}$  with  $K_X$  by this natural isomorphism. By Grauert's direct image theorem,  $\pi_* \mathcal{O}(mK_X) = \pi_* \mathcal{O}(mK_{X,\pi})$  is a coherent sheaf, and is thus generated by

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its global sections over  $\Delta$ . The rank of the space of sections  $\Gamma(X_t, mK_{X_t})$  might a priori jump at points  $t \in \Delta$  where there are sections in  $\Gamma(X_t, mK_{X_t})$  which do not extend to sections in  $\Gamma(X, mK_X)$ . The Main Theorem is therefore equivalent to the statement that for every  $t \in \Delta$  and every positive integer  $m$  every element of  $\Gamma(X_t, mK_{X_t})$  can be extended to an element of  $\Gamma(X, mK_X)$ .

The Hermitian metrics of holomorphic line bundles in this paper are allowed to have singularities and may not be smooth. For a Hermitian metric  $h_0 = e^{-\varphi_0}$  of a holomorphic line bundle  $L_0$  over  $X_0$  we denote by  $\mathcal{I}_{h_0}$  or by  $\mathcal{I}_{\varphi_0}$  its multiplier ideal sheaf on  $X_0$  which by definition means the ideal sheaf on  $X_0$  of all local holomorphic function germs  $f$  on  $X_0$  such that  $|f|^2 e^{-\varphi_0}$  is integrable. For the proof of the Main Theorem, only multiplier ideal sheaves on  $X_0$  are considered and no multiplier ideal sheaves on  $X$  are used. In the case of a Hermitian metric  $h = e^{-\varphi}$  of a holomorphic line bundle  $L$  over  $X$ , for notational simplicity we simply use the notation  $\mathcal{I}_h$  or  $\mathcal{I}_\varphi$  to mean the multiplier ideal sheaf for the Hermitian metric  $h|_{X_0} = e^{-\varphi}|_{X_0}$  of the holomorphic line bundle  $L|_{X_0}$  over  $X_0$  and suppress the notation for restriction to  $X_0$ .

The stalk of a sheaf  $\mathcal{F}$  at a point  $P$  is denoted by  $\mathcal{F}_P$ . The structure sheaf of a complex manifold  $Y$  is denoted by  $\mathcal{O}_Y$ . If  $s$  is a global holomorphic section of a holomorphic line bundle  $L$  over a complex manifold  $Y$  and if  $\mathcal{I}$  is an ideal sheaf on  $Y$ , we say that the germ of  $s$  at a point  $P$  belongs to  $\mathcal{I}_P$  if the holomorphic function germ at  $P$  which corresponds to the germ of  $s$  at  $P$  with respect to some local trivialization of  $L$  belongs to  $\mathcal{I}_P$ . We say that  $s$  is everywhere locally contained in  $\mathcal{I}$  if at every point  $P \in Y$  the germ of  $s$  at  $P$  belongs to  $\mathcal{I}_P$ , or equivalently,  $s$  belongs to the subset  $\Gamma(Y, \mathcal{I} \otimes L)$  of  $\Gamma(Y, L)$ . If  $E$  is another holomorphic line bundle over  $Y$  with a Hermitian metric  $e^{-\lambda}$ , we say that  $|s|^2 e^{-\lambda}$  is locally uniformly bounded (respectively locally integrable) on  $Y$  if at every point  $P \in Y$  with respect to some local trivializations of  $L$  and  $E$  on some open neighborhood  $U$  of  $P$  the function on  $U$  corresponding to  $|s|^2 e^{-\lambda}$  is uniformly bounded (respectively integrable) on  $U$ . We say that  $s$  is locally  $L^2$  with respect to  $e^{-\lambda}$  on  $Y$  if  $|s|^2 e^{-\lambda}$  is locally integrable on  $Y$ .

*History and Sketch of the Proof of the Main Theorem.* Iitaka [I69-71] proved the special case of the invariance of the plurigena in a family of surfaces. His method works only for surfaces because it uses much of the information from the classification of surfaces. Levine [L83, L85] proved that for every positive integer  $m$  every element of  $\Gamma(X_0, mK_{X_0})$  can be extended to the fiber of  $X$  over the double point of  $\Delta$  at  $t = 0$ . So far there is no way to continue the process to get an extension to the

fiber of  $X$  over a point of  $\Delta$  at  $t = 0$  of any finite order. Nakayama [Nak86] pointed out that if the relative case of the minimal model program can be carried out for a certain dimension, the conjecture of the invariance of the plurigenera for that dimension would follow directly from it. Thus the invariance of the plurigenera for threefolds is a consequence of the completion of the relative case of the minimal model program for the case of threefolds by Kollár and Mori [KM92].

