Very Ampleness Part of Fujita's Conjecture and Multiplier Ideal Sheaves of Kohn and Nadel

Yum-Tong Siu [1]

Let X be a compact complex manifold of complex dimension n with canonical line bundle K_X and L be a positive holomorphic line bundle over X. Fujita [F87] made a two-part conjecture. One part concerns effective freeness which says that for $m \geq n+1$ the line bundle $mL + K_X$ is generated by its global holomorphic sections. Another part concerns effective very ampleness which says that for $m \geq n+2$ the line bundle $mL + K_X$ is very ample. Very ampleness of $mL+K_X$ means that a basis $s_0, \cdots, s_N$ of $\Gamma\left(X, mL + K_X\right)$ over $\mathbf{C}$ defines a holomorphic embedding of X into the complex projective space $\mathbf{P}_N$ of complex dimension N.

There have been quite a number of results on the freeness part of Fujita's conjecture, including its full verification for $n \leq 4$ ([R88, Kol93, EL93, F93, Ka97]) and its partial verification for general n which requires m to be greater than a constant of order n^2 ([AS94, H97, H99]). For the very ampleness part, the results in the literature so far give only the very ampleness of $mL + 2K_X$ for $m \geq m_n$ with some m_n depending on n and is of the order of a^n for some positive constant a ([D93, D96, ELN96, S95, S96b]). These very ampleness results use methods which cannot be applied to remove the coefficient 2 in $mL + 2K_X$.

This article contains four sections. The first section outlines a method of proving the very ampleness of $mL+K_X$ for $m \geq m_n$ with m_n depending only on n. To illustrate the method, we give only the details in the case $n = 2$. The basic idea of the method is essentially the same for both the case $n = 2$ and the case of general $n > 2$. However, in the case of general $n > 2$ the formulation used in a detailed proof is much more involved. We will work out and present the details of the case of general $n > 2$ elsewhere.

The other three sections (§2, §3, §4) consist of discussions, of a conjectural nature, on the relation of the very ampleness part of Fujita's conjecture to

---

[1] Partially supported by a grant from the National Science Foundation.

Published in *Complex Analysis and Geometry* (Columbus, OH, 1999), pp.171–191, *Ohio State Univ. Math. Res. Inst. Publ.*, **9**, de Gruyter, Berlin, 2001.

L^2 estimates involving derivatives and on Kohn's [Koh79] and Nadel's [N89] multiplier ideal sheaves. These discussions present ideas and problems, rather than results, in the topics under discussion.

The second section relates the very ampleness part of Fujita's conjecture to a regularity problem of $\bar{\partial}$ for locally Stein domains with L^2 estimates involving first-order derivatives.

The third section discusses the orders of the "spikes" of a domain associated to a singular metric of a holomorphic line bundle and the application to the question of the rationality of Lelong numbers [L68, S74] which arises from the problem of the finite generation of the canonical ring of a compact manifold of general type.

The fourth section discusses the relation between Kohn's multiplier ideal sheaf [Koh79] and Nadel's multiplier ideal sheaf [N89].

§1. *Techniques for the Very Ampleness Part of Fujita's Conjecture.*

The usual method of using multiplier ideal sheaves to obtain Fujita's conjecture type results is to attempt to construct, for points P_j of X and nonnegative integer q_j $(1 \leq j \leq k)$, a singular metric h for the line bundle mL over X so that

(i) the curvature current of h dominates some strictly positive smooth $(1,1)$-form on X in the sense of currents, and

(ii) the multiplier ideal sheaf $\mathcal{I}$ of the metric h (in the sense of Nadel [N89]) has the property that its zero-set is isolated at each P_j and it is contained in $\mathbf{m}_{P_j}^{q_j+1}$ $(1 \leq j \leq k)$.

Here for a point P of X, $\mathbf{m}_P$ means the maximum ideal of X at P. The multiplier ideal sheaf $\mathcal{I}$ of h (in the sense of Nadel [N89]) means the sheaf of all holomorphic function germs f on X with $h\,|f|^2$ locally integrable on X. For convenience of description we refer to a singular metric satisfying Property (i) as an *admissible singular metric.*

Once we have an admissible singular metric h, Property (i) implies the vanishing of $H^1\left(X, \mathcal{I}\left(mL + K_X\right)\right)$ by Nadel's vanishing theorem [N89]. (In the special case when the singular metric is algebraic geometrically defined, Nadel's vanishing theorem is the same as the theorem of Kawamata-Viehweg [Ka82, V82].) From the exactness of

$$\Gamma\left(X, mL+K_X\right) \to \oplus_{j=1}^{k}\left(\mathcal{O}_X/\mathbf{m}_{P_j}^{q_j+1}\right) \to H^1\left(X, \mathcal{I}\left(mL+K_X\right)\right)$$

it follows that any prescribed q_j-jets of mL at P_j $(1 \leq j \leq k)$ can be simultaneously achieved by some global holomorphic section of mL over X.

The usual way of constructing the singular metric h is to use a finite number of multivalued holomorphic sections $s_1, \cdots, s_N$ of mL over X so that

$$h = \frac{1}{\sum_{j=1}^{N}\left|s_j\right|^2}.$$

Here a multivalued holomorphic section s of aL over X means that, for some positive integer p, its p-th power s^p is a holomorphic section of paL over X. We say that the multivalued holomorphic section s vanishes to order γ at a point P of X when s^p vanishes to order $p\gamma$ at P. This definition of a multivalued holomorphic section of aL makes sense even when a is just a positive rational number. To construct the multivalued holomorphic sections $s_1, \cdots, s_N$ of mL over X, the usual way is to apply the theorem of Riemann-Roch [Hi66] either directly to X or to a subvariety of X and then form products whose factors are multivalued holomorphic sections on X and multivalued holomorphic sections extended from subvarieties of X which satisfy certain vanishing order requirements.

The reason why by this method the very ampleness part of Fujita's conjecture is so much harder to handle than its freeness part can be illustrated by the following simple calculus lemma.

(1.1) *Lemma.* Let a, b, c are positive numbers. Then

$$\int_{0<x<1,0<y<1} \frac{dxdy}{x^a(x^b+y^c)}$$

is infinite if and only if $\max\left(a, a+b\left(1-\frac{1}{c}\right)\right) \geq 1$.

Proof. If $a \geq 1$, then clearly

$$\int_{0<x<1,0<y<1} \frac{dxdy}{x^a(x^b+y^c)} \geq C\int_{0<x<1} \frac{dx}{x^a} = +\infty,$$

where C is a positive constant. Now assume that $a < 1$. We distinguish among the following three cases.

