

NONEQUIDIMENSIONAL VALUE DISTRIBUTION THEORY AND MEROMORPHIC CONNECTIONS

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This is a sequel to the paper [Si2] in which we introduced the use of meromorphic connections to handle nonequidimensional value distribution theory. The meromorphic connection is chosen so that the divisor has zero second fundamental form with respect to the meromorphic connection. When we have more than one divisor, they must have zero second fundamental form with respect to *the same* meromorphic connection. Such a condition is so stringent that it is difficult to find examples to which the method can be applied and which cannot be handled by other means. One example is a collection of Fermat curves in \mathbb{P}_2 of the form $a_0 w_0^d + a_1 w_1^d + a_2 w_2^d = 0$ for *the same* homogeneous coordinate system $[w_0, w_1, w_2]$. Shiffman [Sh2] told me that for such a collection of Fermat curves defect relations can be obtained by using a technique of H. Cartan [C2] (see §9), whereas the general case of allowing Fermat curves with respect to *different* homogeneous coordinates cannot be handled by any of the known methods. The only known result on the defect of general divisors is obtained by using the Veronese map and is given in [Sh1] in the following form. Let $f: \mathbb{C} \rightarrow \mathbb{P}_n$ be a holomorphic curve such that $f(\mathbb{C})$ is not contained in any algebraic hypersurface in \mathbb{P}_n . Let $\{S_j\}$ be a collection of algebraic hypersurfaces of degree d in \mathbb{P}_n . The sum of the defects of f for the collection of divisors $\{S_j\}$ is bounded from above by $\binom{n+d}{n}$ provided that the S_j 's satisfy a suitable general position assumption. When applied to Fermat curves of degree d in \mathbb{P}_2 in different coordinate systems, the upper bound of the sum of the defects is of the order d^2 . For Fermat curves of degree d in \mathbb{P}_2 in the same coordinate system, this is much greater than the known upper bound $6/d$ for the sum of the defects.

For the method of meromorphic connections introduced in [Si2], to consider *different* homogeneous coordinates necessitates the use of *different* meromorphic connections. One has to modify the method of [Si2] and consider what can be called a value distribution theory for meromorphic connections. In this paper we carry out such a modification. As a result of the modification done here, we can handle the case of divisors which have zero second fundamental forms with respect to meromorphic connections with the *same* pole order. In particular, one can handle the situation of a collection of Fermat curves with respect to *different* homogeneous coordinates. An upper bound of the sum of defects for Fermat curves of degree d in

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\mathbb{P}_2 in different coordinate systems is $1603/d$. This is still more than the upper bound of $6/d$ for Fermat curves of degree d in \mathbb{P}_2 in the same coordinate system but is far better than the upper bound of the order of d^2 for Fermat curves of degree d in \mathbb{P}_2 in different coordinate systems obtained by using the Veronese map. Besides the example of Fermat curves, this method can be applied to families of divisors of degree d in \mathbb{P}_n with the property that for some fixed e and for every divisor X there exists a homogeneous coordinate system $[w_0, \dots, w_n]$ in \mathbb{P}_n which may depend on X such that X is defined by a polynomial of the form $\sum_{v=0}^n P_v(w_0, \dots, w_n)w_v^{d-e}$, where P_v is a polynomial. There exists a meromorphic connection of pole order no more than $(n + 1)e + 3$ with respect to which the second fundamental form of X vanishes. We get an upper bound $C(n, e)/d$ for the sum of the defects of a family of such divisors where $C(n, e)$ is a constant depending only on n and k . This class of divisors will be discussed in §8. The above results proved in this paper were announced in [Si3].

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§0. Preliminaries.

(0.1) To fix the normalizing constants, we introduce some notations.

$$\mathcal{A}_r(g) = \int_{\theta=0}^{2\pi} g(re^{i\theta}) d\theta$$

denotes 2π times the average of the function g over the circle of radius r centered at 0. For a function or a $(1,1)$ -current η on the disc of radius r centered at 0,

$$\mathcal{T}_r(\eta) = \int_{\rho=0}^r \frac{d\rho}{\rho} \int_{|z|<\rho} \eta$$

denotes the repeated integral. When we have a discrete point-set S in \mathbb{C} , the notation $\mathcal{T}_r(S)$ means the integral when S is naturally regarded as a $(1,1)$ -current on \mathbb{C} . Green’s

formula can be rewritten as

$$\mathcal{F}_r(\partial_z \partial_{\bar{z}} g) = \frac{1}{4} \mathcal{A}_r(g) - \frac{\pi}{2} g(0).$$

(0.2) We use the standard notation $\|$ in value distribution theory that results from the so-called Calculus Lemma. The Calculus Lemma states the following. If $S(r)$ is a positive function for $r > 0$ such that $rS'(r)$ is positive and increasing and $S'(r)$ is continuous, then, given positive ε and δ ,

$$\frac{1}{r} \frac{d}{dr} (rS'(r)) \leq r^\varepsilon (S(r))^{1+\delta} \quad \|_{\varepsilon, \delta}$$

where $\|_{\varepsilon, \delta}$ means that the inequality holds for all r outside an open set $I_{\varepsilon, \delta}$ depending on ε, δ such that $\int_{I_{\varepsilon, \delta}} dt/t$ is finite. When there is no risk of confusion, we drop ε, δ from the notation $\|_{\varepsilon, \delta}$ and simply write $\|$ instead.

By applying the Calculus Lemma twice to $\mathcal{F}_r(\eta)$, we have

$$\log \mathcal{A}_r(\eta) \leq \varepsilon \log r + (1 + \delta) \log \mathcal{F}_r(\eta) \quad \|_{\varepsilon, \delta}.$$

(0.3) Definition of Defect. Suppose $f: \mathbb{C} \rightarrow M$ is a holomorphic map from \mathbb{C} to a compact complex manifold M . Suppose L is a holomorphic line bundle over M and L carries a Hermitian metric whose curvature form θ_L is positive definite. To fix the constants, we use the following convention. If the pointwise square norm $|\sigma|^2$ of a local section σ of L with respect to the Hermitian metric is given by $|\sigma|^2 = |\sigma_0|^2 e^{-\varphi}$, where σ_0 is a function representing σ in a local holomorphic trivialization of L , then the curvature form θ_L is given by $(\sqrt{-1/2\pi}) \partial \bar{\partial} \varphi$. Suppose s is a non identically zero holomorphic section of L over M and Zs is the divisor of the section s . The *characteristic function* of f with respect to the line bundle L is $\mathcal{F}_r(f^* \theta_L)$. We denote this characteristic function also by $T(f, r)$. The *counting function* of the divisor Zs for f is $\mathcal{F}_r(f^* Zs)$, when $f^* Zs$ is naturally regarded as a $(1, 1)$ -current on \mathbb{C} . We denote this counting function also by $N(f, r, Zs)$. By applying Green's formula to the Poincaré-Lelong formula $(\sqrt{-1/2\pi}) \partial \bar{\partial} \log |f^* s|^2 = f^* Zs - f^* \theta_L$, we obtain

$$\begin{aligned} \mathcal{F}_r(f^* Zs) - \mathcal{F}_r(f^* \theta_L) &= \mathcal{F}_r \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f^* s|^2 \right) \\ &= \frac{1}{4\pi} \mathcal{A}_r(\log |f^* s|^2) - \frac{1}{2} \log |f^* s|^2(0). \end{aligned}$$

The defect is defined as

$$\delta_f(Zs) = \liminf_{r \rightarrow \infty} \left(1 - \frac{\mathcal{F}_r(f^* Zs)}{\mathcal{F}_r(f^* \theta_L)} \right)$$

which is equal to

$$\liminf_{r \rightarrow \infty} \frac{\mathcal{A}_r \left(\frac{1}{4\pi} \log |f^* s|^2 \right)}{\mathcal{T}_r(f^* \theta_L)} .$$

An upper bound of the defect means a lower bound of $\mathcal{A}_r((1/4\pi) \log |f^* Zs|^2)$ in terms of the characteristic function $\mathcal{T}_r(f^* \theta_L)$. Note that here the normalizing constants used for the definitions of $\mathcal{A}_r(g)$ and $\mathcal{T}_r(\eta)$ are different from those used in [Si2], because we are using different normalizing constants for the curvature form of a line bundle.

(0.4) Defects for hyperplanes. Consider the case where M is the complex projective space \mathbb{P}_n and L is the hyperplane section line bundle H . Let s_j ($1 \leq j \leq q$) be holomorphic sections of L over \mathbb{P}_n so that their divisors Zs_j are in general position. Then the Second Main Theorem implies that for any $\varepsilon > 0$ there is a subset E of $(0, \infty)$ of finite logarithmic measure (i.e., the integral of dt/t over E is finite) and there is a constant C such that

$$(q - n - 1 - \varepsilon)\mathcal{T}_r(f^* \theta_H) + C \leq \sum_{j=1}^q \mathcal{T}_r(f^* Zs_j)$$

for r not in E . (On the left-hand side there is an additional *nonnegative* term $\sum_{k=0}^{n-1} (n - k)\mathcal{T}_r(S_k)$, where S_k is the zero set of the map $F \wedge (d/dz)F \wedge (d^2/dz^2)F \wedge \cdots \wedge (d^k/dz^k)F$ for the k^{th} associated curve when $F: \mathbb{C} \rightarrow (\mathbb{C}^{n+1} - 0)$ induces the map $f: \mathbb{C} \rightarrow \mathbb{P}_n$.) Hence for any $\varepsilon > 0$ there is a subset E of $(0, \infty)$ of finite logarithmic measure and there is a constant C' such that

$$\sum_{j=1}^q \frac{1}{4\pi} \mathcal{A}_r \left(\frac{1}{4\pi} \log |f^* Zs_j|^2 \right) \geq -C' - (n + 1 + \varepsilon)\mathcal{T}_r(f^* \theta_H)$$

for r not in E .

(0.5) Meromorphic connections. Let M be a compact complex manifold. Let $\Gamma_{\beta\gamma}^\alpha$ be a (not necessarily symmetric) connection for the tangent bundle T_M of M . Let F be a positive holomorphic line bundle over M . We say that $\Gamma_{\beta\gamma}^\alpha$ is a meromorphic connection with pole order F (or pole order ℓ in the case when $M = \mathbb{P}_n$ and F is the ℓ^{th} power of the hyperplane section line bundle of \mathbb{P}_n) if there exists a non identically zero holomorphic section t of F over M such that $t\Gamma_{\beta\gamma}^\alpha$ is holomorphic in all local coordinate systems. Here and throughout this paper all local coordinate systems are assumed holomorphic.