For the proof of the Main Theorem here we use a strategy completely different from those used by the others in the past. We now sketch our strategy and leave out the less essential technical details. There are some unavoidable technical inaccuracies in the sketch due to the suppression of precise details. There are three ingredients in our proof: Nadel's multiplier ideal sheaves [Nad89], Skoda's result on the generation of ideals with  $L^2$  estimates with respect to a pluri-subharmonic weight [Sk72], and the extension theorem of Ohsawa-Takegoshi-Manivel for holomorphic top-degree forms which are  $L^2$  with respect to a plurisubharmonic weight [OT87,M93]. The extension theorem of Ohsawa-Takegoshi-Manivel is for the setting of a Stein domain or manifold and a global plurisubharmonic function as weight. Here we adapt it to the case of a projective family of compact complex manifolds and a Hermitian metric of a line bundle with nonnegative curvature current. The adaptation is done by restricting to a Stein Zariski open subset on which the line bundle is globally trivial, because  $L^2$  bounds automatically extend the domain of definition from the Zariski open subset to the family of compact manifolds.

We take the  $m$ th roots of basis elements of  $\Gamma(X_0, mK_{X_0})$  for every positive integer  $m$  to use them in an infinite series to construct a Hermitian metric  $e^{-\varphi_0}$  for  $K_{X_0}$ . We also take the  $m$ th roots of basis elements of  $\Gamma(X, mK_X)|_{X_0}$  for every positive integer  $m$  to use them in an infinite series to construct a Hermitian metric  $e^{-\varphi}$  for  $K_{X_0}$ . By construction, we have  $\varphi \leq \varphi_0$  and  $\varphi$  might have more poles than  $\varphi_0$ , since there might be sections in  $\Gamma(X_0, mK_{X_0})$  which do not extend to  $X$ . However, Skoda's result can be used in combination with the extension theorem of Ohsawa-Takegoshi-Manivel to show that the singularity of  $\varphi$  on  $X_0$  is not much worse than the singularity of  $\varphi_0$ . This is made by comparing the multiplier ideal sheaves  $\mathcal{I}_{\ell\varphi+\psi}$ ,  $\mathcal{I}_{\ell\varphi_0+\psi_0}$ , with small corrections  $\psi$ ,  $\psi_0$  on  $\varphi$ , and  $\varphi_0$  to make sure that the curvature is positive definite. At this point, an induction on  $\ell$  is used. Finally, we raise everything to the power  $1/\ell$ , in such a way that the contribution of the extra terms  $\psi$  and  $\psi_0$  becomes negligible, using Hölder's inequality in the  $L^2$  estimates. A new

application of the Ohsawa-Takegoshi-Manivel extension theorem then shows that every of  $mK_{X_0}$ -valued form on  $X_0$  extends as an  $mK_X$ -valued form on  $X$ .

*Preliminary Tools.* Positivity of the curvature of  $K_X$  and of related line bundles will be exploited through Propositions 1 and 2 below. Proposition 1 establishes a uniform global generation property of multiplier ideal sheaves. Proposition 2 will allow us to extend pluri-canonical sections from  $X_0$  to  $X$  under suitable conditions.

**Proposition 1.** *Let  $E$  be an ample holomorphic line bundle over an  $n$ -dimensional compact complex manifold  $Y$  such that for every point  $P$  of  $Y$  there are a finite number of elements of  $\Gamma(Y, E)$  which all vanish to order at least  $n + 1$  at  $P$  and which do not simultaneously vanish outside  $P$ . Then for every holomorphic line bundle  $L$  over  $Y$  with a Hermitian metric locally of the form  $e^{-\xi}$ ,  $\xi$  plurisubharmonic, of associated multiplier ideal sheaf  $\mathcal{I}_\xi$  the space of global sections  $\Gamma(Y, \mathcal{I}_\xi \otimes (L + E + K_Y))$  generates the sheaf  $\mathcal{I}_\xi \otimes (L + E + K_Y)$  at every point of  $Y$ .*

*Proof.* The key ingredient is the following result of Skoda [Sk72, Th.1, pp. 555–556].