Case 1. $c < 1$. We have

$$\int_{0<x<1,0<y<1} \frac{dxdy}{x^a(x^b + y^c)} \leq \left(\int_{0<x<1} \frac{dx}{x^a}\right)\left(\int_{0<y<1} \frac{dy}{y^c}\right) < +\infty.$$

In this case,

$$a + b\left(1 - \frac{1}{c}\right) \leq a < 1$$

and

$$\max\left(a, a + b\left(1 - \frac{1}{c}\right)\right) < 1.$$

Case 2. $c = 1$. We have

$$\begin{aligned} \int_{0<x<1,0<y<1} \frac{dxdy}{x^a(x^b + y)} &= \int_{0<x<1} \left[\frac{\log(x^b + y)}{x^a}\right]_{y=0}^{y=1} dx \\ &= \int_{0<x<1} \frac{\log(1 + \frac{1}{x^b})}{x^a}\, dx < +\infty. \end{aligned}$$

In this case,

$$\max\left(a, a + b\left(1 - \frac{1}{c}\right)\right) = a < 1.$$

Case 3. $c > 1$. We have

$$\begin{aligned} &\int_{0<x<1,0<y<1} \frac{dxdy}{x^a(x^b + y^c)} = +\infty \\ \Leftrightarrow &\int_{0<x<1,0<y<1} \frac{dxdy}{x^a\left(x^{\frac{b}{c}} + y\right)^c} = +\infty \\ \Leftrightarrow &\frac{-1}{c-1}\int_{0<x<1} \left[\frac{1}{x^a\left(x^{\frac{b}{c}} + y\right)^{c-1}}\right]_{y=0}^{y=1} dx = +\infty \\ \Leftrightarrow &\frac{-1}{c-1}\int_{0<x<1} \frac{dx}{x^a\left(x^{\frac{b}{c}} + 1\right)^{c-1}} + \frac{1}{c-1}\int_{0<x<1} \frac{dx}{x^a\left(x^{\frac{b}{c}}\right)^{c-1}} = +\infty \end{aligned}$$

$$\Leftrightarrow \int_{0<x<1} \frac{dx}{x^a \left(x^{b\left(1-\frac{1}{c}\right)}\right)} = +\infty$$

$$\Leftrightarrow a + b\left(1 - \frac{1}{c}\right) \geq 1.$$

Q.E.D.

(1.2) *Corollary.* Let p and q be nonnegative integers. Let a, b, c be positive numbers. Then

$$\int_{0<x<1,0<y<1} \frac{x^p y^q dx dy}{x^a (x^b + y^c)}$$

is infinite if and only if

$$\max\left(\frac{a}{p+1}, \frac{a}{p+1} + \frac{b}{p+1}\left(1 - \frac{q+1}{c}\right)\right) \geq 1.$$

Proof. We use the substitution $u = x^{p+1}$ and $v = y^{q+1}$ need only replace a by $\frac{a}{p+1}$, b by $\frac{b}{p+1}$, and c by $\frac{c}{q+1}$ in Lemma (1.1). Q.E.D.

(1.3) *Lemma.* Let a, b, and c be positive numbers such that a and $c\left(1 - \frac{\lceil a\rceil - a}{b}\right)$ are non-integers and $\lceil a\rceil - a < b < 1$. Let $p_0 = \lceil a - 1\rceil$ and $q_0 = \left\lfloor c\left(1 - \frac{\lceil a\rceil - a}{b}\right)\right\rfloor$. Then on $\mathbf{C}^2$ with coordinates z, w, the multiplier ideal sheaf for the singular metric

$$\frac{1}{|z|^{2a}\left(|z|^{2b} + |w|^{2c}\right)}$$

is generated by z^{p_0+1} and $z^{p_0}w^{q_0}$. Here $\lfloor\cdot\rfloor$ denotes the round-down (*i.e.,* the largest integer not exceeding that number) and $\lceil\cdot\rceil$ denotes the round-up (*i.e.,* the smallest integer not less than that number.

Proof. Let $x = |z|^2$ and $y = |w|^2$. Let

$$d\Lambda = \frac{1}{|z|^{2a}\left(|z|^{2b} + |w|^{2c}\right)} \left(\frac{\sqrt{-1}}{\pi} dz \wedge d\bar{z}\right) \wedge \left(\frac{\sqrt{-1}}{\pi} dw \wedge d\bar{w}\right).$$

Let p and q be nonnegative integers. The integration of

(1.3.1) $$|z^p w^q|^2 \, d\Lambda$$

over the circle $|z| = 1$ and $|w| = 1$ yields

$$\frac{x^p y^q dxdy}{x^a \left(x^b + y^c\right)}$$

so that by Lemma (1.2) the local integrability of (1.3.1) at 0 is equivalent to

$$\begin{cases} a < p + 1 \\ \frac{a}{p+1} + \frac{b}{p+1}\left(1 - \frac{q+1}{c}\right) < 1. \end{cases}$$

In other words, the local integrability of (1.3.1) at 0 is equivalent to

$$\begin{cases} p > a - 1 \\ q > c\left(1 - \frac{p+1-a}{b}\right) - 1. \end{cases}$$

We now distinguish among the following three cases.

Case 1. $p \geq p_0 + 1 = \lceil a - 1 \rceil + 1$. Since a is not an integer, in this case we have $\lceil a - 1 \rceil + 1 > a - 1 + 1 = a$ and, in particular, $p > a - 1$. Moreover, in this case

$$\begin{aligned} & c\left(1 - \frac{p+1-a}{b}\right) - 1 \\ \leq {} & c\left(1 - \frac{\lceil a-1 \rceil - (a-1) + 1}{b}\right) - 1 \\ < {} & c\left(1 - \frac{1}{b}\right) - 1 < -1, \end{aligned}$$

because $1 - \frac{1}{b} < 0$ from $b < 1$ and $c > 0$. Hence the condition

$$q > c\left(1 - \frac{p+1-a}{b}\right) - 1$$

is always satisfied for any nonnegative integer q. Thus (1.3.1) is always locally integrable at 0 when $p \geq p_0 + 1$.

Case 2. $p = p_0 = \lceil a - 1 \rceil$. In this case we also have $p > a - 1$, because a is not an integer. Since

$$c\left(1 - \frac{p+1-a}{b}\right) = c\left(1 - \frac{\lceil a \rceil - a}{b}\right),$$

for any nonnegative integer q, the condition

$$q > c\left(1 - \frac{p+1-a}{b}\right) - 1$$

is equivalent to

$$q \geq \left\lfloor c\left(1 - \frac{\lceil a \rceil - a}{b}\right)\right\rfloor,$$

because $c\left(1 - \frac{\lceil a \rceil - a}{b}\right)$ is not an integer. Thus (1.3.1) is locally integrable at 0 when $p = p_0$ and $q \geq q_0$.

Case 3. $p < p_0$. Then $p + 1 \leq \lceil a - 1 \rceil < a$, because a is not an integer. Thus (1.3.1) is not locally integrable at 0 when $p < p_0$.

Combining the three cases, we conclude that (1.3.1) is locally integrable at 0 if either $p > p_0$ or

$$\begin{cases} p = p_0 \\ q \geq q_0 \end{cases}$$

Let $f(z, w)$ be a holomorphic function germ on $\mathbf{C}^2$ at $(z, w) = (0, 0)$. We can write

$$f = g_1(z, w) + z^{p_0} g_2(w) + z^{p_0} w^{q_0} h_1(z, w) + z^{p_0+1} h_2(z, w),$$

where

$$g_1(z, w) = \sum_{p=0}^{p_0-1} \sum_{q=0}^{\infty} a_{p,q} z^p w^q,$$
$$g_2(w) = \sum_{q=0}^{q_0-1} a_{p_0,q} w^q,$$

with $a_{p,q} \in \mathbf{C}$, and $h_1(z, w)$ and $h_2(z, w)$ are holomorphic function germs on $\mathbf{C}^2$ at $(z, w) = (0, 0)$. To finish the proof of the lemma, it suffices to show that $|f(z, w)|^2 \, d\Lambda$ is locally integrable at 0 if and only if both $g_1(z, w)$ and $g_2(w)$ are identically zero. Since $|z^{p_0+1}|^2 \, d\Lambda$ and $|z^{p_0} w^{q_0}|^2 \, d\Lambda$ are both locally integrable at 0, without loss of generality we can assume that both $h_1(z, w)$ and $h_2(z, w)$ are identically zero. Assume that one of $g_1(z, w)$ and $g_2(w)$ is not identically zero. We are going to show that $|f(z,w)|^2\,d\Lambda$ is not locally integrable at 0. We distinguish between the following two cases.