(0.6) Second fundamental form. Let Z be a hypersurface of M locally defined by a holomorphic function s . Assume that the connection $\Gamma_{\beta\gamma}^\alpha$ is meromorphic with pole order F so that $t\Gamma_{\beta\gamma}^\alpha$ is holomorphic in all local coordinate systems for some non

identically zero holomorphic section t of F . We say that the second fundamental form of Z with respect to the connection $\Gamma_{\beta\gamma}^\alpha$ is zero if $t\mathbf{D}_\alpha\partial_\beta s = A_\alpha\partial_\beta s + B_\beta\partial_\alpha s + C_{\alpha\beta}s$ for some local smooth functions A_α , B_β , and $C_{\alpha\beta}$, where \mathbf{D}_α is the covariant differentiation with respect to the connection $\Gamma_{\beta\gamma}^\alpha$, and ∂_α is the partial differential operator with respect to the α^{th} local coordinate.

(0.7) Nondegeneracy condition for a holomorphic map. Let $f: \mathbb{C} \rightarrow M$ be a holomorphic map from \mathbb{C} to a compact complex manifold M of complex dimension n . Let z be the coordinate of \mathbb{C} and f_z be the derivative of f with respect to z which is a map from \mathbb{C} to the tangent bundle T_M of M . Let F be a positive line bundle over M . We say that f is nondegenerate with respect to the line bundle F if we cannot find a holomorphic section s of $F^{n(n-1)/2} \otimes K_M^{-1}$ over M (where K_M is the canonical line bundle of M) and a meromorphic connection $\Gamma_{\beta\gamma}^\alpha$ with pole order F such that $f_z \wedge (t\mathbf{D}_z)f_z \wedge \cdots \wedge (t\mathbf{D}_z)^{n-1}f_z + f^*s = 0$, where t is a holomorphic section of F so that $t\Gamma_{\beta\gamma}^\alpha$ is holomorphic in all local coordinate systems and \mathbf{D}_z is the covariant differentiation with respect to the connection $\Gamma_{\beta\gamma}^\alpha$, and the coordinate z of \mathbb{C} . The exterior product $f_z \wedge (t\mathbf{D}_z)f_z \wedge \cdots \wedge (t\mathbf{D}_z)^{n-1}f_z$ is taken in the tangent bundle T_M of M so that $f_z \wedge (t\mathbf{D}_z)f_z \wedge \cdots \wedge (t\mathbf{D}_z)^{n-1}f_z$ is a holomorphic map from \mathbb{C} to the line bundle $F^{n(n-1)/2} \otimes K_M^{-1}$ of M .

There is only a finite degree of freedom in choosing s and $\Gamma_{\beta\gamma}^\alpha$. The condition $f_z \wedge (t\mathbf{D}_z)f_z \wedge \cdots \wedge (t\mathbf{D}_z)^{n-1}f_z + f^*s = 0$ is an ordinary differential equation in \mathbb{C} of order $n - 1$. This kind of nondegeneracy condition is different from the kind normally used in the equidimensional case and the case where the target manifold is the complex projective space with hyperplane divisors. For known defect relations in the equidimensional case the nondegeneracy condition required is that the map is open at some point. It is conjectured that the nondegeneracy condition in the equidimensional case should be the condition that the image is not contained in some algebraic hypersurface, which is the same as saying that we cannot find any non identically zero holomorphic section s of some positive line bundle such that $f^*s = 0$. In the case of the target space being the complex projective space with hyperplane divisors the nondegeneracy condition for defect relations is that the image is not contained in some hyperplane, which is the same as saying that $f^*s = 0$ cannot hold for any non identically zero holomorphic section s of the hyperplane section line bundle of the complex projective space. We can regard $f^*s = 0$ as an algebraic equation which can be chosen with a finite degree of freedom. Another way of expressing the nondegeneracy condition in the case of the target space being the complex projective space with hyperplane divisors is that $f_z \wedge (\mathbf{D}_z f_z) \wedge \cdots \wedge (\mathbf{D}_z^{n-1} f_z) = 0$ does not hold for the Levi Civita connection \mathbf{D} of the Fubini-Study metric of the complex projective space. Our present nondegeneracy condition of the failure of $f_z \wedge (t\mathbf{D}_z)f_z \wedge \cdots \wedge (t\mathbf{D}_z)^{n-1}f_z + f^*s = 0$ is, in a way, a combination of the two equations $f^*s = 0$ and $f_z \wedge (\mathbf{D}_z f_z) \wedge \cdots \wedge (\mathbf{D}_z^{n-1} f_z) = 0$.

(0.8) General position condition for divisors. Let M be a compact complex manifold. Let E be a holomorphic line bundle of M . Let $\{s_\nu\}$ be a set of distinct holomorphic sections of E over M . Let e_0, \dots, e_m be a basis of $\Gamma(M, E)$ over M and

let $s_\nu = \sum_{j=0}^m c_{\nu j} e_j$ with $c_{\nu j} \in \mathbb{C}$. The set $\{s_\nu\}$ is said to be in general position if the hyperplanes $\sum_{j=1}^m c_{\nu j} z_j = 0$ in \mathbb{P}_m with homogeneous coordinates $[z_0, \dots, z_m]$ are in general position.

Let $\{Z_\nu\}$ be a family of complex hypersurfaces in M . Let F be a positive line bundle with the property that for every ν there exists a meromorphic connection $\Gamma_{\beta_\nu}^\alpha(\nu)$ with pole order F so that the second fundamental form of Z_ν , with respect to $\Gamma_{\beta_\nu}^\alpha(\nu)$ is zero. Let t_ν be a non identically zero holomorphic section of F so that $t_\nu \Gamma_{\beta_\nu}^\alpha(\nu)$ is holomorphic in all local coordinate systems. Let $\mathbf{D}^{(\nu)}$ be the covariant differentiation with respect to the connection $\Gamma_{\beta_\nu}^\alpha(\nu)$. (We assume that the index ν does not take on the value 0.) Let F_0 be a positive line bundle and $\Gamma_{\beta_0}^\alpha(0)$ be any meromorphic connection with pole order F_0 . Let t_0 be a non identically zero holomorphic section of F_0 so that $t_0 \Gamma_{\beta_0}^\alpha(0)$ is holomorphic in all local coordinate systems. The difference $t_0 t_\nu (\Gamma_{\beta_\nu}^\alpha(\nu) - \Gamma_{\beta_0}^\alpha(0))$ is a holomorphic section of $T_M \otimes T_M^* \otimes T_M^* \otimes F \otimes F_0$ over M . The pair $(t_\nu, t_0 t_\nu (\Gamma_{\beta_\nu}^\alpha(\nu) - \Gamma_{\beta_0}^\alpha(0)))$ is a holomorphic section of the vector bundle $F \oplus (T_M \otimes T_M^* \otimes T_M^* \otimes F \otimes F_0)$ over M . We say that $\{Z_\nu\}$ is in general position of order k with respect to F and F_0 if for some $t_0, \Gamma_{\beta_0}^\alpha(0), t_\nu,$ and $\Gamma_{\beta_\nu}^\alpha(\nu)$ all the distinct sections $(t_\nu, t_0 t_\nu (\Gamma_{\beta_\nu}^\alpha(\nu) - \Gamma_{\beta_0}^\alpha(0)))$ are in general position as holomorphic sections of $F \oplus (T_M \otimes T_M^* \otimes T_M^* \otimes F \otimes F_0)$ over M and, moreover, no more than k of the hypersurfaces Z_ν with the same \mathbf{D}_ν have a common point of intersection. In the case when $M = \mathbb{P}_n$ we will take as $\Gamma_{\beta_0}^\alpha(0)$ the connection which is identically zero on the affine part \mathbb{C}^n of \mathbb{P}_n with respect to the Euclidean coordinates of \mathbb{C}^n . We will take as F_0 the hyperplane section line bundle of \mathbb{P}_n . We will take as t_0 the section of F_0 whose divisor is the infinity part of \mathbb{P}_n . The verification that $t_0 \Gamma_{\beta_0}^\alpha(0)$ is holomorphic in all local coordinate systems of \mathbb{P}_n will be given in §7.

§1. Statement of results. First we state our result for the special case of a defect relation for Fermat curves in different coordinates in \mathbb{P}_2 .

(1.1) THEOREM. *Let H be the positive hyperplane section line bundle of \mathbb{P}_2 . Suppose $f: \mathbb{C} \rightarrow \mathbb{P}_2$ is a holomorphic map and for each ν the curve Z_ν is a Fermat curve $\zeta_{\nu 0}^d + \zeta_{\nu 1}^d + \zeta_{\nu 2}^d = 0$ of degree d in some homogeneous coordinate system $[\zeta_{\nu 0}, \zeta_{\nu 1}, \zeta_{\nu 2}]$ of \mathbb{P}_2 . Suppose $\{Z_\nu\}$ is in general position of order k with respect to H^3 and H , and f is nonsingular with respect to the line bundle H^4 . Then one has the defect estimate $\Sigma_\nu \delta_f(Z_\nu) \leq (1603/d)k$.*

We now state our result in a more general setting.

(1.2) THEOREM. *Let M be a compact complex manifold of complex dimension n . Let F_0 be a positive line bundle such that some meromorphic connection $\mathbf{D}^{(0)}$ with pole order F_0 exists for M . Let L be a positive line bundle and $\{s_\nu\}$ a collection of holomorphic sections of L over M . (Assume that the index ν does not take on the value 0.) Let F be a positive line bundle with the following property: for each ν a meromorphic connection $\mathbf{D}^{(\nu)}$ exists for M with pole order F such that the second fundamental form of the zero-set Zs_ν of s_ν with respect to $\mathbf{D}^{(\nu)}$ is zero. Let t_ν be a non identically zero holomorphic section of F such that $t_\nu \mathbf{D}^{(\nu)}$ is holomorphic. Let $p = \dim_{\mathbb{C}} \Gamma(M, F)$ and*

$m = \dim \Gamma_{\mathbb{C}}(M, T_M^* \otimes T_M^* \otimes T_M \otimes F \otimes F_0)$. Let v be a positive number such that $(F \otimes F_0)^{n(n-1)/2} \otimes K_M^{-1}$ is generated over M by v global holomorphic sections of $(F \otimes F_0)^{n(n-1)/2} \otimes K_M^{-1}$ over M . Let α be a nonnegative number such that $\alpha\theta_L$ minus the curvature of $(F \otimes F_0)^{n(n-1)/2} \otimes K_M^{-1}$ is everywhere nonnegative on M . Suppose f is a holomorphic map from \mathbb{C} to M . Suppose $\{Z_{s_v}\}$ is in general position of order k with respect to F and F_0 , and f is nonsingular with respect to the line bundle $F \otimes F_0$. Then the defect of f is no more than $(\alpha \binom{b}{c} + \alpha v)k$, where $b = n(n-1)/2 + p + m - 1$ and $c = n(n-1)/2$.

(1.3) COROLLARY. For the special case $M = \mathbb{P}_n$ with $L = H^d$ (where H is the hyperplane section line bundle of \mathbb{P}_n), let H^ℓ be the pole order of the meromorphic connections $\mathbf{D}^{(v)}$ for a collection $\{s_v\}$ of holomorphic sections of L so that the second fundamental form of the zero-set Z_{s_v} of s_v is zero with respect to $\mathbf{D}^{(v)}$. Suppose $\{Z_{s_v}\}$ is in general position of order k with respect to H^3 and H and f is nonsingular with respect to the line bundle $H^{\ell+1}$. Then the defect is no more than

$$\left(\binom{b}{c} + n + 1 \right) \frac{(\ell + 1)^{n(n+1)/2} + n + 1}{d} k,$$

where $b = \frac{n(n-1)}{2} + \binom{\ell+n}{n} + (n+1)^2(n+\ell+1) \binom{n+\ell-1}{n-1} - 1$ and $c = \frac{n(n-1)}{2}$.