Let  $\Omega$  be a pseudoconvex domain in  $\mathbf{C}^n$  and  $\psi$  be a plurisubharmonic function on  $\Omega$ . Let  $g_1, \dots, g_p$  be holomorphic functions on  $\Omega$ . Let  $\alpha > 1$  and  $q = \inf(n, p - 1)$ . Then for every holomorphic function  $f$  on  $\Omega$  such that

$$\int_{\Omega} |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda < \infty ,$$

there exist holomorphic functions  $h_1, \dots, h_p$  on  $\Omega$  such that

$$f = \sum_{j=1}^p g_j h_j$$

and

$$\int_{\Omega} |h|^2 |g|^{-2\alpha q} e^{-\psi} d\lambda \leq \frac{\alpha}{\alpha - 1} \int_{\Omega} |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda ,$$

where

$$|g| = \left( \sum_{j=1}^p |g_j|^2 \right)^{\frac{1}{2}}, \quad |h| = \left( \sum_{j=1}^p |h_j|^2 \right)^{\frac{1}{2}},$$

and  $d\lambda$  is the Euclidean volume element of  $\mathbf{C}^n$ .

Fix arbitrarily  $P \in Y$ . Take an arbitrary element  $s$  of  $(\mathcal{F}_\xi)_P$ . Let  $z = (z_1, \dots, z_n)$  be a local coordinate system on some open neighborhood  $U$  of  $P$  with  $z(P) = 0$  such that  $L|_U$  is trivial. Let  $\rho$  be a cut-off function centered at  $P$  so that  $\rho$  is a smooth nonnegative-valued function with compact support in  $U$  which is identically 1 on some Stein open neighborhood  $\Omega$  of  $P$ . Choose  $u_1, \dots, u_N \in \Gamma(Y, E)$  whose common zero-set consists of the single point  $P$  and which all vanish to order at least  $n + 1$  at  $P$ . Let  $h_E$  be a smooth Hermitian metric of  $E$  whose curvature form is strictly positive at every point of  $Y$ . Let  $0 < \eta < \frac{1}{n+1}$ . By the standard techniques of  $L^2$  estimates of  $\bar{\partial}$ , we can solve the equation

$$\bar{\partial}\sigma = s\bar{\partial}\rho$$

for a smooth section  $\sigma$  of  $L + E + K_Y$  over  $Y$  which is  $L^2$  with respect to the Hermitian metric

$$\frac{e^{-\xi}(h_E)^\eta}{\left(\sum_{j=1}^N |u_j|^2\right)^{1-\eta}}$$

of  $L + E$ . Then  $\rho s - \sigma$  is an element of  $\Gamma(Y, \mathcal{F}_\xi \otimes (L + E + K_Y))$ . Since  $s\bar{\partial}\rho$  is identically zero on  $\Omega$ , it follows that  $\sigma$  is holomorphic on  $\Omega$ . We now apply Skoda's result to the case  $g_j = z_j$  ( $1 \leq j \leq n$ ) with  $q = n - 1$  and  $\alpha = \frac{(1-\eta)(n+1)-1}{n-1} > 1$  and  $\psi = \xi$ . (For the case  $n = 1$  we simply choose  $\alpha$  be any number greater than 1, because in that case  $\alpha q$  is always zero.) Let  $|z| = \left(\sum_{j=1}^n |z_j|^2\right)^{\frac{1}{2}}$ . Since  $u_1, \dots, u_N$  all vanish to order at least  $n + 1$  at  $P$ , it follows that

$$\int_{\Omega} |\sigma|^2 e^{-\xi} |z|^{-2\alpha q - 2} = \int_{\Omega} |\sigma|^2 e^{-\xi} |z|^{-2(1-\eta)(n+1)} < \infty .$$

By Skoda's result

$$\sigma = \sum_{j=1}^n \tau_j z_j$$

locally at  $P$  for some  $\tau_1, \dots, \tau_n \in (\mathcal{F}_\xi)_P$ .