Case (i). $g_1(z,w)$ is not identically zero. Suppose $|f|^2\,d\Lambda$ is integrable on some open neighborhood U of 0 in $\mathbf{C}^2$ and we are going to derive a contradiction. Let p_1 be the smallest integer $0 \leq p < p_0$ such that $\sum_{q=0}^{\infty} a_{p,q} w^q$ is not identically zero. Then there exists an open neighborhood W of some point of $U \cap \{z = 0,\ w \neq 0\}$ in U such that

$$|f(z,w)| \geq A\,|z|^{p_1}$$

on W, where A is some positive number. Since $p_1 < p_0 = \lceil a-1 \rceil$, it follows that $p_1 < a-1$, because the inequality $p_1 \geq a-1$ would have implied $p_1 \geq \lceil a-1 \rceil$ from the fact that p_1 is an integer. Hence

$$\frac{|z^{p_1}|^2}{|z|^{2a}} \left(\frac{\sqrt{-1}}{\pi} dz \wedge d\bar{z} \right) \wedge \left(\frac{\sqrt{-1}}{\pi} dw \wedge d\bar{w} \right)$$

is not integrable on W and, in particular, $|z^{p_1}|^2\,d\Lambda$ is not integrable on W. It follows that $|f|^2\,d\Lambda$ is not integrable on W, which contradicts $W \subset U$.

Case (ii). $g_1(z,w)$ is identically zero. Then $g_2(w)$ is not identically zero. Let q_2 be the smallest integer $0 \leq q < q_0$ such that $a_{p_0,q}$ is nonzero. Then

$$|f(z,w)| \geq B\,|z|^{p_0}\,|w|^{q_2}$$

on some open neighborhood of 0 in $\mathbf{C}^2$, where B is some positive number. Since $|z^{p_0} w^{q_2}|^2\,d\Lambda$ is not locally integrable at 0 from $q_2 < p_0$, it follows that $|f|^2\,d\Lambda$ is not locally integrable at 0. Q.E.D.

In Lemma (1.3) the multiplier ideal sheaf of the singular metric

$$\frac{1}{|z|^{2a}\left(|z|^{2b} + |w|^{2c}\right)}$$

can also be described as being generated by monomials $z^p w^q$ so that (λ, ν) is no less than (p_0, q_0) in the lexicographical ordering. For convenience of description we refer to such a kind of ideal sheaf as a *lexicographical ideal sheaf*. We also use the corresponding description *lexicographical ideal* for an ideal instead of an ideal sheaf.

To continue our explanation why by such a method the very ampleness part is harder than the freeness part, for the sake of simplicity we confine ourselves to the case of $n = 2$. Fix a point P_0 in X. We first use the theorem of Riemann-Roch to get a multivalued holomorphic section s of some multiple of L over X which vanishes to a certain order at P_0. Let C be the 1-dimensional component of the zero-set of the multiplier ideal sheaf of $\frac{1}{|s|^2}$. We need only consider the case when C contains P_0. We apply the theorem of Riemann-Roch to get a multivalued section $\tilde{t}$ of some multiple of $L|C$ vanishing to a certain order at P_0 on C. Then we extend $\tilde{t}$ to a multivalued holomorphic section of the same multiple of L over all of X. The vanishing order of t at P_0 on X in general may be a very small positive number even though the vanishing order of $\tilde{t}$ at P_0 on C is very large. For example, for a local coordinate system (z, w) of X centered at P_0, the curve C can be locally given by $z = 0$ and t may be locally given by $\left(w^{Np} + z\right)^{\frac{1}{N}}$ so that the vanishing order of $t|C$ is p at P_0 on C but the vanishing order of t at p on X is only $\frac{1}{N}$. The multiplier ideal sheaf of the singular metric $\frac{1}{|st|^2}$ resembles the multiplier ideal sheaf of the singular metric

$$\frac{1}{|z|^{2a}\left(|z|^{2b} + |w|^{2c}\right)}$$

with $b > 0$ very small. So we end up with a multiplier ideal sheaf which behaves like a lexicographical ideal sheaf at P_0. To make the support of a lexicographical ideal sheaf isolated at a point, it is possible if we require only that the ideal sheaf is contained in $\mathbf{m}_{P_0}$, but it is impossible in general if we require that the ideal sheaf is contained in $\mathbf{m}_{P_0}^q$ for some $q \geq 2$. For this reason this method can give results on the freeness but not the very ampleness part.

To handle the very ampleness part of Fujita's conjecture, in the method of this paper we consider not just the multiplier ideal sheaf $\mathcal{I}$ of an admissible singular metric, but also other coherent ideal sheaves constructed as the middle term of a short exact sequence of multiplier ideal sheaves of singular metrics. By applying Nadel's vanishing theorem [N89] to multiplier ideal sheaves of admissible singular metrics, from the short exact sequences we get also the corresponding vanishing of cohomology for the coherent ideal sheaves which are the middle terms of short exact sequences. These coherent ideal sheaves can be constructed in such a way that their zero-sets are isolated at the prescribed points and they are contained in the prescribed powers of the maximum ideals at those prescribed points. The construction is essentially modelled on recovering a high power of the maximum ideal of a regular local ring from the lexicograhical ideals.

For the case of complex dimension two, the method yields the following theorem.

(1.4) *Theorem.* Let X be a compact complex surface and L be a positive holomorphic line bundle over X. Let $P_1, \cdots, P_\kappa$ be a finite number of distinct points in X and $q_1, \cdots, q_\kappa$ be positive numbers. Let $m \geq 1 + \left(2 + \sum_{j=1}^{\kappa} q_j\right)^2$. Then, for any prescribed q_j-jet at P_j for $1 \leq j \leq \kappa$, there exists an element $\Gamma\left(X, mL + K_X\right)$ whose q_j-jet at P_j is the prescribed one for $1 \leq j \leq \kappa$.

We will carry out the proof only for the case $\kappa = 1$, because the modification needed to get the case of general κ is completely straightforward and is obvious from the proof for $p = 1$.

For $n = 2$ the construction of the desired coherent ideal sheaf needs only one step. The following trivial lemma highlights why the construction of the desired coherent ideal sheaf is so much simpler. In the case of general n the zero dimension condition has to be replaced by the condition of complex codimension at least two and additional conditions on the quotient sheaf will be needed.

(1.5) *Lemma.* Let $\mathcal{I} \subset \mathcal{J}$ be coherent ideal sheaves on a compact complex surface X such that the support of the quotient $\mathcal{J}/\mathcal{I}$ has dimension zero. Let L be a holomorphic line bundle over X. If $H^p\left(X, \mathcal{I}\left(L + K_X\right)\right) = 0$ for $p \geq 1$, then $H^p\left(X, \mathcal{J}\left(L + K_X\right)\right) = 0$ for $p \geq 1$.

Proof. From the exact sequence

$$0 \to \mathcal{I}\left(L + K_X\right) \to \mathcal{J}\left(L + K_X\right) \to \left(\mathcal{J}/\mathcal{I}\right)\left(L + K_X\right) \to 0$$

one has the exact sequence

$$H^p\left(X, \mathcal{I}\left(L + K_X\right)\right) \to H^p\left(X, \mathcal{J}\left(L + K_X\right)\right) \to H^p\left(X, \left(\mathcal{J}/\mathcal{I}\right)\left(L + K_X\right)\right).$$

Since the support of the quotient $\mathcal{J}/\mathcal{I}$ has dimension zero, it follows that

$$H^p\left(X, \left(\mathcal{J}/\mathcal{I}\right)\left(L + K_X\right)\right) = 0$$

for $p \geq 1$. The conclusion of the lemma follows. Q.E.D.