For notational simplicity, in this paper we will present our proof of the above results only for the case $n = 2$, because the proof for the case of a general n differs from the case $n = 2$ only in the complexity of notations, which makes the essential arguments less transparent. The sections §2 through §7 below are devoted to the proof of the results stated above.

§2. Main inequality.

(2.1) Let M be a compact complex manifold of complex dimension two. We recall the main inequality in [Si2, p. 241, (8.5)] which is derived by using the vanishing of the second fundamental form and linear algebra. It is

$$(2.1.1) \quad \mathcal{F}_r \left(\left(\frac{|f_z \wedge t\mathbf{D}_z f_z|^{2\lambda}}{|s|^2(\log|s|^2)^2} \right)^\mu \right) \leq \text{constant} \cdot (\mathcal{F}_r(f^*\theta_L)^{16\lambda\mu} + r^4).$$

Here $\mu > 0$ and $\lambda - 1 > 0$ are both sufficiently small. The meanings of the notations in this inequality are as follows. The covariant differentiation \mathbf{D} comes from a meromorphic connection $\Gamma_{\beta\gamma}^\alpha$ on M and t is a non identically zero holomorphic section of the positive line bundle F so that $t\Gamma_{\beta\gamma}^\alpha$ is holomorphic in all local coordinate systems. The section s is a holomorphic section of L so that the second

fundamental form of the zero set Z_s of s with respect to the meromorphic connection $\Gamma_{\beta\gamma}^\alpha$ is zero. The nondegeneracy condition of the nonvanishing of $f_z \wedge tD_z f_z$ is assumed.

By the Calculus Lemma and the concavity of logarithm this main inequality yields

$$\mathcal{A}_r \left(\log \frac{|f_z \wedge tD_z f_z|^{2\lambda}}{|s|^2 (\log|s|^2)^2} \right) = O(\log \mathcal{T}_r(f^* \theta_L) + \log r).$$

This is to be coupled with the following inequality in [Si2, p. 241, (*)] derived from Green’s formula

$$\frac{1}{4\pi} \mathcal{A}_r \left(\log \frac{1}{|s|^2 (\log|s|^2)^2} \right) \geq (1 - \varepsilon) \mathcal{T}_r(f^* \theta_L) - \mathcal{T}_r(Zf^* s)$$

for $0 < \varepsilon < 1$ and for r sufficiently large. Thus

$$\begin{aligned} & \frac{1}{4\pi} \mathcal{A}_r \left(\log \frac{|f_z \wedge tD_z f_z|^{2\lambda}}{|s|^2 (\log|s|^2)^2} \right) \\ & \geq \frac{1}{4\pi} \mathcal{A}_r(\log|f_z \wedge tD_z f_z|^{2\lambda}) + (1 - \varepsilon) \mathcal{T}_r(f^* \theta_L) - \mathcal{T}_r(Zf^* s) \end{aligned}$$

and

$$\begin{aligned} & (1 - \varepsilon) \mathcal{T}_r(f^* \theta_L) - \mathcal{T}_r(Zf^* s) \\ & \leq -\frac{1}{4\pi} \mathcal{A}_r(\log|f_z \wedge tD_z f_z|^{2\lambda}) + O(\log \mathcal{T}_r(f^* \theta_L) + \log r) \end{aligned}$$

for $\lambda - 1 > 0$ sufficiently small, yielding the following estimate for the defect relation.

$$\delta_f(Zs) \leq \liminf_{r \rightarrow \infty} \frac{-\frac{1}{4\pi} \mathcal{A}_r(\log|f_z \wedge tD_z f_z|^2)}{\mathcal{T}_r(f^* \theta_L)}.$$

(2.2) Suppose we have a finite number of holomorphic sections s_ν of L so that the second fundamental form of the zero-set Z_{s_ν} of s_ν with respect to the same meromorphic connection $\Gamma_{\alpha\beta}^\gamma$ is zero. Assume that no more than k of Z_{s_ν} have a common point of intersection. We use the argument of [Si2, p. 242, (9.2)] and get

$$\frac{1}{\prod_\nu (|s_\nu|^2 (\log|s_\nu|^2)^2)} \leq C \left(\sum_{\nu_1, \dots, \nu_k \text{ distinct}} \frac{1}{\prod_{j=1}^k (|s_{\nu_j}|^2 (\log|s_{\nu_j}|^2)^2)} \right),$$

where C is a constant. Thus by (2.1.1)

$$\begin{aligned} & \mathcal{F}_r \left(\left(\frac{|f_z \wedge t\mathbf{D}_z f_z|^{2k\lambda}}{\prod_v (|s_v|^2 (\log |s_v|^2)^2)} \right)^\mu \right) \\ & \leq C' \mathcal{F}_r \left(\sum_{v_1, \dots, v_k \text{ distinct}} \left(\frac{|f_z \wedge t\mathbf{D}_z f_z|^{2k\lambda}}{\prod_{j=1}^k (|s_{v_j}|^2 (\log |s_{v_j}|^2)^2)} \right)^\mu \right) \\ & \leq C'' \sum_v \mathcal{F}_r \left(\left(\frac{|f_z \wedge t\mathbf{D}_z f_z|^{2\lambda}}{(|s_v|^2 (\log |s_v|^2)^2)} \right)^{k\mu} \right) = o(\mathcal{F}_r(f^* \theta_L)) \end{aligned}$$

for $\mu > 0$ and $\lambda - 1 > 0$ both sufficiently small by (2.1.1), where C' and C'' are constants. As in (2.1) we get the following defect estimate in this case

$$(2.2.1) \quad \sum_v \delta_f(Z_{s_v}) \leq \liminf_{r \rightarrow \infty} \frac{-\frac{k}{4\pi} \mathcal{A}_r(\log |f_z \wedge t\mathbf{D}_z f_z|^2)}{\mathcal{F}_r(f^* \theta_L)}.$$

(2.3) We now look at the general case when we have a finite number of holomorphic sections s_v of L and a meromorphic connection $\Gamma_{\alpha\beta}^\gamma(v)$ for each v so that the second fundamental form of the zero-set Z_{s_v} of s_v with respect to $\Gamma_{\alpha\beta}^\gamma(v)$ is zero. We assume that the pole order of all $\Gamma_{\alpha\beta}^\gamma(v)$ is the same positive line bundle F . Let t_v be a non identically zero holomorphic section of F over M so that $t_v \Gamma_{\alpha\beta}^\gamma(v)$ is holomorphic in all local coordinate systems. Assume that among all the indices v with the same $\Gamma_{\alpha\beta}^\gamma(v)$ no more than k of Z_{s_v} have a common point of intersection. By summing (2.2.1) over all distinct connections $\Gamma_{\beta\gamma}^\alpha(v)$, we have the following estimate of the sum of defects

$$(2.3.1) \quad \sum_v \delta_f(Z_{s_v}) \leq \liminf_{r \rightarrow \infty} \frac{-\frac{k}{4\pi} \mathcal{A}_r(\log \prod'_v |f_z \wedge t_v \mathbf{D}_z^{(v)} f_z|^2)}{\mathcal{F}_r(f^* \theta_L)},$$

where the product \prod'_v is taken over all *distinct* meromorphic connections $\mathbf{D}^{(v)}$. We have to obtain the estimate for $\mathcal{A}_r(\log \prod'_v |f_z \wedge t_v \mathbf{D}_z^{(v)} f_z|^2)$ from below. This estimate will be obtained by considering the defect relation for the space of all meromorphic connections with the same pole order.

§3. Value distribution theory of meromorphic connections.

(3.1) We use the notations of (2.3) and suppose that all the assumptions in (2.3) are satisfied. We use \sum'_v to mean summation over only those v with $\mathbf{D}^{(v)}$ distinct. We also assume that the index v does not take on the value 0. Let F_0 be a positive

line bundle and $\Gamma_{\beta\gamma}^\alpha(0)$ be some meromorphic connection so that for some non identically zero holomorphic section t_0 of F , $t_0\Gamma_{\beta\gamma}^\alpha(0)$ is holomorphic in all local coordinate systems. Let $\mathbf{D}^{(0)}$ be the differentiation with respect to $\Gamma_{\beta\gamma}^\alpha(0)$. Suppose with respect to the line bundle $F \otimes F_0$ the map f satisfies the nondegeneracy condition in the sense of (0.7). Also suppose that with respect to F and F_0 the collection $\{Z_{S_v}\}$ satisfies the general position condition in the sense of (0.8). Each $f_z \wedge t_v \mathbf{D}_z^{(v)} f_z$ is a holomorphic section of $f^*F \otimes K_M^{-1}$ over \mathbb{C} . We claim the following.

(3.1.1) There exists a finite number of holomorphic sections ψ_0, \dots, ψ_u of $f^*(F \otimes F_0 \otimes K_M^{-1})$ over M so that each $t_0 f_z \wedge t_v \mathbf{D}_z^{(v)} f_z$ is a linear combination $\sum_{j=0}^u a_{vj} \psi_j$ of ψ_0, \dots, ψ_u with $a_{vj} \in \mathbb{C}$. Here u depends only on F and F_0 and M and is independent of f . However, ψ_0, \dots, ψ_u will depend on f .

We now proceed to prove the claim. Suppose we have a meromorphic connection $\Gamma_{\beta\gamma}^\alpha$ whose covariant differential operator is \mathbf{D} and suppose t is a non identically zero holomorphic section of F such that $t\Gamma_{\beta\gamma}^\alpha$ is holomorphic in all local coordinate systems. Let $A_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha - \Gamma_{\beta\gamma}^\alpha(0)$. Then $A_{\beta\gamma}^\alpha$ is a meromorphic tensor and

$$\mathbf{D}_z f_z^\alpha = \mathbf{D}_z^{(0)} f_z^\alpha + A_{\beta\gamma}^\alpha f_z^\beta f_z^\gamma.$$

Here and for the rest of this paper the summation convention is being used.

For a tensor B with components $B_{\beta\gamma}^\alpha$ we use the notation $B(f_z \otimes f_z)$ to denote $B_{\beta\gamma}^\alpha f_z^\beta f_z^\gamma$ which is a section of the pullback of the tangent bundle of M by f . We use the same notation when B is a section of $T_M^* \otimes T_M^* \otimes T_M \otimes E$ for some line bundle E over M instead of $T_M^* \otimes T_M^* \otimes T_M$.

Let τ_1, \dots, τ_p be a basis of $\Gamma(M, F)$ over \mathbb{C} . We have $t = \sum_{i=1}^p \gamma_i \tau_i$ for some $\gamma_i \in \mathbb{C}$ and we can write $t_0 t f_z \wedge \mathbf{D}_z^{(0)} f_z$ as $\sum_{i=1}^p \gamma_i \tau_i t_0 f_z \wedge \mathbf{D}_z^{(0)} f_z$.