Let  $J$  be the ideal at  $P$  generated by elements of  $\Gamma(Y, \mathcal{I}_\xi \otimes (L + E + K_Y))$  over  $(\mathcal{O}_Y)_P$ . It follows from

$$\rho s - \sigma \in \Gamma(Y, \mathcal{I}_\xi \otimes (L + E + K_Y))$$

that

$$s \in J + \mathfrak{m}_P(\mathcal{I}_\xi)_P ,$$

where  $\mathfrak{m}_P$  is the maximum ideal of  $Y$  at  $P$ . Since  $s$  is an arbitrary element of  $(\mathcal{I}_\xi)_P$ , it follows that

$$(\mathcal{I}_\xi)_P \subset J + \mathfrak{m}_P(\mathcal{I}_\xi)_P .$$

Clearly we have  $J \subset (\mathcal{I}_\xi)_P$ . Thus

$$(\mathcal{I}_\xi)_P/J \subset \mathfrak{m}_P((\mathcal{I}_\xi)_P/J) .$$

By Nakayama's lemma,

$$(\mathcal{I}_\xi)_P/J = 0$$

and  $J = (\mathcal{I}_\xi)_P$ .

Q.E.D.

The next Proposition is an extension statement which is an adaptation of the extension theorem of Ohsawa-Takegoshi-Manivel.

**Proposition 2.** *Let  $\gamma : Y \rightarrow \Delta$  be a projective family of compact complex manifolds parametrized by the open unit 1-disk  $\Delta$ . Let  $Y_0 = \gamma^{-1}(0)$  and let  $n$  be the complex dimension of  $Y_0$ . Let  $L$  be a holomorphic line bundle with a Hermitian metric which locally is represented by  $e^{-\chi}$  such that  $\sqrt{-1}\partial\bar{\partial}\chi \geq \omega$  in the sense of currents for some smooth positive  $(1, 1)$ -form  $\omega$  on  $Y$ . Let  $0 < r < 1$  and  $\Delta_r = \{t \in \Delta \mid |t| < r\}$ . Then there exists a positive constant  $A_r$  with the following property. For any holomorphic  $L$ -valued  $n$ -form  $f$  on  $Y_0$  with*

$$\int_{Y_0} |f|^2 e^{-\chi} < \infty ,$$

*there exists a holomorphic  $L$ -valued  $(n + 1)$ -form  $\tilde{f}$  on  $\gamma^{-1}(\Delta_r)$  such that  $\tilde{f}|_{Y_0} = f \wedge \gamma^*(dt)$  at points of  $Y_0$  and*

$$\int_Y |\tilde{f}|^2 e^{-\chi} \leq A_r \int_{Y_0} |f|^2 e^{-\chi} .$$

Here no metrics of the tangent bundles of  $Y_0$  and  $Y$  are needed to define the integrals of the absolute-value squares of top-degree holomorphic forms  $f$  and  $\tilde{f}$  respectively on  $Y_0$  and  $Y$ .

*Proof.* The proof can be easily adapted in the following way from the techniques given in [Si96] for the alternative proof there of the theorem of Ohsawa-Takegoshi. (Proofs can also be obtained by modifying those in [OT83, M93].) Let  $v$  be a meromorphic section of  $L$  over  $Y$  so that neither the pole-set nor the zero-set of  $v$  contains  $Y_0$ . Choose a complex hypersurface  $Z$  in  $Y$  containing the zero-set and the pole-set of  $v$  such that  $Z$  does not contain  $Y_0$  and  $Y - Z$  is Stein. For every positive integer  $\nu$  let  $\Omega_\nu$  be a relatively compact Stein open subset of  $Y - Z$  with smooth strictly pseudoconvex boundary such that  $\cup_{\nu=1}^\infty \Omega_\nu = Y - Z$  and  $\Omega_\nu$  is relatively compact in  $\Omega_{\nu+1}$ . On  $Y - Z$  under the isomorphism defined by division by  $v$  the line bundle  $L|_{Y - Z}$  is globally trivial. We let  $\tilde{\chi}$  be the plurisubharmonic function  $-\log(|v|^2 e^{-\chi})$  on  $Y - Z$ .