(1.6) *Proof of Theorem (1.4) for* $p = 1$. To prove Theorem (1.4) for the case $p = 1$, we denote P_1 by P_0 and $q_1 + 1$ by q. Choose arbitrarily a rational number $0 < \eta < \frac{1}{2}$ which will later be required to be very small and whose smallness we will specify toward the end of this proof. Let s be a multivalued holomorphic section of L over X such that the vanishing order ord_{s,P_0} of s at P_0 is at least $1 - \eta$. Let

$$\hat{q} = \frac{q+1}{\mathrm{ord}_{s,P_0}}.$$

Let $h_\alpha = \frac{1}{|s|^{2\alpha}}$. Let $\mathcal{I}_\alpha$ be the multiplier ideal sheaf of the metric h_α of the $\mathbf{Q}$-bundle αL for $0 \leq \alpha \leq \hat{q}$. There exist

(i) a finite subset F of X and

(ii) a finite number of complex curves $C_1, \cdots, C_\ell$ in X such that

(a) on $X - F$ the ideal sheaf $\mathcal{I}_\alpha$ is the ideal sheaf of a divisor of the form $\sum_{j=1}^{\ell} a_{\alpha,j} C_j$ for $0 \leq \alpha \leq \hat{q}$, where $a_{\alpha,j}$ $(1 \leq j \leq \ell$, $0 \leq \alpha \leq \hat{q})$ are nonnegative integers,

(b) $a_{\alpha,j} \leq a_{\beta,j}$ for $0 \leq \alpha < \beta \leq \hat{q}$ and $1 \leq j \leq \ell$, and

(c) $a_{\hat{q},j} > 0$ for $1 \leq j \leq \ell$.

There exists a sequence

$$0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq \hat{q}$$

such that

$$\sum_{j=1}^{\ell} a_{\alpha,j} = \begin{cases} 0 & \text{for } 0 < \alpha < \alpha_1 \\ \nu & \text{for } \alpha_\nu \leq \alpha < \alpha_{\nu+1} \text{ and } 1 \leq \nu < k\,. \\ k & \text{for } \alpha_k \leq \alpha \leq \hat{q} \end{cases}$$

As a current

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log |s|^{2\hat{q}} \geq \sum_{j=1}^{\ell} a_{\hat{q},j} \, [C_j] \,,$$

where $[C_j]$ is the closed positive $(1,1)$-current on X defined by integration over the regular part of C_j. Since the Lelong number of the current

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|s|^{2\hat q}$$

at P_0 is $q+1$, it follows that

$$\sum_{j=1}^{\ell} a_{\hat q,j}\left(\mathrm{mult}_{P_0}C_j\right)\leq q+1$$

and, in particular,

$$\sum_{j=1}^{\ell}\mathrm{mult}_{P_0}C_j\leq q+1,$$

where $\mathrm{mult}_{P_0}C_j$ is the multiplicity of C_j at P_0.

Let Δ be the open disk in $\mathbf{C}$ of radius 1 centered at the origin. Take an open neighborhood U of P_0 in X and a local holomorphic family of nonsingular complex lines E_λ in U parametrized by $\lambda\in\Delta$ such that

(i) E_λ contains P_0 when $\lambda=0$,

(ii) $E_\lambda\cap\left(\bigcup_{j=1}^{\ell}C_j\right)$ is 0-dimensional for $\lambda\in\Delta$,

(iii) E_λ intersects $\bigcup_{j=1}^{\ell}C_j$ only at the regular points of $\bigcup_{j=1}^{\ell}C_j$ and all the intersections are normal crossing for $\lambda\in\Delta-0$, and

(iv) the number of points in $E_\lambda\cap\left(\bigcup_{j=1}^{\ell}C_j\right)$ is no more than $q+1$ for $\lambda\in\Delta-0$.

For $\lambda\in\Delta$ by using the theorem of Riemann-Roch we can find a multivalued holomorphic section $\tilde t_\lambda$ of $(q+2)L|\left(\bigcup_{j=1}^{\ell}C_j\right)$ over $\left(\bigcup_{j=1}^{\ell}C_j\right)$ so that the sum of the vanishing order of t_λ at all the points of $E_\lambda\cap\left(\bigcup_{j=1}^{\ell}C_j\right)$ is at least $1-\eta$ for $\lambda\in\Delta-0$ and the dependence of t_λ on the parameter λ is holomorphic for $\lambda\in\Delta$. We can extend $\tilde t_\lambda$ to a multivalued holomorphic section t_λ of $(q+1)L$ over X so that the dependence of t_λ on the parameter λ is holomorphic for $\lambda\in\Delta$. Let $t=t_0$.

Let $\sigma_1,\cdots,\sigma_{N_0}$ be holomorphic sections of m_0L over X without common zeros in X for some large positive integer m_0. Let $\sigma=\sum_{j=1}^{N_0}|\sigma_j|^{\frac{2}{m_0}}$. We fix some positive rational number ϵ, δ. Consider the metric

$$h_{\alpha,\beta}=\frac{1}{|s|^{2(\alpha-\epsilon)}\sigma^{\epsilon}\left(|s|^{2\delta}\sigma^{\beta(q+1)-\delta}+|t|^{2\beta}\right)}$$

of the $\mathbf{Q}$-bundle $\left(\alpha+\beta(q+1)\right)L$ over X. The range of α being considered for $h_{\alpha,\beta}$ is $0\leq\alpha\leq\hat{q}$. Let $\mathcal{J}_{\alpha}$ be the multiplier ideal sheaf of $h_{\alpha,\beta}$ when $\beta=\frac{q}{1-2\eta}$. Let $\mathcal{I}'_{\alpha_\nu}$ be the ideal sheaf of $\sum_{1\leq j\leq\ell}a_{\alpha_\nu,j}C_j$ for $0\leq\nu\leq k$. For some sufficiently small appropriately chosen ϵ,δ we have

$$\mathcal{J}_{\alpha_\nu}\subset\mathbf{m}_{P_0}^{q}+\mathcal{I}'_{\alpha_\nu}$$

and

$$\mathcal{J}_{\alpha_\nu}\subset\mathcal{I}_{\alpha_{\nu-1}}$$

for $1\leq\nu\leq k$. Let

$$\mathcal{J}'_{\alpha_\nu}=\mathcal{J}_{\alpha_\nu}+\mathcal{I}'_{\alpha_\nu}.$$

From

$$\mathcal{I}_{\alpha_\nu}\subset\mathcal{I}'_{\alpha_\nu}$$

it follows that

$$\mathcal{I}'_{\alpha_\nu}\subset\mathcal{J}'_{\alpha_\nu}\subset\mathbf{m}_{P_0}^{q}+\mathcal{I}'_{\alpha_\nu}$$

and

$$\mathcal{J}'_{\alpha_\nu}=\left(\mathcal{J}'_{\alpha_\nu}\cap\mathbf{m}_{P_0}^{q}\right)+\mathcal{I}'_{\alpha_\nu}.$$

From

$$\mathcal{J}'_{\alpha_\nu}\subset\mathcal{I}_{\alpha_{\nu-1}}\subset\mathcal{I}'_{\alpha_{\nu-1}}$$

it follows that

$$\mathcal{J}'_{\alpha_\nu}\subset\left(\mathcal{I}'_{\alpha_{\nu-1}}\cap\mathbf{m}_{P_0}^{q}\right)+\mathcal{I}'_{\alpha_\nu}.$$

Fix an integer $m>\hat{q}+\frac{q}{1-2\eta}(q+1)$. Since dimension of the support of the quotient

$$\left(\left(\mathcal{I}'_{\alpha_{\nu-1}}\cap\mathbf{m}_{P_0}^{q}\right)+\mathcal{I}'_{\alpha_\nu}\right)/\mathcal{J}_{\alpha_\nu}$$

is zero, it follows from

$$H^p\left(X,\mathcal{J}_{\alpha_\nu}\left(mL+K_X\right)\right)=0$$

for $p\geq 1$ that

$$H^p\left(X,\left(\left(\mathcal{I}'_{\alpha_{\nu-1}}\cap\mathbf{m}_{P_0}^{q}\right)+\mathcal{I}'_{\alpha_\nu}\right)\left(mL+K_X\right)\right)=0 \tag{1.6.1}$$

for $p \geq 1$ and $1 \leq \nu \leq k$.