The section $t_0 t A_{\beta\gamma}^\alpha$ of $T_M^* \otimes T_M^* \otimes T_M \otimes F \otimes F_0$ over M is holomorphic. Let $B_{\beta\gamma}^\alpha(\kappa)$, $1 \leq \kappa \leq m$, be a basis of $\Gamma(M, T_M^* \otimes T_M^* \otimes T_M \otimes F \otimes F_0)$ over \mathbb{C} . Then $t_0 t A_{\beta\gamma}^\alpha = \sum_{\kappa=1}^m c_\kappa f_z \wedge B_{\beta\gamma}^\alpha(\kappa)$ for some $c_\kappa \in \mathbb{C}$ and

$$t_0 t f_z \wedge (\mathbf{D}_z f_z - \mathbf{D}_z^{(0)} f_z) = \sum_{\kappa=1}^m c_\kappa f_z \wedge B(\kappa)(f_z \otimes f_z).$$

Let $u = p + m - 1$ and define the holomorphic section ψ_0, \dots, ψ_u of $f^*(F \otimes F_0 \otimes K_M^{-1})$ over \mathbb{C} by $\psi_j = \tau_{j+1} t_0 f_z \wedge \mathbf{D}_z^{(0)} f_z$ ($0 \leq j < p$) and $\psi_j = f_z \wedge B(j - p + 1)(f_z \otimes f_z)$ ($p \leq j < p + m$). Our claim (3.1.1) is proved with $t_0 f_z \wedge t \mathbf{D}_z f_z = \sum_{j=0}^u a_j \psi_j$ with $a_j = \tau_{j+1}$ ($0 \leq j < p$), $a_j = c_{j-p+1}$ ($p \leq j < p + m$).

When \mathbf{D} is replaced by $\mathbf{D}^{(v)}$ and t by t_v , we have constants $a_{vj} = \tau_{v,j+1}$ ($0 \leq j < p$), $a_{vj} = c_{v,j-p+1}$ ($p \leq j < p + m$) corresponding to $a_j = \tau_{j+1}$ ($0 \leq j < p$), $a_j = c_{j-p+1}$ ($p \leq j < p + m$).

(3.2) Let v be a positive number such that $F \otimes F_0 \otimes K_M^{-1}$ is generated over M by v global holomorphic sections $\sigma_1, \dots, \sigma_v$ of $F \otimes F_0 \otimes K_M^{-1}$ over M . This means

that $\sigma_1, \dots, \sigma_v$ have no common zeroes. Let $\psi_{u+j} = f^* \sigma_j$ for $1 \leq j \leq v$. Let $N = u + v$. The pullbacked sections ψ_{u+j} ($1 \leq j \leq v$) are introduced to make sure that the sections ψ_0, \dots, ψ_N of $f^*(F \otimes F_0 \otimes K_M^{-1})$ over \mathbb{C} have no common zeroes so that they can serve as homogeneous coordinates for a holomorphic map from \mathbb{C} to \mathbb{P}_N . Denote by Ψ the holomorphic map from \mathbb{C} to \mathbb{P}_N defined by the homogeneous coordinates $[\psi_0, \dots, \psi_N]$. Let $[w_0, \dots, w_N]$ be the homogeneous coordinates of \mathbb{P}_N . From (3.1) each $t_0 f_z \wedge t_v \mathbf{D}_z^{(v)} f_z$ is of the form $\sum_{j=0}^N a_{vj} \psi_j$ with constants $a_{vj} = \tau_{v,j+1}$ ($0 \leq j < p$), $a_{vj} = c_{v,j-p+1}$ ($p \leq j < p + m$), $a_{vj} = 0$ ($p + m \leq j \leq N$). Consider the holomorphic section S_v of the hyperplane section line bundle H of \mathbb{P}_N whose divisor is the hyperplane $\sum_{j=0}^N a_{vj} w_j = 0$ in \mathbb{P}_N . We only consider the collection of all S_v so that the meromorphic connections $\mathbf{D}^{(v)}$ are *distinct* and we denote the collection by $\{S_v\}'$. We are going to use the Cartan-Ahlfors-Weyl-Weyl theory [C2, A, W-W] to handle the value distribution theory of the map Ψ and the hyperplanes $\sum_{j=0}^N a_{vj} w_j = 0$ in \mathbb{P}_N .

§4. Estimation of characteristic functions.

(4.1) To apply the Cartan-Ahlfors-Weyl-Weyl theory, first we have to compute the characteristic function $T(\Psi, r)$ of the map Ψ . We have to estimate the growth of the characteristic function of the map Ψ in terms of the growth of the characteristic function $T(f, r)$ of the original map f . Let α be a nonnegative number such that $\alpha \theta_L$ minus the curvature of $F \otimes F_0 \otimes K_M^{-1}$ is everywhere nonnegative on M . We are going to verify that $T(\Psi, r) \leq \alpha T(f, r) + O(\log r) + O(\log T(f, r))$.

We recall the following. Suppose E is a holomorphic line bundle over a compact complex manifold X and E carries a Hermitian metric h along its fibers so that the curvature form $\theta_{E,h}$ of the Hermitian metric h is positive definite at every point of \mathbb{P}_n . Suppose F is a holomorphic map from \mathbb{C} to X . The characteristic function $T(F, r)$ is defined as $\mathcal{T}_r(F^* \theta_{E,h})$. This definition determines $T(F, r)$ up to an additive term $O(1)$. The reason is as follows. Suppose g is another Hermitian metric along the fibers of E . Then $\theta_{E,h} - \theta_{E,g} = (\sqrt{-1}/2\pi) \partial \bar{\partial} \varphi$ for some smooth function φ on X and $\mathcal{T}_r(\partial_z \bar{\partial}_z F^* \varphi) = \frac{1}{4} \mathcal{A}_r(F^* \varphi) - (\pi/2) F^* \varphi(0) = O(1)$. Suppose e_1, \dots, e_v are holomorphic sections of E over M which are not simultaneously zero at any point of X . Then $e_j \bar{e}_j$ is a section of $E \otimes \bar{E}$ and $(\sum_{j=1}^v e_j \bar{e}_j)^{-1}$ is a section of $E^* \otimes \bar{E}^*$ (where \bar{E} is the complex conjugate of E and \bar{E}^* is the complex conjugate of the dual bundle E^* of E) and defines a Hermitian metric along the fibers of E . We can use this Hermitian metric to calculate the characteristic function $T(F, r)$ up to an additive term $O(1)$. The characteristic function $T(F, r)$ is equal to $\mathcal{T}_r((\sqrt{-1}/2\pi) \partial_z \bar{\partial}_z \log \sum_{j=1}^v e_j \bar{e}_j)$ up to an additive term $O(1)$.

We now apply this observation to the case at hand. Our map $\Psi: \mathbb{C} \rightarrow \mathbb{P}_N$ is defined by the $N + 1$ holomorphic sections ψ_j ($0 \leq j \leq N$) of $f^*(F \otimes F_0 \otimes K_M^{-1})$ over \mathbb{C} with $\psi_j = \tau_{j+1} t_0 f_z \wedge \mathbf{D}_z^0 f_z$ ($0 \leq j < p$), $\psi_j = f_z \wedge B(j - p + 1)(f_z \otimes f_z)$ ($p \leq j < p + m$), and $\psi_j = f^* \sigma_{j-p-m+1}$ ($p + m \leq j \leq N$). Let $[w_0, \dots, w_N]$ be the homogeneous coordinates of \mathbb{P}_N . Then $\psi_j = \Psi^* w_j$ ($0 \leq j \leq N$). Each homogeneous coordinate w_j can be naturally regarded as a holomorphic section of the hyperplane section line bundle

H of \mathbb{P}_N . Hence

$$(4.1.1) \quad \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{j=0}^N \psi_j \bar{\psi}_j \right) = \Psi^* \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{j=0}^N w_j \bar{w}_j \right) \right),$$

where $\psi_j \bar{\psi}_j$ (respectively $w_j \bar{w}_j$) is a section of $(f^*(F \otimes F_0 \otimes K_M^{-1})) \otimes (f^*(F \otimes F_0 \otimes K_M^{-1}))$ over \mathbb{C} (respectively $H \otimes \bar{H}$ over M). The left-hand side and the right-hand side of (4.1.1) are computed respectively by using local trivializations of the line bundles $(f^*(F \otimes F_0 \otimes K_M^{-1}))$ and H , and the result is independent of the local trivializations chosen. The characteristic function $T(\Psi, r)$ of Ψ is given by

$$\begin{aligned} & \mathcal{F}_r \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{j=0}^N \psi_j \bar{\psi}_j \right) \right) \\ &= \mathcal{F}_r \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{j=0}^N \|\psi_j\|^2 \right) \right) + \mathcal{F}_r \left(\frac{\sqrt{-1}}{2\pi} f^* \theta_{F \otimes F_0 \otimes K_M^{-1}} \right) \\ &\leq \mathcal{F}_r \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{j=0}^N \|\psi_j\|^2 \right) \right) + \alpha \mathcal{F}_r \left(\frac{\sqrt{-1}}{2\pi} f^* \theta_L \right) \\ &= \frac{1}{4\pi} \mathcal{A}_r \left(\log \left(\sum_{j=0}^N \|\psi_j\|^2 \right) \right) - \frac{1}{2} \log \left(\sum_{j=0}^N \|\psi_j(0)\|^2 \right) + \alpha T(f, r). \end{aligned}$$

Here, in order to emphasize the difference between the section $\psi_j \bar{\psi}_j$ of the line bundle $(f^*(F \otimes F_0 \otimes K_M^{-1})) \otimes (f^*(F \otimes F_0 \otimes K_M^{-1}))$ over \mathbb{C} and the square norm $\|\psi_j\|^2$ of ψ_j with respect to the Hermitian metric along the fibers of the line bundle $F \otimes F_0 \otimes K_M^{-1}$, we temporarily deviate from our convention and use $\|\psi_j\|^2$ instead of $|\psi_j|^2$ to denote the square norm.

(4.2) We claim that $\mathcal{A}_r(\log(\sum_{j=0}^N \|\psi_j\|^2)) \leq O(\log r) + O(\log T(f, r))$. This claim actually corresponds to the logarithmic derivative lemma.

Since

$$\mathcal{A}_r \left(\log \left(\sum_{j=0}^N \|\psi_j\|^2 \right) \right) \leq \sum_{j=0}^N \mathcal{A}_r(\log(1 + \|\psi_j\|^2)),$$

it suffices to show that $\mathcal{A}_r(\log(1 + \|\psi_j\|^2)) \leq O(\log r) + O(\log T(r, f))$. When $p + m \leq j \leq N$, $\|\psi_j\| = \|\sigma_{j-p-m+1}\|$ and $\sup_{\mathbb{C}} \|\psi_j\| \leq \sup_M \|\sigma_{j-p-m+1}\|$ is bounded and the inequality clearly holds for $p + m \leq j \leq N$.