We now apply the techniques in [Si96] of the alternative proof of the theorem of Ohsawa-Takegoshi to extend, after multiplication by  $\gamma^*(dt)$ , the top-degree holomorphic form  $\frac{f}{v}$  on  $\Omega_\nu \cap Y_0$  which is  $L^2$  on  $\Omega_\nu \cap Y_0$  with respect to  $e^{-\tilde{\chi}}$  to a top-degree holomorphic form  $G_\nu$  on  $\gamma^{-1}(\Delta_r) \cap \Omega_\nu$  whose  $L^2$  norm on  $\gamma^{-1}(\Delta_r) \cap \Omega_\nu$  with respect to  $e^{-\tilde{\chi}}$  is bounded by a finite constant independent of  $\nu$ . When we apply the techniques of the alternative proof of the theorem of Ohsawa-Takegoshi, we have to use holomorphic tangent vector fields of the Stein manifold  $\Omega_{\nu+1}$  to get a sequence of smooth plurisubharmonic functions on  $\Omega_\nu$  which approach the plurisubharmonic function  $\tilde{\chi}$  on  $\Omega_\nu$ . The extension  $\tilde{f}$  is obtained as the limit of  $G_\nu v$  as  $\nu$  goes to infinity. The smooth positive  $(1, 1)$ -form  $\omega$  in the assumption is needed for the  $\nu$ -independent *a priori* estimates for the solution of the modified  $\bar{\partial}$  equation on  $\gamma^{-1}(\Delta_r) \cap \Omega_\nu$  in the techniques of the alternative proof of the theorem of Ohsawa-Takegoshi. Q.E.D.

*Construction of Hermitian metrics on  $K_{X_0}$ .* One of the technical details is that the Hermitian metric on  $K_X$  has to be chosen to make sure that its curvature current dominates a smooth positive  $(1, 1)$ -form in order to apply the  $L^2$  existence and extension theorems of Skoda and Ohsawa-Takegoshi-Manivel. For that modification the Kodaira technique of writing some high multiple of a big line bundle as an effective divisor plus an ample line bundle is used.

**Lemma 3.** *There is a positive integer  $a$  such that  $aK_X = D + F$ , where  $D$  is an effective divisor on  $X$  not containing  $X_0$  and  $F$  is a positive line bundle on  $X$ .*

*Proof.* Let  $F$  be a positive line bundle over  $X$ , and let  $r_a$  be the generic rank of  $\Gamma(X_t, aK_{X_t} - F)$ , which is achieved for  $t \in \Delta - S_a$  in the complement of a suitable discrete set  $S_a \subset \Delta$ . Fix  $t_1 \in \Delta - S_a$ . Since  $X_{t_1}$  is of general type, we know that  $h^0(X_{t_1}, aK_{X_{t_1}}) \geq ca^n$ , hence  $h^0(X_{t_1}, aK_{X_{t_1}} - F) \geq c'a^n$  for suitable  $c, c' > 0$  and  $a$  large enough. Thanks to the choice of  $t_1$ , any non zero section in  $\Gamma(X_{t_1}, aK_{X_{t_1}} - F)$  extends to a section  $s$  in  $\Gamma(X, aK_X - F)$ , thus  $aK_X = D + F$  where  $D$  is the divisor of  $s$ . If necessary, we can remove  $X_0$  from the components in  $D$  after dividing  $s$  by a suitable power of  $t$ . Q.E.D.

The next step is to construct Hermitian metrics on  $K_{X_0}$  and  $K_X$ , respectively, and to compare their related multiplier ideal sheaves. For every positive integer  $m$  we choose a basis

$$s_{0,1}^{(m)}, \dots, s_{0,q_{m,0}}^{(m)} \in \Gamma(X_0, mK_{X_0})$$

and we choose

$$s_1^{(m)}, \dots, s_{q_m}^{(m)} \in \Gamma(X, mK_X)$$

so that

$$s_1^{(m)}|_{X_0}, \dots, s_{q_m}^{(m)}|_{X_0}$$

is a basis of  $\Gamma(X, mK_X)|_{X_0}$  and  $s_v^{(m)} = s_{0,v}^{(m)}$  for  $1 \leq v \leq q_m$ . We can choose a sequence of positive numbers  $\theta_m$  so that

$$\sum_{m=1}^{\infty} \theta_m \left( \sum_{v=1}^{q_{m,0}} |s_{0,v}^{(m)}|^{\frac{2}{m}} \right)$$

converges uniformly on compact subsets of  $X_0$  to a Hermitian metric of  $-K_{X_0}$  and

$$\sum_{m=1}^{\infty} \theta_m \left( \sum_{v=1}^{q_m} |s_v^{(m)}|^{\frac{2}{m}} \right)$$

converges uniformly on compact subsets of  $X$  to a Hermitian metric of  $-K_X$ .