We have the isomorphism

$$\left(\mathcal{I}'_{\alpha_{\nu-1}} \cap \mathbf{m}^q_{P_0}\right) / \left(\mathcal{I}'_{\alpha_\nu} \cap \mathbf{m}^q_{P_0}\right) \to \left(\left(\mathcal{I}'_{\alpha_{\nu-1}} \cap \mathbf{m}^q_{P_0}\right) + \mathcal{I}'_{\alpha_\nu}\right) / \mathcal{I}'_{\alpha_\nu}, \tag{1.6.2}$$

because

$$\left(\mathcal{I}'_{\alpha_{\nu-1}} \cap \mathbf{m}^q_{P_0}\right) \cap \mathcal{I}'_{\alpha_\nu} = \mathcal{I}'_{\alpha_\nu} \cap \mathbf{m}^q_{P_0}.$$

Since

$$\mathcal{I}_{\alpha_\nu} \subset \mathcal{I}'_{\alpha_\nu}$$

and the support of

$$\mathcal{I}'_{\alpha_\nu} / \mathcal{I}_{\alpha_\nu}$$

has dimension zero, it follows from

$$H^p\left(X, \mathcal{I}_{\alpha_\nu}\left(mL + K_X\right)\right) = 0$$

for $p \geq 1$ and $0 \leq \nu \leq k$ that

$$H^p\left(X, \mathcal{I}'_{\alpha_\nu}\left(mL + K_X\right)\right) = 0 \tag{1.6.3}$$

for $p \geq 1$ and $0 \leq \nu \leq k$. By (1.6.1) and (1.6.3), we have

$$H^p\left(X, \left(\left(\left(\mathcal{I}'_{\alpha_{\nu-1}} \cap \mathbf{m}^q_{P_0}\right) + \mathcal{I}'_{\alpha_\nu}\right) / \mathcal{I}'_{\alpha_\nu}\right)\left(mL + K_X\right)\right) = 0$$

for $p \geq 1$ and $1 \leq \nu \leq k$. From the isomorphism (1.6.2) it follows that

$$H^p\left(X, \left(\left(\mathcal{I}'_{\alpha_{\nu-1}} \cap \mathbf{m}^q_{P_0}\right) / \left(\mathcal{I}'_{\alpha_\nu} \cap \mathbf{m}^q_{P_0}\right)\right)\left(mL + K_X\right)\right) = 0 \tag{1.6.4}$$

for $p \geq 1$ and $1 \leq \nu \leq k$. We have

$$\mathcal{I}_{\alpha_k} \cap \mathbf{m}^q_{P_0} = \mathcal{I}_{\alpha_k},$$

because

$$\mathcal{I}_{\alpha_k} \subset \mathbf{m}^q_{P_0}.$$

Hence

$$H^p\left(X, \left(\mathcal{I}_{\alpha_k} \cap \mathbf{m}^q_{P_0}\right)\left(mL + K_X\right)\right) = 0$$

for $p \geq 1$ and $0 \leq \nu \leq k$. Since

$$\mathcal{I}_{\alpha_k} \cap \mathbf{m}_{P_0}^q \subset \mathcal{I}'_{\alpha_k} \cap \mathbf{m}_{P_0}^q$$

and the support of

$$\left(\mathcal{I}'_{\alpha_k} \cap \mathbf{m}_{P_0}^q\right) / \left(\mathcal{I}_{\alpha_k} \cap \mathbf{m}_{P_0}^q\right)$$

is of dimension zero, it follows that

$$H^p\left(X, \left(\mathcal{I}'_{\alpha_k} \cap \mathbf{m}_{P_0}^q\right)(mL + K_X)\right) = 0 \tag{1.6.5}$$

for $p \geq 1$ and $0 \leq \nu \leq k$. By using the exact sequence

$$0 \to \mathcal{I}'_{\alpha_\nu} \cap \mathbf{m}_{P_0}^q \to \mathcal{I}'_{\alpha_{\nu-1}} \cap \mathbf{m}_{P_0}^q \to \left(\mathcal{I}'_{\alpha_{\nu-1}} \cap \mathbf{m}_{P_0}^q\right) / \left(\mathcal{I}'_{\alpha_\nu} \cap \mathbf{m}_{P_0}^q\right) \to 0$$

and descending induction on $1 \leq \nu \leq k$, we conclude from (1.6.4) and (1.6.5) that

$$H^p\left(X, \left(\mathcal{I}'_{\alpha_\nu} \cap \mathbf{m}_{P_0}^q\right)(mL + K_X)\right) = 0 \tag{1.6.6}$$

for $p \geq 1$ and $0 \leq \nu \leq k$. Since $\mathcal{I}'_{\alpha_0} = \mathcal{O}_X$, the case $\nu = 0$ of (1.6.6) yields

$$H^p\left(X, \mathbf{m}_{P_0}^q(mL + K_X)\right) = 0 \tag{1.6.7}$$

for $p \geq 1$. Now choose the positive number η to be so small that

$$\hat{q} + \frac{q}{1 - 2\eta}(q+1) \leq \frac{1}{1 - 2\eta}(q+1)^2 < 1 + (q+1)^2.$$

From the exact sequence

$$0 \to \mathbf{m}_{P_0}^q \to \mathcal{O}_X \to \mathcal{O}_X / \mathbf{m}_{P_0}^q \to 0$$

and (1.6.7) it follows that $\Gamma(X, mL + K_X)$ generate all the $(q-1)$-jets at P_0 for $m \geq 1 + (q+1)^2$. This finishes the proof of Theorem (1.4) for the case $\kappa = 1$.

Remark. In the proof of Theorem (1.4) there is no attempt to give a sharp lower bound for m. Even without introducing any new techniques, the proof in (1.6) could be easily modified to improve somewhat the lower bound for m given in (1.4).

§2. *Relation Between Very Ampleness and L^2 Estimates Involving Derivatives.*

The very ampleness part of Fujita's conjecture is related, in the following way, to a regularity problem of $\bar\partial$ for locally Stein domains with L^2 estimates involving derivatives. Now we no longer restrict ourselves to the surface case and allow n to be any integer ≥ 2.

Let P_0 and Q_0 be two distinct points of X. For the freeness part of Fujita's conjecture one produces an admissible singular metric $h = e^{-\varphi}$ of $m_0 L$ whose multiplier ideal sheaf $\mathcal{I}$ has an isolated zero at P_0 and Q_0 and is contained in $\mathbf{m}_{P_0}$ and $\mathbf{m}_{Q_0}$. Then we use Nadel's vanishing theorem to get the vanishing of $H^1\left(X, \mathcal{I}\left(mL + K_X\right)\right)$ for $m \geq m_0$. It means that for any $\bar\partial$-closed mL-valued $(n,1)$-form f on X with

$$\int_X |f|^2 e^{-\varphi} < \infty,$$

we can solve the $\bar\partial$-equation $\bar\partial u = f$ on X for an mL-valued $(n,0)$-form u on X with L^2 estimates

$$\int_X |u|^2 e^{-\varphi} < \infty$$

with weight $e^{-\varphi}$. Here $|f|^2$ is the pointwise L^2-norm square with respect to any smooth metric of X.