For the remaining two types of sections $\psi_j = \tau_{j+1} t_0 f_z \wedge \mathbf{D}_z^0 f_z$ ($0 \leq j < p$), $\psi_j = f_z \wedge B(j - p + 1)(f_z \otimes f_z)$ ($p \leq j < p + m$) of $f^*(F \otimes F_0 \otimes K_M^{-1})$ over \mathbb{C} , it suffices

to show that

$$(4.2.1) \quad \begin{aligned} \mathcal{A}_r(\log(1 + |f_z|^2)) &\leq O(\log r) + O(\log T(f, r)), \\ \mathcal{A}_r(\log(1 + |\mathbf{D}_z^{(0)}f_z|^2)) &\leq O(\log r) + O(\log T(f, r)). \end{aligned}$$

Here we return to our convention of using $|f_z|$ and $|\mathbf{D}_z^{(0)}f_z|$ to denote norms calculated with respect to the Hermitian metric of M .

From the Calculus Lemma and the concavity of the logarithm it follows that

$$\mathcal{A}_r(\log(1 + |f_z|^2)) \leq \varepsilon \log r + (1 + \delta) \log \mathcal{T}_r(1 + |f_z|^2) \quad \|\varepsilon, \delta.$$

On the other hand we have $\mathcal{T}_r(1 + |f_z|^2) \leq \text{constant} \cdot (\log r + \mathcal{T}_r(f^*\theta_H))$, because θ_H is positive definite on M . Hence we get

$$\mathcal{A}_r(\log(1 + |f_z|^2)) = \leq O(\log r) + O(\log T(f, r)) \quad \|\varepsilon, \delta.$$

To prove (4.2.1), again by using the Calculus Lemma and the concavity of the logarithm, we have

$$(4.2.2) \quad \mathcal{A}_r(\log((1 + |\mathbf{D}_z^{(0)}f_z|^2)^\mu)) \leq \varepsilon \log r + (1 + \delta) \log \mathcal{T}_r((1 + |\mathbf{D}_z^{(0)}f_z|^2)^\mu) \quad \|\varepsilon, \delta$$

for $\mu > 0$. Let $h_{\alpha\bar{\beta}}$ be a Kähler metric of M with curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$. From [p. 238, Si2] we have

$$\begin{aligned} \mathcal{T}_r\left(\frac{h_{\alpha\bar{\beta}}(\mathbf{D}_z^{(0)}f_z^\alpha)(\overline{\mathbf{D}_z^{(0)}f_z^\beta})}{(1 + h_{\alpha\bar{\beta}}f_z^\alpha\overline{f_z^\beta})^2}\right) &\leq \frac{\varepsilon}{4} \log r + \frac{1 + \delta}{4} \log \mathcal{T}_r(1 + h_{\alpha\bar{\beta}}f_z^\alpha\overline{f_z^\beta}) \\ &\quad + C\mathcal{T}_r\left(\frac{(h_{\alpha\bar{\beta}}f_z^\alpha\overline{f_z^\beta})^2}{(1 + h_{\alpha\bar{\beta}}f_z^\alpha\overline{f_z^\beta})}\right), \end{aligned}$$

where

$$C = \sup_{M, \xi} \frac{R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^\alpha\bar{\xi}^\beta\xi^\gamma\bar{\xi}^\delta}{(h_{\alpha\bar{\beta}}\xi^\alpha\bar{\xi}^\beta)^2}.$$

Hence

$$c\mathcal{T}_r\left(\frac{h_{\alpha\bar{\beta}}(\mathbf{D}_z^{(0)}f_z^\alpha)(\overline{\mathbf{D}_z^{(0)}f_z^\beta})}{(1 + h_{\alpha\bar{\beta}}f_z^\alpha\overline{f_z^\beta})^2}\right) \leq \mathcal{T}_r(f^*\theta_L) + r^2,$$

where c is some positive constant. So

$$c\mathcal{T}_r((|\mathbf{D}_z^{(0)}f|^2)^\mu) \leq \mathcal{T}_r(f^*\theta_L)^{4\mu} + r^4$$

for $\mu > 0$ sufficiently small and

$$(4.2.3) \quad c\mathcal{T}_r((1 + |\mathbf{D}_z^{(0)}f|^2)^\mu) \leq \mathcal{T}_r(f^*\theta_L)^{4\mu} + r^4.$$

We now can conclude that (4.2.1) is a consequence of (4.2.2) and (4.2.3).

§5. Final derivation of the defect relation. Now we come back to our original situation of the theory of value distribution theory for meromorphic connections. We have to get a lower bound for $(1/4\pi)\sum'_v \mathcal{A}_r(\log |f_z \wedge t_0 t_v \mathbf{D}_z^{(v)} f_z|^2)$. Recall that \sum'_v means summation over all indices v with *distinct* meromorphic connections $\mathbf{D}^{(v)}$. By our definition of $\Psi = (\psi_j)_{0 \leq j \leq N}$, we have

$$f_z \wedge t_0 t_v \mathbf{D}_z^{(v)} f_z = \sum_{j=0}^N a_{vj} \psi_j$$

and $\psi_j = f^* \sigma_{j-p-m+1}$ for $p + m \leq j \leq N$. Since $\sigma_1, \dots, \sigma_v$ generate $F \otimes F_0 \otimes K_M^{-1}$, we can use the section $(\sum_{j=1}^v \sigma_j \bar{\sigma}_j)^{-1}$ of $(F \otimes F_0 \otimes K_M^{-1})^* \otimes (F \otimes F_0 \otimes K_M^{-1})^*$ as a Hermitian metric along the fibers of $F \otimes F_0 \otimes K_M^{-1}$. Hence

$$\begin{aligned} \|f_z \wedge t_0 t_v \mathbf{D}_z^{(v)} f_z\|^2 &= \frac{(\sum_{j=0}^N a_{vj} \psi_j)(\overline{\sum_{j=0}^N a_{vj} \psi_j})}{\sum_{j=p+m}^N \psi_j \bar{\psi}_j} \\ &\geq \frac{(\sum_{j=0}^N a_{vj} \psi_j)(\overline{\sum_{j=0}^N a_{vj} \psi_j})}{\sum_{j=0}^N \psi_j \bar{\psi}_j} \\ &= \Psi^* \left(\frac{(\sum_{j=0}^N a_{vj} \psi_j)(\overline{\sum_{j=0}^N a_{vj} \psi_j})}{\sum_{j=0}^N w_j \bar{w}_j} \right) \\ &= \Psi^* \left\| \sum_{j=0}^N a_{vj} w_j \right\|^2. \end{aligned}$$

Here again we temporarily deviate from our convention and use $\|\cdot\|$ instead of $|\cdot|$ to denote norms with respect to the Hermitian metrics along the fibers of a line bundle in order to emphasize the difference between $\|\cdot\|$ and the sections $\psi_j \bar{\psi}_j$ of $(F \otimes F_0 \otimes K_M^{-1}) \otimes (\overline{F \otimes F_0 \otimes K_M^{-1}})$ and $w_j \bar{w}_j$ of $H \otimes \bar{H}$. The sum $\sum_{j=0}^N a_{vj} w_j$ is naturally regarded as a holomorphic section of the hyperplane section H of \mathbb{P}_N and the norm $\|\sum_{j=0}^N a_{vj} w_j\|$ is with respect to the Hermitian metric along the fibers of H . Now by the defect relation for hyperplanes in the complex projective space (0.4)

$$\begin{aligned} \frac{1}{4\pi} \sum'_v \mathcal{A}_r \left(\log \left\| \sum_{j=0}^N a_{vj} w_j \right\|^2 \right) &\geq \sum'_v (N(\Psi, r, S_v) - T(\Psi, r)) + O(1) \\ &\geq -(N + 1 + o(1))T(\Psi, r). \end{aligned}$$

The nondegeneracy conditions for the map Ψ required by the Cartan-Ahlfors-Weyl-Weyl theory is satisfied, because with respect to the line bundle $F \otimes F_0$ the map f is assumed to satisfy the nondegeneracy condition in the sense of (0.7). Moreover, the general position condition for $\{S_v\}$ required by the Cartan-Ahlfors-Weyl-Weyl theory is satisfied, because the collection $\{Z_{S_v}\}$ is assumed to be in general position in the sense of (0.8). Hence

$$\begin{aligned} \frac{1}{4\pi} \sum'_v \mathcal{A}_r(\log|f_z \wedge t_0 t_v \mathbf{D}_z^{(v)} f_z|^2) &\geq -(N + 1 + o(1))T(\Psi, r) \\ &\geq -(N + 1)\alpha T(r, f) + o(T(r, f)). \end{aligned}$$

From (2.3.1) we have the defect relation $\sum_v \delta_f(Z_{S_v}) \leq (N + 1)\alpha k = (p + m + v)\alpha k$ which is the defect stated in Theorem (1.2) for the case of complex dimension n equal to 2.

§6. Estimation of dimension of cohomology groups.

(6.1) In order to get Corollary (1.3), we now apply Theorem (1.2) to the case when M is the complex projective space \mathbb{P}_n with $n \geq 2$ and when $L = H^d$, where H is the hyperplane section line bundle of \mathbb{P}_n . To calculate the dimension m in Theorem (1.2), we need to get an upper estimate of $\dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell))$. For this we look at the Euler sequence

$$(6.1.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}_n} \xrightarrow{\sigma} (\mathcal{O}_{\mathbb{P}_n}(1))^{n+1} \xrightarrow{\tau} T_{\mathbb{P}_n} \rightarrow 0,$$

where

- (i) $\mathcal{O}_{\mathbb{P}_n}(1)^{n+1}$ denotes the set of $(n + 1)$ -tuples $g = (g_0(w), \dots, g_n(w))$ of linear functions of the homogeneous coordinates $w = [w^0, \dots, w^n]$ of \mathbb{P}_n ,
- (ii) the map τ maps the vector field $\sum_{j=0}^n g_j(w)(\partial/\partial w^j)$ on \mathbb{C}^{n+1} to the vector field $\tau(g)$ on \mathbb{P}_n under the natural quotient map $\mathbb{C}^{n+1} - 0 \rightarrow \mathbb{P}_n$,
- (iii) the map σ identifies $\mathcal{O}_{\mathbb{P}_n}$ with the subsheaf of $\mathcal{O}_{\mathbb{P}_n}(1)^{n+1}$ generated by the Euler vector field $\sum_{j=0}^n w^j(\partial/\partial w^j)$.