Locally on  $X_0$  we define

$$\varphi_0 = \log \sum_{m=1}^{\infty} \theta_m \left( \sum_{v=1}^{q_{m,0}} \left| s_{0,v}^{(m)} \right|^{\frac{2}{m}} \right)$$

so that  $e^{-\varphi_0}$  is a Hermitian metric of  $K_{X_0}$ . Locally on  $X$  we define

$$\varphi = \log \sum_{m=1}^{\infty} \theta_m \left( \sum_{v=1}^{q_m} \left| s_v^{(m)} \right|^{\frac{2}{m}} \right)$$

so that  $e^{-\varphi}$  is a Hermitian metric of  $K_X$ . By construction, we have  $\varphi \leq \varphi_0$  on  $X_0$ .

After possibly passing to a multiple in Lemma 3, we can choose an integer  $a \geq 2$  such that  $aK_X = D + F$ , where  $D$  is an effective divisor on  $X$  not containing  $X_0$  and  $F$  is such a high multiple of a positive line bundle on  $X$  that

- (i) for every point  $P \in X_0$ , there exist a finite number of elements of  $\Gamma(X, F - 2K_X)|_{X_0}$  whose common zero-set consists only of the single point  $P$  and which all vanish to order at least  $n + 1$  at  $P$  and
- (ii) a basis of  $\Gamma(X, F)|_{X_0}$  embeds  $X_0$  as a complex submanifold of some complex projective space.

Let  $s_D$  be the canonical section of the holomorphic line bundle  $D$  so that the divisor of  $s_D$  is  $D$ . Let

$$u_1, \dots, u_N \in \Gamma(X, F)$$

such that

$$u_1 \mid X_0, \dots, u_N \mid X_0$$

form a basis of  $\Gamma(X, F)|_{X_0}$ . Since  $s_D u_j \in \Gamma(X, aK_X)$  ( $1 \leq j \leq N$ ), we get a Hermitian metric

$$e^{-\psi} = \left( \frac{1}{|s_D|^2 \sum |u_j|^2} \right)^{\frac{1}{a}}$$

for the line bundle  $K_X$ . We can multiply the sections  $u_j$  by small coefficients in such a way that the sections  $s_D u_j$  are linear combinations with small coefficients of the basis sections  $(s_v^{(m)})$ ,  $m = a$ . Then we can achieve that  $\psi \leq \varphi$  over  $X$ , possibly after shrinking  $\Delta$  a little bit. Moreover

$$\psi = \frac{1}{a} \left( \log |s_D|^2 + \chi \right)$$

where  $\chi = \log(\sum |u_j|^2)$  defines a smooth Hermitian metric of positive curvature on  $F$ . Thus

**Lemma 4.** *The Hermitian metric  $e^{-\psi}$  has positive definite curvature and  $\psi \leq \varphi \leq \varphi_0$  on  $X_0$ .*

Our crucial argument is the following comparison result for multiplier ideal sheaves over  $X_0$ . (As stated at the beginning of the paper, in the statement below and for the rest of the paper, the notation  $\mathcal{I}_{(\ell+a-\epsilon)\varphi+\epsilon\psi}$  denotes an ideal sheaf on  $X_0$  and not an ideal sheaf on  $X$ ).

**Proposition 5.** *Choose  $0 < \epsilon < 1$  so small that  $e^{-\epsilon\psi}$  is locally integrable on  $X$  (possibly after shrinking  $\Delta$  a little bit) and  $e^{-\epsilon\psi}|_{X_0}$  is locally integrable on  $X_0$ . Then*

$$\mathcal{I}_{(\ell-\epsilon)\varphi_0+(a+\epsilon)\psi} \subset \mathcal{I}_{(\ell-1+a-\epsilon)\varphi+\epsilon\psi}$$

for all integers  $\ell \geq 1$ .