If we can solve the $\bar\partial$-equation $\bar\partial u = f$ on X with L^2 estimates with weight $e^{-\varphi}$ for derivatives up to order 1, then we can conclude the very ampleness of $mL + K_X$. More precisely, if for any $\bar\partial$-closed mL-valued $(n,1)$-form f on X with

$$\int_X \left(|f|^2 + |\nabla f|^2\right) e^{-\varphi} < \infty, \tag{2.1}$$

we can solve the $\bar\partial$-equation $\bar\partial u = f$ on X for an mL-valued $(n,0)$-form u on X such that

$$\int_X \left(|u|^2 + |\nabla u|^2\right) e^{-\varphi} < \infty, \tag{2.2}$$

then we can conclude the very ampleness of $mL + K_X$, where ∇ is the covariant derivative in the direction of $(1,0)$ with respect to any smooth metric of

X. (Note that the finiteness condition of the abve two integrals is independent of the choice of the smooth metric of X.) The reason is as follows.

Take any open neighborhood U of P_0 in X and any $g \in \Gamma(U, mL + K_X)$. Let $0 \leq \rho \leq 1$ be a smooth function supported on U which is identically 1 on some open neighborhood U' of P_0 in U. Let $f = \bar{\partial}(\rho g)$. Clearly f satisfies (2.1). If we have a solution u of $\bar{\partial} u = f$ which satisfies (2.2), then u is holomorphic on U' and belongs to $\mathbf{m}_{P_0}^2$. Hence $g - u$ is an element of $\Gamma(X, mL + K_X)$ whose 1-jet at P_0 agrees with the 1-jet of g at P_0. Thus we can conclude the very ampleness of $mL + K_X$.

Let $e^{-\psi}$ be a smooth metric of L with strictly positive curvature on X. There is a technique introduced in [Koh73] to get L^2 estimates involving derivatives on pseudoconvex domains in $\mathbf{C}^n$. It uses an additional weight function to take care of the error terms arising from the commutation of $\bar{\partial}$ and ∇ and the commutation of $\bar{\partial}^*$ and ∇. The reason why in the above discussion we increase m_0 to m is that we hope to use $e^{-(m-m_0)\psi}$ to take the place of the additional weight function in the technique of [Koh73].

One difficulty in applying the technique of [Koh73] to our problem at hand is that the process of taking the commutation of $\bar{\partial}$ and ∇ (and also of $\bar{\partial}^*$ and ∇) introduces the differentiation of φ. We have no way of estimating the derivatives of φ. One way of avoiding the difficulty of differentiating φ is as follows.

We consider the total bundle space of the dual L^* of L and introduce the open subset Ω consisting of all elements of L^* whose length with respect to the metric h^{-1} is < 1. Let $\pi : L^* \to X$ be the bundle projection and let w be its fiber coordinate. Let $\tilde{f} = (\pi^* f) \wedge \frac{dw}{w}$. Then $\tilde{f}$ is a scalar-valued $(n+1, 1)$-form on L^*. Since f is an $(n, 1)$-form with value in mL, we know that $\tilde{f}$ vanishes to order $m - 1$ at the zero-section Z of the line bundle $\pi : L^* \to X$. Instead of the $\bar{\partial}$-equation $\bar{\partial} u = f$ on X, we use the $\bar{\partial}$-equation $\bar{\partial} \tilde{u} = \tilde{f}$ on Ω to solve for the $(n+1, 0)$-form $\tilde{u}$ on Ω with

$$\int_{\Omega} |u|^2 < \infty.$$

Since $\tilde{f}$ and $\tilde{u}$ are scalar-valued forms instead of line-bundle-valued forms, we no longer have to use any weight function from the metric of the line bundle. The role of the original weight function $e^{-\varphi}$ is now replaced by the volume of the fiber disk inside Ω. The intersection of Ω with a fiber is the disk $|w|^2 < e^{-\varphi}$ and

$$\frac{1}{2p\pi}\int_{|w|^2<e^{-\varphi}}\left|w^{p-1}\right|^2\sqrt{-1}dw\wedge d\bar{w}=e^{-\varphi} \tag{2.3}$$

for any positive integer p.

To be able to solve the $\bar{\partial}$-equation $\bar{\partial}\tilde{u}=\tilde{f}$ on Ω with L^2 estimates involving derivatives, the technique of [Koh73] requires an additional weight and we need a strictly plurisubharmonic function on Ω or at least on $\Omega - Z$. To get such a function we introduce $\tau = |w|^2 e^{\psi}$, where $e^{-\psi}$ is the smooth metric of L over X with strictly positive curvature on X. We use τ to help us get the extra weight function to try to solve the $\bar{\partial}$-equation $\bar{\partial}\tilde{u}=\tilde{f}$ on Ω with L^2 estimates for derivatives up to order 1 (where the derivative is computed with respect to the local coordinate system of L^* consisting of w and the pullback of a local coordinate system of X).

One hopes that for the estimate involving derivatives of order up to 1 some appropriate weight function constructed from τ can take care of the commutation terms by using the techniques of [Koh73]. The function τ vanishes at the zero-section Z of the line bundle $\pi : L^* \to X$ and is strictly plurisubharmonic only outside Z. The vanishing of $\tilde{f}$ to order m at Z will be used to take care of the vanishing order and the lack of plurisubharmonicity of τ at Z. The main difficulty is how to make sure that m can be chosen to depend only on n.

Once one gets a solution $\tilde{u}$, one takes its power series expansion with respect to the fiber coordinate w and single out the term $uw^{m-1}dw$. Then u is an mL-valued $(n,0)$-form on X which solves the the $\bar{\partial}$-equation $\bar{\partial}u = f$ on X. The L^2 estimate of u with weight $e^{-m\varphi}$ involving derivatives up to order 1 comes from (2.3). The singularity of $e^{-\varphi}$ is now reflected in the non-relative-compactness of Ω in L^*. The domain Ω has infinite "spikes" at the points of X where φ becomes $-\infty$.

The idea of introducing Ω to transform line-bundle-valued forms to scalar-valued forms traces back to Grauert [G62].

§3. *Spikes and Rationality of Lelong Numbers.*

Consider a metric

$$h = \frac{1}{\sum_{j=1}^{N} |s_j|^2}$$

of a holomorphic line bundle L over a compact complex manifold X of complex dimension n, where s_j $(1 \leq j \leq N)$ are multivalued holomorphic sections of L over X. There are two ways in which h may fail to be a smooth metric with positive curvature. The first one is that $s_1, \cdots, s_N$ all vanish at some point of X. The second one is that at some point P of X there exists no subset of $n+1$ elements in $\{s_1, \cdots, s_N\}$ which can serve as a local homogeneous coordinate system at P. Let Ω be the domain in the dual bundle L^* of L consisting of all elements whose length is < 1 with respect to the metric h^{-1} of L^*. The first way of failure corresponds to the base point set of mL when ${s_1}^m, \cdots, {s_N}^m$ form a basis of $\Gamma(X, mL)$ over $\mathbf{C}$. It also corresponds to the "spikes" of the domain Ω in L^*. The second way of failure corresponds to the non-strictly pseudoconvex points in the boundary of Ω.

Non-strictly pseudoconvex boundary points, in the context of the regularity of the Kohn solution, have been extensively studied (for example, [B84, B92, BS90, BS91, BS93, C82, C83, C87, Ch91, Chr96, D'AK99, FKM90, Ki91, Koh73, Koh79, Koh97, Koh98, Koh00]). However, no theory has yet been developed for "spikes". We outline here a problem which could be used as a guide for the development of a theory for "spikes".