From (6.1.1) we have the exactness of

$$(6.1.2) \quad 0 \rightarrow \Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell)) \rightarrow \Gamma(\mathbb{P}_n, (\mathcal{O}_{\mathbb{P}_n}(\ell + 1))^{n+1}) \rightarrow \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}(\ell)) \rightarrow 0,$$

because $H^1(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell)) = 0$. We now use the sequence (6.1.2) to compute the dimension of $\Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}(\ell))$ over \mathbb{C} . The dimension of $\Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell))$ is $\binom{n + \ell}{\ell}$. So for $\ell \geq -1$, the dimension of $\Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}(\ell))$ over \mathbb{C} is

$$(n + 1) \cdot \binom{n + \ell + 1}{\ell + 1} - \binom{n + \ell}{\ell} = (n + \ell + 2) \binom{n + \ell}{n - 1}.$$

This can also be obtained directly from Bott's Formula for the dimension of $H^q(\mathbb{P}_n, \Omega_{\mathbb{P}_n}^p(\rho))$ which is

- (i) $\binom{\rho + n - p}{\rho} \binom{\rho - 1}{p}$ for $q = 0, 0 \leq p \leq n$, and $\rho > p$,
- (ii) 1 for $\rho = 0, 0 \leq p = q \leq n$,
- (iii) $\binom{-\rho + p}{-\rho} \binom{-\rho - 1}{n - p}$ for $q = n, 0 \leq p \leq n$, and $\rho < p - n$, and
- (iv) 0 otherwise.

By duality, Bott's formula for the case $q = n$ and $\rho = -n - 1 - \ell$ yields $\dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}(\ell)) = (n + \ell + 2) \binom{n + \ell}{n - 1}$.

(6.2) The dual of the sequence (6.1.1) yields the exact sequence

$$0 \rightarrow T_{\mathbb{P}_n}^*(\ell) \rightarrow (\mathcal{O}_{\mathbb{P}_n}(\ell - 1))^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}_n}(\ell) \rightarrow 0$$

which gives rise to the exact sequences

$$0 \rightarrow T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell) \xrightarrow{\xi} (T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell - 1))^{n+1} \rightarrow T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell) \rightarrow 0,$$

$$0 \rightarrow T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}^*(\ell) \xrightarrow{\eta} (T_{\mathbb{P}_n}^*(\ell - 1))^{n+1} \rightarrow T_{\mathbb{P}_n}^*(\ell) \rightarrow 0.$$

From the injectivity of ξ we get

$$\dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell)) \leq \dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, (T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell - 1))^{n+1})$$

and from the injectivity of η we get

$$\dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell)) \leq \dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, (T_{\mathbb{P}_n}(\ell - 1))^{n+1}).$$

From $\dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}(\ell)) = (\ell + n + 2) \binom{n + \ell}{n - 1}$ for $\ell \geq -1$, it follows that $\dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell)) \leq (n + 1)(n + \ell + 1) \binom{n + \ell - 1}{\ell}$ for $\ell \geq 0$ and $\dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell)) \leq (n + 1)^2(n + \ell) \binom{n + \ell - 2}{\ell - 1}$ for $\ell \geq 1$.

(6.3) Remark. The referee suggested using the vanishing theorem of Bott to precisely compute $\dim_{\mathbb{C}} \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell))$ instead of merely estimating it so that the defect relation can be correspondingly improved.

Let V be the holomorphic vector bundle $\Omega_{\mathbb{P}^n}^1(1)$. Then $\Gamma(\mathbb{P}^n, T_{\mathbb{P}^n}^* \otimes T_{\mathbb{P}^n}^* \otimes T_{\mathbb{P}^n}(\ell))$ is the same as $\Gamma(\mathbb{P}^n, V \otimes V \otimes V^*(\ell - 1))$. Since we have the pairing from $\Omega_{\mathbb{P}^n}^1(1) \times \bigwedge^{n-1}(\Omega_{\mathbb{P}^n}^1(1))$ to $\bigwedge^n(\Omega_{\mathbb{P}^n}^1(1)) \approx \Omega_{\mathbb{P}^n}^n(n) \approx \mathcal{O}_{\mathbb{P}^n}(-1)$ (due to $\Omega_{\mathbb{P}^n}^n \approx \mathcal{O}_{\mathbb{P}^n}(-n-1)$), it follows that $V^* \approx \bigwedge^{n-1}(V) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ and $\Gamma(\mathbb{P}^n, V \otimes V \otimes V^*(\ell + 1))$ is the same as $\Gamma(\mathbb{P}^n, V \otimes V \otimes \bigwedge^{n-1}(V)(\ell))$.

We now use the Littlewood-Richardson rule. For a partition $I = (i_1, \dots, i_r)$ (which means an r -tuple of nonnegative integers for some nonnegative integer r with $i_1 \geq i_2 \geq \dots \geq i_r$) we define the Young tableau to be the set of points $\{(v, i) | 1 \leq i \leq i_v, 1 \leq v \leq r\}$. Here we follow the convention in [M, p. 2] that, as with matrices, the first coordinate v (the row index) increases as one goes downwards, and the second coordinate i (the column index) increases as one goes from left to right. Let $|I| = \sum_{v=1}^r i_v$. For a vector space E of dimension n we define $S_I E$ to be the set of all multilinear functionals ℓ on the product of $|I|$ copies of the dual E^* of E such that when the summands E are indexed by (v, i) with $1 \leq i \leq i_v, 1 \leq v \leq r$ the following two conditions are satisfied:

- (i) for every fixed $1 \leq v \leq r$ the multilinear functional ℓ is symmetric in the i_v variables corresponding to $(v, 1), \dots, (v, i_v)$
- (ii) for every fixed $1 \leq i \leq i_r$ the multilinear functional ℓ is skew-symmetric in the variables (v, i) for $i \leq i_v$.

Another way of describing $S_I E$ is the set of all elements of $E^{\otimes |I|}$ which are symmetric in the variables on the same row of the Young tableau and skew-symmetric in the variables on the same column of the Young tableau.

The Littlewood-Richardson rule [Li, p. 94] tells us how to decompose $S_I E \otimes S_J E$ into direct sums of summands of the form $S_K E$. Let $J = (j_1, \dots, j_s)$. From the Young tableau of I the Young tableau of K is built by adding j_λ points represented by identical symbols α_λ ($1 \leq \lambda \leq s$) subject to the following two conditions:

- (i) after the addition of each set of identical symbols we must have a Young tableau with no two identical symbols in the same row and the Young tableau must be regular in the sense that the lengths of the rows of the Young tableau must be nonincreasing from the top down,
- (ii) if the total set of added symbols is read from right to left in the consecutive rows of the final tableau we have a permutation of $\alpha_1^{j_1} \alpha_2^{j_2} \dots \alpha_s^{j_s}$ which is a lattice permutation [Li, p. 67] in the sense that for $1 \leq t \leq |J|$ among the first t terms of the permutation the number of times α_μ occurs is no less than the number of times $\alpha_{\mu+1}$ occurs for $1 \leq \mu < s$.

The Littlewood-Richardson rule is given in [La, p. 209] as the first formula of Pieri.

We now apply the Littlewood-Richardson rule twice: the first time to $V \otimes \bigwedge^{n-1} V$ and the second time to the tensor product of V with the summands $S_K V$ of $V \otimes \bigwedge^{n-1} V$. Now $\bigwedge^{n-1} V$ is the same as $S_J V$ with $J = (1, \dots, 1)$ where there are $n - 1$ entries of 1 in J . We use the convention of [Li] and write $J = (1^{n-1})$. There are two summands $S_{K_\mu} V$ ($\mu = 1, 2$) in $V \otimes \bigwedge^{n-1} V$ with $K_1 = (2, 1^{n-2})$ and $K_2 = (1^n)$. We have three summands in $V \otimes S_{K_1} V$ corresponding to the partitions $(3, 1^{n-2}), (2, 2, 1^{n-3}), (2, 1^{n-1})$ and we have only one summand $S_p V$ in $V \otimes S_{K_2} V$ corresponding

to the partition $(2, 1^{n-1})$. The vector space $S_P V$ is isomorphic to $(\bigwedge^n V) \otimes S_Q V$ where Q is obtained by subtracting 1 from each of the n entries of the partition P (see e.g., [La, p. 208]). In other words, in our case $S_P V$ is isomorphic to $(\bigwedge^n V) \otimes V$, because $Q = (1)$. We know that $\bigwedge^n V \approx \bigwedge^n (\Omega_{\mathbb{P}^n}^1(1)) \approx \Omega_{\mathbb{P}^n}^n(n) \approx \mathcal{O}_{\mathbb{P}^n}(-1)$. Hence $S_P V$ is isomorphic to $V(-1)$. Thus $V \otimes V \otimes \bigwedge^{n-1} V$ is isomorphic to $V(-1) \oplus V(-1) \oplus (S_{(3, 1^{n-2})} V) \oplus (S_{(2, 2, 1^{n-1})} V)$ and $V \otimes V \otimes \bigwedge^{n-1} V(\ell)$ is isomorphic to $V(\ell - 1) \oplus V(\ell - 1) \oplus (S_{(3, 1^{n-2})} V)(\ell) \oplus (S_{(2, 2, 1^{n-1})} V)(\ell)$.

We now apply Bott's vanishing theorem to compute the dimension of the space of holomorphic sections of $V(\ell - 1) \oplus V(\ell - 1) \oplus (S_{(3, 1^{n-2})} V)(\ell) \oplus (S_{(2, 2, 1^{n-1})} V)(\ell)$ over \mathbb{P}^n . We take the dual of the Euler sequence and get the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n}^1(1) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0.$$

Let $W = \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \approx \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)})$. We apply Bott's theorem in the form presented in [La, p. 231, Th. 5.1(i)] with the following values: $J = (\ell - 1)$ or (ℓ) , $n(I, J) = 0$, $q = 1$, and $Q = \mathcal{O}_{\mathbb{P}^n}(1)$. We get

$$\Gamma(\mathbb{P}^n, V(\ell - 1)) \approx S_{(\ell-1, 1)} W,$$

$$\Gamma(\mathbb{P}^n, (S_{(3, 1^{n-2})} V)(\ell)) \approx S_{(\ell, 3, 1^{n-2})} W,$$

$$\Gamma(\mathbb{P}^n, (S_{(2, 2, 1^{n-3})} V)(\ell)) \approx S_{(\ell, 2, 2, 1^{n-3})} W.$$

Note that in [La] the entries in a partition are nondecreasing rather than non-increasing and we use here the convention of [Li] where the entries in a partition are nonincreasing. Finally we conclude that $\Gamma(\mathbb{P}^n, T_{\mathbb{P}^n}^* \otimes T_{\mathbb{P}^n}^* \otimes T_{\mathbb{P}^n}(\ell))$ is isomorphic to

$$(S_{(\ell-1, 1)} W) \oplus (S_{(\ell-1, 1)} W) \oplus (S_{(\ell, 3, 1^{n-2})} W) \oplus (S_{(\ell, 2, 2, 1^{n-3})} W).$$

The computation of $\dim_{\mathbb{C}} \Gamma(\mathbb{P}^n, T_{\mathbb{P}^n}^* \otimes T_{\mathbb{P}^n}^* \otimes T_{\mathbb{P}^n}(\ell))$ is reduced to the computation of the dimensions of $S_{(\ell-1, 1)} W$, $S_{(\ell, 3, 1^{n-2})} W$, and $S_{(\ell, 2, 2, 1^{n-3})} W$.