*Proof.* We argue by induction on  $\ell$ . For  $\ell = 1$ , Lemma 4 implies

$$(1 - \epsilon)\varphi_0 + (a + \epsilon)\psi \leq (1 - \epsilon)\varphi_0 + a\varphi + \epsilon\psi \leq C + (a - \epsilon)\varphi + \epsilon\psi$$

since  $\varphi_0$  is locally bounded from above by a constant  $C$ . We thus get

$$\mathcal{I}_{(1-\epsilon)\varphi_0+(a+\epsilon)\psi} \subset \mathcal{I}_{(a-\epsilon)\varphi+\epsilon\psi} ,$$

as desired.

Now, assume that the inclusion has been proved for  $\ell$ . Take a germ of function  $f$  in the ideal sheaf  $\mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\epsilon)\psi}$ , defined on a small neighborhood  $U$  of a point  $P \in X_0$ . Take a non vanishing section  $e$  of  $(\ell + a)K_{X_0}$  over  $U$ . Then  $s = fe$  is a section of

$$\mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\epsilon)\psi} \otimes (\ell + a)K_{X_0}$$

over  $U$ . Note that, for an arbitrary plurisubharmonic function  $\xi$ , we have

$$\mathcal{I}_{\xi+\log |s_D|^2} = \mathcal{I}_\xi \otimes \mathcal{O}(-D) .$$

Writing  $aK_{X_0} = (D + F)|_{X_0}$  by Lemma 1 and  $a\psi = \log |s_D|^2 + \chi$  by definition of  $\psi$ , we can reinterpret  $s$  as a section of

$$\mathcal{I}_{(\ell+1-\epsilon)\varphi_0+\epsilon\psi+\log |s_D|^2+\chi} \otimes ((D + F)|_{X_0} + \ell K_{X_0})$$

$$= \mathcal{I}_{(\ell+1-\epsilon)\varphi_0+\varepsilon\psi} \otimes (E|_{X_0} + (\ell + 2)K_{X_0})$$

where  $E = F - 2K_X$  (as  $\chi$  is smooth,  $\chi$  does not contribute to the multiplier ideal sheaf). Observe that  $(\ell + 1 - \epsilon)\varphi_0 + \varepsilon\psi$  defines a Hermitian metric of positive curvature on  $(\ell + 1)K_{X_0}$ . By assumption (i) above and by Proposition 1 applied with  $Y = X_0$  and  $L = (\ell + 1)K_{X_0}$ , we conclude that

$$\mathcal{I}_{(\ell+1-\epsilon)\varphi_0+\varepsilon\psi} \otimes (E|_{X_0} + (\ell + 2)K_{X_0})$$

is generated by its global sections on  $X_0$ . Hence, without loss of generality, we may restrict ourselves to germs  $f$  such that  $fe$  coincides on  $U$  with a global section

$$s \in \Gamma(X_0, \mathcal{I}_{(\ell+1-\epsilon)\varphi_0+\varepsilon\psi} \otimes (E|_{X_0} + (\ell + 2)K_{X_0})) .$$

Reversing the computations above, we find

$$\begin{aligned} &\Gamma(X_0, \mathcal{I}_{(\ell+1-\epsilon)\varphi_0+\varepsilon\psi} \otimes (E|_{X_0} + (\ell + 2)K_{X_0})) \\ &= \Gamma(X_0, \mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\varepsilon)\psi} \otimes (\ell + a)K_{X_0}) \\ &\subset \Gamma(X_0, \mathcal{I}_{(\ell-\epsilon)\varphi_0+(a+\varepsilon)\psi} \otimes (\ell + a)K_{X_0}) \\ &\subset \Gamma(X_0, \mathcal{I}_{(\ell+a-1-\epsilon)\varphi+\varepsilon\psi} \otimes (\ell + a)K_{X_0}) , \end{aligned}$$

where the first inclusion is obtained by forgetting one term  $\varphi_0$  in the weight, and the second one is a consequence of the induction hypothesis for  $\ell$ . Now,  $(\ell + a - 1 - \epsilon)\varphi + \varepsilon\psi$  defines a Hermitian metric of positive definite curvature on  $L = (\ell + a - 1)K_X$ , and Proposition 2 implies that  $s$  extends to a global section  $\tilde{s} \in \Gamma(X, (\ell + a)K_X)$  (possibly after shrinking  $\Delta$  a little bit). The definition of  $\varphi$  implies  $|\tilde{s}|^2 \leq Ce^{(\ell+a)\varphi}$ , hence