One long outstanding conjecture in algebraic geometry is the finite generation of the canonical ring $R(X, K_X)$ of a compact complex projective algebraic manifold X of general type. General type means that the dimension of $\Gamma(X, mK_X)$ dominates $c\,m^n$ when m is sufficiently large, where c is a positive constant (independent of m) and n is the complex dimension of X. For a holomorphic line bundle L over X the ring $R(X, L)$ is defined by

$$R(X, L) = \bigcup_{m=1}^{\infty} \Gamma(X, mL).$$

Let

$$s_1^{(m)}, \cdots, s_{q_m}^{(m)} \in \Gamma(X, mL)$$

be a basis over $\mathbf{C}$ and let $\epsilon_m > 0$ $(1 \leq m < \infty)$ be a sequence of positive numbers chosen to be so rapidly decreasing that

$$\Phi_L := \sum_{m=1}^{\infty} \epsilon_m \sum_{j=1}^{q_m} \left| s_j^{(m)} \right|^{\frac{2}{m}}$$

is convergent on X as a metric of L^*. If the ring $R(X, L)$ is finitely generated, the vanishing orders of Φ_L must be rational numbers, or more precisely, the Lelong numbers of the closed positive $(1,1)$-current

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Phi_L$$

must be rational numbers (see [L68, S74] for the theory of Lelong numbers and closed positive currents).

One possible approach to obtaining the rationality of Lelong numbers for the case $L = K_X$ is to use a property of Kohn's multiplier ideal sheaf involving certain differential operators. The setting of Kohn's multiplier ideal sheaf [Koh79] is a bounded domain D in $\mathbf{C}^n$ with smooth weakly pseudoconvex boundary defined by $r < 0$ so that dr is nowhere zero on the boundary ∂D of D. The stalk, at a point P, of Kohn's multiplier ideal sheaf $\mathcal{I}^{(1)}$ for $(0,1)$-forms consists of all smooth function germs F at P such that there exist an open neighborhood U of P in $\mathbf{C}^n$ and positive numbers ϵ and C (both depending on F) satisfying

$$|||Fg|||_\epsilon^2 \leq C\left(\|\bar{\partial}g\|^2 + \|\bar{\partial}^* g\|^2 + \|g\|^2\right)$$

for all $(0,1)$-form g supported on $U \cap \bar{D}$ which is in the domain of $\bar{\partial}$ and $\bar{\partial}^*$. Here $|||\cdot|||_\epsilon$ is the L^2 norm on D involving derivatives up to order ϵ in the boundary tangential direction of D, $\|\cdot\|$ is the usual L^2 norm on D without involving any derivatives, and $\bar{\partial}^*$ is the actual adjoint of $\bar{\partial}$ with respect to $\|\cdot\|$. Likewise, the stalk, at a point P, of Kohn's multiplier module sheaf $\mathcal{M}^{(1)}$ for $(0,1)$-forms consists of all smooth germs of $(1,0)$-forms ω at P such that there exist an open neighborhood U of P in $\mathbf{C}^n$ and positive numbers ϵ and C (both depending on ω) satisfying

$$|||g\cdot\omega|||_\epsilon^2 \leq C\left(\|\bar{\partial}g\|^2 + \|\bar{\partial}^* g\|^2 + \|g\|^2\right)$$

for all $(0,1)$-form g supported on $U \cap \bar{D}$ which is in the domain of $\bar{\partial}$ and $\bar{\partial}^*$. Here $g \cdot \omega$ is the pointwise contraction of the $(0,1)$-form g with the $(1,0)$-form

ω. There are a number of properties of $\mathcal{I}^{(1)}$ and $\mathcal{M}^{(1)}$. For example, r is in $\mathcal{I}^{(1)}$; the contraction of $\partial\bar\partial r$ with any smooth $(1,0)$-form orthogonal to ∂r is in $\mathcal{M}^{(1)}$; and any smooth function germ whose absolute value is dominated by the absolute value of some positive power of an element of $\mathcal{I}^{(1)}$ is itself in $\mathcal{I}^{(1)}$.

There are two important properties of $\mathcal{I}^{(1)}$ and $\mathcal{M}^{(1)}$ which are most relevant to our discussion. The first one is that the ∂ of an element of $\mathcal{I}^{(1)}$ is in $\mathcal{M}^{(1)}$. The second one is that, if $\omega_1, \cdots, \omega_n$ belong to $\mathcal{M}^{(1)}$, then the coefficient of the $(n,0)$-form $\omega_1 \wedge \cdots \wedge \omega_n$ belongs to $\mathcal{I}^{(1)}$. These two properties together give us the property that some nonlinear differentiation process involving a number of elements of $\mathcal{I}^{(1)}$ again yields an element of $\mathcal{I}^{(1)}$.

One hopes that this particular property of $\mathcal{I}^{(1)}$ involving a nonliear differention process would be able to give us a system of linear equations with coefficients in $\mathbf{Q}$ whose unknowns are the vanishing orders of $\mathcal{I}^{(1)}$. When the system of linear equations is nondegenerate enough, one could conclude the rationality of such vanishing orders.

Kohn's multiplier ideal sheaves are for non-strictly pseudoconvex boundary points. The analogous argument for their counterpart for "spikes" might yield the rationality of the Lelong numbers of

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \Phi_L$$

(or the vanishing orders of Φ_L) when $L = K_X$. One guide for the development of a theory for the "spikes" analogous to the theory for the non-strictly psuedoconvex points is to prove the rationality of the Lelong numbers of

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \Phi_L$$

when $L = K_X$.

This argument could work only for the case $L = K_X$, because of the process of using $\omega_1 \wedge \cdots \wedge \omega_n$ for $\omega_1, \cdots, \omega_n \in \mathcal{M}^{(1)}$. This corresponds to the fact that if we take $(1,0)$ covariant differentiation of smooth sections $t_1, \cdots, t_n$ of mK_X over X to get sections $Dt_1, \cdots, Dt_n$ of $K_X \otimes T_X^*$ over X, the exterior product $Dt_1 \wedge \cdots \wedge Dt_n$ is a section of $(nm+1)K_X$. Here T_X^* is the cotangent bundle of X.

So far as the behavior at the base point set of X is concerned, $\log \Phi_L$ for $L = K_X$ may be linked in some way to the potential in the Monge-Ampére equation for the "Kähler-Einstein" current for X.

§4. *Relation Between Kohn's and Nadel's Multipliers.*

Nadel's multiplier ideal sheaf is defined so that the $\bar\partial$-equation $\bar\partial u = f$ could be solved when f belongs to Nadel's multiplier ideal sheaf. On the other hand, Kohn's multiplier ideal sheaf consists of multipliers so that, after a "test function" for the estimate needed for the solution of the $\bar\partial$-equation is multiplied by the multiplier, the product satisfies the estimate. Nadel's multiplier ideal sheaf is for the right-hand side of the $\bar\partial$-equation and Kohn's multiplier ideal sheaf is for the "test function" of the $\bar\partial$-equation. It is natural to expect some form of duality between the two kinds of multiplier ideal sheaves.

Consider the weakly pseudoconvex domain D in $\mathbf{C}^{n+1}$ with coordinates $(w, z) = (w, z_1, \cdots, z_n)$ whose boundary is given by

$$\operatorname{Re} w = \sum_{j=1}^{k} |f_j(z)|^2 ,$$

where $f_j(z)$ are holomorphic functions in z $(1 \leq j \leq k)$. Suppose we apply to $\mathbf{C}^n$ (with coordinates z) a monoidal transformation with a regular center and get $\pi : Y \to \mathbf{C}^n$. For example, we can blow up a single point in $\mathbf{C}^n$. We consider the weakly pseudoconvex domain $\tilde D$ in $\mathbf{C} \times Y$ whose boundary is defined by

$$\operatorname{Re} w = \sum_{j=1}^{k} |f_j \circ \pi|^2 .$$

When the boundary of D does not contain any positive-dimensional complex-analytic subvarieties, the boundary of $\tilde D$ may contain positive-dimensional complex-analytic subvarieties. One hopes that, by considering the pullback to $\tilde D$ of $(0,1)$-forms from D, in the situation of complex-analytic subvarieties in the boundary one can arrive at a general formulation of the condition for the right-hand side for the $\bar\partial$-equation for regularity to hold. Such a formulation would mix the concept of Kohn's multiplier with the concept of Nadel's multiplier and might yield a better understanding of the relation between the two.