The dimension of $S_I E$ is computed in the following way. The general linear group $GL(n, \mathbb{C})$ acts on E and as a result acts diagonally on the product of $|I|$ copies of E^* . So it acts on $S_I E$. This representation is precisely the irreducible representation of $GL(n, \mathbb{C})$ corresponding to the partition $I = (i_1, \dots, i_r)$. For $r < v \leq n$ define $i_v = 0$. By the result of Schur the value of the character of this representation on the diagonal matrix with diagonal elements x_ν ($1 \leq \nu \leq n$) is precisely the Schur function [M, p. 24]

$$s_I(x_1, \dots, x_n) = \frac{\det(x_\mu^{i_\nu + n - \nu})_{1 \leq \mu, \nu \leq n}}{\det(x_\mu^\nu)_{1 \leq \mu, \nu \leq n}}.$$

A reference for this relation between the Schur function and the character of the representation is [W, p. 203, Th. (7.6B)] and the argument following [W, p. 202, Lemma (7.6A)] shows that the expression in [W, p. 203, Th. (7.6B)] is identical to the expression in [W, p. 201, Th. (7.5B)] which is precisely the Schur function. The dimension of $S_r E$ is simply equal to the value of the character of the representation at the identity matrix of $GL(n, \mathbb{C})$ which is the same as the value of the Schur function $s_I(x_1, \dots, x_n)$ at $x_1 = \dots = x_n = 1$. That value is computed in [M, p. 28, Example 4] as follows. The reference [M, p. 28, Example 4] for the computation of that value was given to me by P. Diaconis. For (v, i) with $1 \leq v \leq i_v$ by the *hook length* $h(v, i)$ of (v, i) we mean $i_v - v - i + 1 + \max\{\mu | i \leq i_\mu\}$ [M, p. 9, Example 1]. By the *content* $c(v, i)$ of (v, i) we mean $i - v$. The value of $s_I(x_1, \dots, x_n)$ at $x_1 = \dots = x_n = 1$ is equal to

$$\prod_{1 \leq v \leq r, 1 \leq i \leq i_v} \frac{n + c(v, i)}{h(v, i)}.$$

This is the general formula combining the following two special cases. One is the number of ways of selecting r objects from n objects allowing repetition, and it corresponds to the dimension of the space of symmetric elements in the tensor product of r copies of a vector space of dimension n . The result is the product of $n, n + 1, \dots, n + r - 1$ divided by the product of $r, r - 1, \dots, 1$. The other one is the number of ways of selecting r objects from n objects disallowing repetition, and it corresponds to the dimension of the space of skew-symmetric elements in the tensor product of r copies of a vector space of dimension n . The result is the product of $n, n - 1, \dots, n - r + 1$ divided by the product of $r, r - 1, \dots, 1$. So for symmetric elements we increase the factor of the numerator by 1 every time we add a factor, but for skew-symmetric elements we decrease the factor of the numerator by 1 every time we add a factor. That is the reason why the content $c(v, i)$ appears in the numerator. The hook length $h(v, i)$ is obtained by counting the number of elements in the Young tableau to the left and below the given element (v, i) and it corresponds to one of the factors $r, r - 1, \dots, 1$ in the denominator in the purely symmetric or purely skew-symmetric cases.

Since the dimension of W is $n + 1$, the above formula applied to our case gives

$$\dim S_{(\ell-1, 1)} W = \frac{(n + \ell - 1)!}{(n - 1)!(\ell - 2)!\ell}.$$

$$\dim S_{(\ell, 3, 1^{n-2})} W = \frac{(n + \ell)!(n + 1)(n + 2)}{(\ell + n - 1)\ell(\ell - 1)[(\ell - 3)!]2(n + 1)[(n - 2)!]}.$$

$$\dim S_{(\ell, 2, 2, 1^{n-3})} W = \frac{(n + \ell)!n(n + 1)}{(\ell + n - 1)(\ell + 1)[(\ell - 2)!]2n(n - 1)[(n - 3)!]}.$$

where I_{n-1} is the identity matrix of order $n - 1$. The matrix $(\partial\eta^\lambda/\partial\xi^\alpha)$ with $0 \leq \lambda \leq n - 1$ as the row index and $1 \leq \alpha \leq n$ as the column index is

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -\frac{1}{(\xi^n)^2} \\ & & & & \frac{\xi^1}{(\xi^n)^2} \\ & & & & \vdots \\ \frac{1}{\xi^n} I_{n-1} & & & & \frac{\xi^{n-1}}{(\xi^n)^2} \end{bmatrix}.$$

For fixed $1 \leq \beta \leq n - 1$ all entries of the matrix $(\partial^2\eta^\lambda/\partial\xi^\beta\partial\xi^\alpha)$ with $0 \leq \lambda \leq n - 1$ as the row index and $1 \leq \alpha \leq n$ as the column index are zero except the α^{th} entry in the last column which is $-1/(\xi^n)^2$. The matrix $(\partial^2\eta^\lambda/\partial\xi^n\partial\xi^\alpha)$ with $0 \leq \lambda \leq n - 1$ as the row index and $1 \leq \alpha \leq n$ as the column index is

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -\frac{2}{(\xi^n)^3} \\ & & & & \frac{2\xi^1}{(\xi^n)^3} \\ & & & & \vdots \\ \frac{-1}{(\xi^n)} I_{n-1} & & & & \frac{2\xi^{n-1}}{(\xi^n)^3} \end{bmatrix}.$$

For fixed $1 \leq \beta \leq n - 1$ the product matrix $\sum_\lambda (\partial\xi^\alpha/\partial\eta^\lambda)(\partial^2\eta^\lambda/\partial\xi^\beta\partial\xi^\gamma)$ with row index α and column index γ has zero entries except the entry on the β^{th} row and the last column which is $-1/\xi^n$. For fixed $\beta = n$ the product matrix $\sum_\lambda (\partial\xi^\alpha/\partial\eta^\lambda)(\partial^2\eta^\lambda/\partial\xi^\beta\partial\xi^\gamma)$ is

$$\begin{bmatrix} & 0 \\ & 0 \\ & \vdots \\ \frac{-1}{\xi^n} I_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & -\frac{2}{\xi^n} \end{bmatrix}.$$

For the transition from the coordinates η^δ on $\{w^n \neq 0\}$ to the coordinates ξ^α on $\{w^0 \neq 0\}$ the term $\sum_\lambda (\partial^2 \eta^\lambda / \partial \xi^\beta \partial \xi^\gamma) (\partial \xi^\alpha / \partial \eta^\lambda)$ adds at most 1 to the pole order in ξ^n . Now we look at the contribution to the pole order from $(\partial \xi^\alpha / \partial \eta^\lambda) (\partial \eta^\mu / \partial \xi^\beta) (\partial \eta^\nu / \partial \xi^\gamma)$. Each element of the matrix $(\partial \xi^\alpha / \partial \eta^\lambda)$ has a factor ξ^n , while each element of the matrices $(\partial \eta^\mu / \partial \xi^\beta)$ and $(\partial \eta^\nu / \partial \xi^\gamma)$ has at most a pole factor $1/(\xi^n)^2$. Hence the factor $(\partial \xi^\alpha / \partial \eta^\lambda) (\partial \eta^\mu / \partial \xi^\beta) (\partial \eta^\nu / \partial \xi^\gamma)$ contributes at most 3 to the pole order in $1/\xi^n$.

We now choose the connection $\Theta_{\mu\nu}^\lambda$ which is zero with respect to the inhomogeneous coordinate system η^λ . Then the pole order of this connection for other inhomogeneous coordinate systems ξ^α comes only from the term $\sum_\lambda (\partial^2 \eta^\lambda / \partial \xi^\beta \partial \xi^\gamma) (\partial \xi^\alpha / \partial \eta^\lambda)$. So we know that the pole order of this connection on all of \mathbb{P}_n is 1 and F_0 can be chosen to be H .

(7.2) We would like to remark that this discussion about the effect on the pole of a meromorphic connection with respect to affine coordinates by the change of inhomogeneous coordinates in \mathbb{P}_n yields right away the verification that the meromorphic connection, with respect to which a Fermat hypersurface has zero second fundamental form, has pole order 3. The reason is that that particular connection $\Theta_{\mu\nu}^\lambda$ with respect to a suitable inhomogeneous coordinate system η^λ is nonzero only when $\lambda = \mu = \nu$ and $\Theta_{\lambda\lambda}^\lambda$ equals a constant times $1/\eta^\lambda$ which is equal to ξ^n (when $\lambda = 0$) and ξ^n/ξ^λ (when $1 \leq \lambda \leq n - 1$). With the factor ξ^n from the matrix $(\partial \xi^\alpha / \partial \eta^\lambda)$ and the pole $1/(\xi^n)^2$ from the matrix $(\partial \eta^\mu / \partial \xi^\beta)$ we conclude that the pole order of the connection is equal to 3.

(7.3) We now return to the proof of Corollary (1.3). We have

$$p = \dim \Gamma(\mathbb{P}_n, F) = \dim \Gamma(\mathbb{P}_n, H^\ell) = \binom{\ell + n}{n}$$

and

$$\begin{aligned} m &= \dim \Gamma(\mathbb{P}_n, T_M^* \otimes T_M^* \otimes T_M \otimes F \otimes F_0) \\ &= \dim \Gamma(\mathbb{P}_n, T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}^* \otimes T_{\mathbb{P}_n}(\ell + 1)) \\ &\leq (n + 1)^2 (n + \ell + 1) \binom{n + \ell - 1}{n - 1}. \end{aligned}$$

Any line bundle $E = H^q$ with $q > 0$ can be generated over \mathbb{P}_n always by $n + 1$ elements of $\Gamma(\mathbb{P}_n, H^q)$, because we can always choose them to be the q^{th} power of a basis of $\Gamma(\mathbb{P}_n, H)$. Hence v is always $n + 1$. The number α is equal to $((\ell + 1)^{n(n+1)/2} + n + 1)/d$. From these values of p, m, a , and α we obtain Corollary (1.3).

Theorem (1.1) for Fermat curves in \mathbb{P}_2 is the special case where $\ell = 3$. Then $b = 1 + \binom{2}{2} + 3^2 \cdot 6 \cdot 4 - 1 = 226$ and $c = 1$. The number $((\ell + 1)^{n(n+1)/2} + n + 1)/d$ is $7/d$. Hence $\binom{2}{2} + n + 1$ times $((\ell + 1)^{n(n+1)/2} + n + 1)/d$ is $(226 + 3) \cdot (7/d)k$.

§8. More general examples.

(8.1) We can apply our result in a more general setting. However, we have to assume that the divisor is defined by a homogeneous polynomial of degree d satisfying the following condition: every term of the polynomial contains the $(d - e)^{\text{th}}$ power of some variable. The defect estimate then depends on e . This is more general than the Fermat surfaces.