$$|\tilde{s}|^2 e^{-(\ell+a)\varphi-\varepsilon\psi} \leq Ce^{-\varepsilon\psi}$$

is integrable on  $X$ . From this we conclude that  $fe = s = \tilde{s}|_{X_0}$  lies in  $\mathcal{I}_{(\ell+a)\varphi+\varepsilon\psi} \otimes (\ell + a)K_{X_0}$ , thus  $f \in \mathcal{I}_{(\ell+a)\varphi+\varepsilon\psi}$ . Step  $\ell + 1$  of the induction is proved. Q.E.D.

*Proof of the Main Theorem.* Take a section  $s \in \Gamma(X_0, mK_{X_0})$ , for any positive integer  $m$ . By definition of  $\varphi_0$ , we locally have  $|s|^2 \leq Ce^{m\varphi_0}$  over  $X_0$ . If  $s_D$  is the canonical section of  $\mathcal{O}(D)$ , we infer that  $s^\ell s_D$  is locally  $L^2$  with respect to the weight  $e^{-\ell m\varphi_0-(a+\varepsilon)\psi}$ , since  $a\psi$  has the same singularity as  $\log|s_D|^2$  and  $e^{-\varepsilon\psi}$  is supposed to be locally integrable on  $X_0$ . Hence, the components of  $s^\ell s_D$  are locally in the ideal

sheaf  $\mathcal{F}_{(\ell m - \varepsilon)\varphi_0 + (a + \varepsilon)\psi}$ . By Proposition 5, they are also in  $\mathcal{F}_{(\ell m - 1 - \varepsilon)\varphi + \varepsilon\psi}$ , i.e.

$$\int_U |s^\ell s_D|^2 e^{-(\ell m - 1 - \varepsilon)\varphi - \varepsilon\psi} < +\infty$$

on every sufficiently small open set  $U$  with  $K_{X_0}|_U$  and  $\mathcal{O}(D)|_U$  trivial. In an equivalent way, we can write

$$\int_U |s|^{2\ell} e^{-(\ell m - 1 - \varepsilon)\varphi + (a - \varepsilon)\psi} < +\infty .$$

Take  $\ell$  so large that  $a/(\ell - 1) \leq \varepsilon$ , in order that  $e^{\varphi - \frac{a}{\ell-1}\psi} \leq Ce^{-\varepsilon\psi}$  be integrable over  $U$ . By Hölder’s inequality with conjugate exponents  $\ell, \ell' = \ell/(\ell - 1)$ , we find

$$\begin{aligned} +\infty &> \left( \int_U |s|^{2\ell} e^{-(\ell m - 1 - \varepsilon)\varphi + (a - \varepsilon)\psi} \right)^{1/\ell} \left( \int_U e^{\varphi - \frac{a}{\ell-1}\psi} \right)^{(\ell-1)/\ell} \\ &\geq \int_U |s|^2 e^{-(m - \frac{1}{\ell} - \frac{\varepsilon}{\ell})\varphi + (\frac{a}{\ell} - \frac{\varepsilon}{\ell})\psi} e^{(1 - \frac{1}{\ell})\varphi - \frac{a}{\ell}\psi} = \int_U |s|^2 e^{-(m - 1 - \delta)\varphi - \delta\psi} \end{aligned}$$

with  $\delta = \varepsilon/\ell$ . We view  $s$  as a section of  $K_{X_0} + (L|_{X_0})$  with  $L = (m - 1)K_X$ . The weight  $(m - 1 - \delta)\varphi - \delta\psi$  defines a Hermitian metric of positive definite curvature on  $L$ , and  $s$  is globally  $L^2$  with respect to this metric. By the Ohsawa-Takegoshi-Manivel extension theorem, we can therefore extend  $s$  to a section  $\tilde{s} \in \Gamma(X, K_X + L) = \Gamma(X, mK_X)$ , possibly after shrinking  $\Delta$ . This concludes the proof of the Main Theorem.

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