One possible application of this kind of consideration is to apply monoidal transformations with nonsingular centers to reduce the set of holomorphic functions $f_1, \cdots, f_k$ to a more standard situation where the Hölder estimates for $\bar{\partial}$-equation can be more easily investigated. At this point $\bar{\partial}$ Hölder estimates for weakly pseudoconvex domains are known only for very special cases such as a simultaneously diagonalizable Levi form ([FKM90]).

References.

[AS94] U. Angehrn and Y.-T. Siu, Effective freeness and separation of points for adjoint bundles. Invent. Math. 122 (1995), 291–308.

[B84] D. Barrett, Irregularity of the Bergman projection on a smooth bounded domain in $\mathbf{C}^2$, Ann. of Math. 119 (1984), 431-436.

[B92] D. Barrett, Behavior of the Bergman projection on the Diederich-Fornaess worm, Acta Math. 168 (1992), 1-10.

[BS90] H. Boas and E. Straube, Equivalence of regularity for the Bergman projection and the $\bar{\partial}$-Neumann operator, Manuscripta Math. 67 (1990), 25-66.

[BS91] H. Boas and E. Straube, Sobolev estimates for the $\bar{\partial}$-Neumann operator oin domains in $\mathbf{C}^n$ admitting a defining function that is plurisubharmonic on the boundary, Math. Zeitschr. 206 (1991), 81-88.

[BS93] H. Boas and E. Straube, De Rham cohomology of manifolds containing the points of infinite type, and Sobolev estimates for the $\bar{\partial}$-Neumann problem, J. Geom. Analysis 3 (1993), 225-235.

[C82] D. Catlin, Global regularity for the $\bar{\partial}$-Neumann problem, Proc. Symp. Pure Math. A.M.S. 41 (1982), 39-49.

[C83] D. Catlin, Necessary conditions for the subellipticity of the $\bar{\partial}$-Neumann problem, Ann. of Math. 117 (1983), 147-171.

[C87] D. Catlin, Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains, Ann. of Math. 126 (1987), 131-192.

[Ch91] S.-C. Chen, Global regularity of the $\bar{\partial}$-Neumann problem in dimension two, Proc. Symp. Pure Math. A.M.S. 52 (1991), 55-61.

[Chr96] M. Christ, Global C^∞ irregularity of the $\bar{\partial}$-Neumann problem for worm domains, J. of Amer. Math. Soc. 9 (1996), 1171-1185.

[D'AK99] J. D'Angelo and J. J. Kohn, Subelliptic estimates and finite type. Several complex variables (Berkeley, CA, 1995–1996), 199–232, Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge, 1999.

[D93] J.-P. Demailly, A numerical criterion for very ampleness, J. Diff. Geom. 37 (1993), 323-374.

[D96] J.-P. Demailly, Effective bounds for very ample line bundles. Invent. Math. 124 (1996), 243–261.

[EL93] L. Ein and R. Lazarsfeld, Global generation of pluricanonical and adjoint linear series on smooth projective threefolds, J. Amer. Math. Soc. 6 (1993), 875-903 .

[ELN96] Ein, Lawrence; Lazarsfeld, Robert; Nakamaye, Michael Zero-estimates, intersection theory, and a theorem of Demailly. Higher-dimensional complex varieties (Trento, 1994), 183–207, de Gruyter, Berlin, 1996.

[FKM90] C. Fefferman, J. J. Kohn, and M. Machedon, Hölder estimates on CR manifolds with a diagonalizable Levi form. Adv. Math. 84 (1990), 1–90.

[F87] T. Fujita, On polarized manifolds whose adjoint bundles are not semipositive. In: Algebraic Geometry, Sendai, Advanced Studies in Pure Math. 10 (1987), 167-178.

[F93] T. Fujita, Remarks on Ein-Lazarsfeld criterion of spannedness of adjoint bundles of polarized threefolds, preprint 1993.

[G62] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen. Math. Ann. 146 (1962), 331–368.

[H97] S. Helmke, On Fujita's conjecture. Duke Math. J. 88 (1997), 201–216.

[H99] S. Helmke, On global generation of adjoint linear systems. Math. Ann. 313 (1999), 635–652.

[Hi66] F. Hirzebruch, Topological Methods in Algebraic Geometry. Berlin Heidelberg New York: Springer 1966.

[Ka82] Y. Kawamata: A generalization of the Kodaira-Ramanujam's vanishing theorem, Math. Ann. 261 (1982), 43-46.

[Ka97] Y. Kawamata, On Fujita's freeness conjecture for 3-folds and 4-folds. Math. Ann. 308 (1997), 491–505.

[Ki91] C. Kiselman, A study of the Bergman projection in certain Hartogs domains, Proceedings of Symposia in Pure Math. 52 (1991), 219-231.

[Koh73] J. J. Kohn, Global regularity for $\bar{\partial}$ on weakly pseudo-convex manifolds. Trans. Amer. Math. Soc. 181 (1973), 273–292.

[Koh79] J. J. Kohn, Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions. Acta Math. 142 (1979), 79–122.

[Koh97] J. J. Kohn, Quantitative estimates for global regularity. Analysis and geometry in several complex variables (Katata, 1997), 97–128, Trends Math., Birkhäuser Boston, Boston, MA, 1999.

[Koh98] J. J. Kohn, Hypoellipticity of some degenerate subelliptic operators. J. Funct. Anal. 159 (1998), 203–216.

[Koh00] J. J. Kohn, Hypoellipticity at points of infinite type. Analysis, geometry, number theory: the mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998), 393–398, Contemp. Math., 251, Amer. Math. Soc., Providence, RI, 2000.

[Kol93] J. Kollár, Effective base point freeness, Math. Ann. 296 (1993), 595-605.

[L68] P. Lelong, Fonctions plurisousharmoniques et formes différentielles positives. Gordon & Breach, Paris-London-New York (Distributed by Dunod éditeur, Paris) 1968 ix+79 pp.

[N89] A. Nadel, Multiplier ideal sheaves and the existence of Kähler-Einstein metrics of positive scalar curvature, Proc. Natl. Acad. Sci. USA, 86, 7299-7300 (1989), and Ann. of Math., 132 (1989), 549-596.

[R88] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math., 127, 309-316 (1988).

[Si74] Y.-T. Siu, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. Invent. Math. 27 (1974), 53–156.

[S95] Y.-T. Siu, Very ampleness criterion of double adjoints of ample line bundles. Modern methods in complex analysis (Princeton, NJ, 1992), 291–318, Ann. of Math. Stud., 137, Princeton Univ. Press, Princeton, NJ, 1995.

[S96a] Y.-T. Siu, The Fujita conjecture and the extension theorem of Ohsawa-Takegoshi, in *Geometric Complex Analysis* ed. Junjiro Noguchi *et al*, World Scientific: Singapore, New Jersey, London, Hong Kong 1996, pp. 577-592.

[S96b] Y.-T. Siu, Effective very ampleness. Invent. Math. 124 (1996), 563–571.

[V82] E. Viehweg, Vanishing theorems, J. reine und angew. Math., 335 (1982), 1-8.

Authors' addresses:

Yum-Tong Siu, Department of Mathematics, Harvard University, Cambridge, MA 02138. *e-mail:* siu@math.harvard.edu.