Let us see how this condition comes in. Suppose a divisor S is defined by a homogeneous polynomial $F(w_0, \dots, w_n)$ of degree d . Let e be a positive integer $< d$. We assume that $F(w_0, \dots, w_n)$ is of the form $\sum_{v=0}^n G_v(w_0, \dots, w_n)w_v^{d-e}$, where G_v is a polynomial. We want to produce a meromorphic connection so that S has zero second fundamental form with respect to it and the pole order of the meromorphic connection depends only on e and n and is independent of d . Consider the affine part of \mathbb{P}_n given by $w_0 \neq 0$ and let $z_v = w_v/w_0$ ($1 \leq v \leq n$) be the inhomogeneous coordinates. Let $P(z_1, \dots, z_n) = (1/w_0^d)F(w_0, \dots, w_n)$. Write

$$P(z_1, \dots, z_n) = Q_0(z_1, \dots, z_n) + \sum_{v=1}^n Q_v(z_1, \dots, z_n)z_v^{d-e},$$

where $Q_v(z_1, \dots, z_n)$, $0 \leq v \leq n$, is a polynomial of degree $\leq e$ in z_1, \dots, z_n . Let $P_{z_\mu z_\nu}$ and P_{z_μ} denote respectively the second partial derivatives of P with respect to z_μ and z_ν and the first partial derivative of P with respect to z_μ . We have

$$P_{z_\mu} = Q_{\mu 0}(z_1, \dots, z_n) + \sum_{v=1}^n Q_{\mu v}(z_1, \dots, z_n)z_v^{d-e-1}$$

$$P_{z_\mu z_\nu} = Q_{\mu \nu 0}(z_1, \dots, z_n) + \sum_{\lambda=1}^n Q_{\mu \nu \lambda}(z_1, \dots, z_n)z_\lambda^{d-e-2},$$

where $Q_{\mu \nu}$ ($\nu > 0$) and $Q_{\mu \nu \lambda}$ ($\lambda > 0$) are polynomials of degree $\leq e$ and $Q_{\mu 0}$ is a polynomial of degree $\leq e - 1$ and $Q_{\mu \nu 0}$ is a polynomial of degree $\leq e - 2$ in z_1, \dots, z_n . We want to write $P_{z_\mu z_\nu}$ as a linear combination of P_{z_1}, \dots, P_{z_n} and P whose coefficients are rational functions in z_1, \dots, z_n with small degree for both the numerator and the denominator. Write

$$P_{z_\mu z_\nu} = Q_{\mu \nu 0}(z_1, \dots, z_n) + \sum_{\lambda=1}^n \frac{Q_{\mu \nu \lambda}(z_1, \dots, z_n)}{z_\lambda} z_\lambda^{d-e-1}.$$

We use Cramer's rule to solve for $1, z_1^{d-e-1}, \dots, z_n^{d-e-1}$ in the system of equations

$$\begin{cases} P = Q_0 + \sum_{v=1}^n (Q_{0v} z_v^{d-e-1}) \\ P_{z_1} = Q_{10} + \sum_{v=1}^n Q_{1v} z_v^{d-e-1} \\ \vdots \\ P_{z_n} = Q_{n0} + \sum_{v=1}^n Q_{nv} z_v^{d-e-1} \end{cases}$$

for $1, z_1^{d-e-1}, \dots, z_n^{d-e-1}$. Let $R(z_1, \dots, z_n)$ be the determinant of $(Q_{\mu\nu})_{0 \leq \mu, \nu \leq n}$, where $Q_{0v} = Q_v$ for $0 \leq v \leq n$. As a polynomial in z_1, \dots, z_n the degree of R is $\leq (n + 1)e$. We get

$$\begin{cases} 1 = \frac{R_{00}}{R} P + \sum_{v=1}^n \frac{R_{0v}}{R} P_{z_v} \\ z_1^{d-e-1} = \frac{R_{10}}{R} P + \sum_{v=1}^n \frac{R_{1v}}{R} P_{z_v} \\ \vdots \\ z_n^{d-e-1} = \frac{R_{n0}}{R} P + \sum_{v=1}^n \frac{R_{nv}}{R} P_{z_v} \end{cases}$$

where $R_{\mu\nu} = R_{\mu\nu}(z_1, \dots, z_n)$ is a polynomial of degree $\leq ne$ in z_1, \dots, z_n , and $R_{\mu\nu}$ ($\mu \neq 0$ or $\nu = 0$) is a polynomial of degree $\leq ne$, and R_{0v} ($v \neq 0$) is a polynomial of degree $\leq ne - 1$. Hence

$$P_{z_\mu z_\nu} = \left(Q_{\mu\nu 0} \frac{R_{00}}{R} + \sum_{\lambda=1}^n \frac{Q_{\mu\nu\lambda}}{z_\lambda} \frac{R_{\lambda 0}}{R} \right) P + \sum_{\ell=1}^n \left(Q_{\mu\nu 0} \frac{R_{0\ell}}{R} + \sum_{\lambda=1}^n \frac{Q_{\mu\nu\lambda}}{z_\lambda} \frac{R_{\lambda\ell}}{R} \right) P_{z_\ell}.$$

So we can use $\Gamma_{\mu\nu}^\ell = Q_{\mu\nu 0}(R_{0\ell}/R) + \sum_{\lambda=1}^n (Q_{\mu\nu\lambda}/z_\lambda)(R_{\lambda\ell}/R)$ as the meromorphic connection. The numerator and denominator of $\Gamma_{\mu\nu}^\ell$ are polynomials of degree $\leq (n + 1)e + 1$. The pole order of the connection $\Gamma_{\mu\nu}^\ell$ on the affine part \mathbb{C}^n of \mathbb{P}_n is at most $(n + 1)e + 1$ in the inhomogeneous coordinates. So the pole order of the connection $\Gamma_{\mu\nu}^\ell$ on all of \mathbb{P}_n is at most $(n + 1)e + 4$.

(8.2) To get other examples of hypersurfaces in \mathbb{P}_n with good defect relations, one can start with a given pole order ℓ first and work backwards to consider the set of all hypersurfaces S_ν of degree d whose second fundamental form is zero with respect to some meromorphic connection of pole order ℓ . To get good defect relations we want a high value for d and a low value for ℓ . It is trivial to get the case with ℓ of the same order as d , but in general it is not easy to get examples with ℓ small and d large. While the problem for nonequidimensional value distribution

theory is to get defect relations for divisors of degree d , the method of meromorphic connections introduced here gets defect relations only for divisors of degree d whose second fundamental form is zero with respect to meromorphic connections of pole order ℓ . For this reason the class of divisors considered by the methods of meromorphic connections is more restrictively formulated than that considered by the general problem for nonequidimensional value distribution theory.

(8.3) Intuitively, the use of meromorphic connections with respect to which the second fundamental form of a divisor vanishes is, in a way, related to the use of the Veronese map. Both methods try to “straighten out” the divisor. An advantage of the method of meromorphic connections is that, to get good defect relations in special cases like the Fermat curves, the method of using the Veronese map forces one to fix some coordinate system (see §9) whereas connections are invariantly defined and are independent of the coordinate system. That is the reason why the method of meromorphic connections enables one to handle the case of Fermat curves in different coordinate systems.

(8.4) Nadel [Na] applied the method of meromorphic connections to construct new hyperbolic surfaces in \mathbb{P}_3 . For this problem there is no divisor and one does not have to worry about the vanishing of any second fundamental form with respect to the meromorphic connections. What is needed is to show that, for suitable meromorphic connections, the failure of the nondegeneracy condition means that the image of the holomorphic map from \mathbb{C} to the surface must lie on a complex curve, and then one shows that such a complex curve does not exist.

§9. Shiffman’s defect relation for Fermat curves in the same coordinates.

(9.1) We present Shiffman’s derivation [Sh2] of the defect relation for Fermat curves in the same coordinate system by using Cartan’s theorem [C2].

Let $C(a)$ be the Fermat curve given by $\sum_{i=0}^2 a_i w_i^d = 0$, where $a = (a_0, a_1, a_2)$ and $[w_0, w_1, w_2]$ are the homogeneous coordinates of \mathbb{P}_2 . Let $a^{(1)}, \dots, a^{(q)}$ be q triples of complex numbers and let $f: \mathbb{C} \rightarrow \mathbb{P}_2$ be a holomorphic map such that $f(\mathbb{C})$ is not contained in $C(a)$ for any $a \in \mathbb{C}^3$. Let us suppose first that

- (a) no point is contained in more than two of the curves $C(a^{(j)})$ and
- (b) no point in any of the coordinate hyperplanes $H_i = \{w_i = 0\}$, $i = 0, 1, 2$, is contained in more than one of the curves $C(a^{(j)})$.

The defect relation is $\sum_j \delta_f(C(a^{(j)})) \leq 6/d$. The proof is as follows.

Let $\xi: \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be given by $\xi(w_0, w_1, w_2) = (w_0^d, w_1^d, w_2^d)$. Let $g = \xi f$ and let $S_j = \{z \in \mathbb{P}_2 \mid \sum_{i=0}^2 a_i^{(j)} z_i = 0\}$. Then we have the relation $\delta_f(C(a^{(j)})) = \delta_g(S_j)$ between the defects of the maps f and g . Let $H_j = \{z \in \mathbb{P}_2 \mid z_j = 0\}$. By applying Cartan’s theorem to the $q + 3$ divisors $S_1, \dots, S_q, H_0, H_1, H_2$, we get

$$(q + 3 - 3)T(g, r) \leq \sum_{j=1}^q N_2(g, r, S_j) + \sum_{j=0}^2 N_2(g, r, H_j) + o(T(g, r))$$

(see [C2, p. 7, (3)]), where $N_p(g, r, S)$ is the counting function for the map g and the divisor S computed by counting a zero of multiplicity m as a zero of multiplicity $\max(m, p)$. For the formula to hold, the p in N_p must be equal to the dimension of the target space \mathbb{P}_2 , which is 2. The reason for adding three divisors H_0, H_1, H_2 is to get the factor $q + 3 - 3$ on the left-hand side instead of just $q - 3$. These three additional divisors are chosen to be H_1, H_2, H_3 because the multiplicity of g^*H_j is d at every point of its support. Since

$$N_2(g, H_i, r) \leq \frac{2}{d} N(g, H_i, r) \leq \frac{2}{d} T(g, r),$$

the above defect relation follows. We now consider the general case. Suppose that

- (a) no point is contained in more than $2k$ of the curves $C(a^{(j)})$ and
- (b) no point in the H_i is contained in more than k of the curves $C(a^{(j)})$.

The defect relation in this case becomes $\sum_j \delta_f(C(a^{(j)})) \leq 6k/d$. This can be generalized to maps into \mathbb{P}_n , $n \geq 2$, and the generalization follows from the following result.

(9.2) THEOREM. *Let F be a family of hyperplanes in \mathbb{P}_2 and let $a: F \rightarrow (0, 1]$ be a map such that for all projective linear subspaces $Q \subset \mathbb{P}_n$ of codimension r , the sum of $a(H)$ for $H \in F$ and $Q \subset H$ does not exceed r for $2 \leq r \leq n$. Let $f: \mathbb{C} \rightarrow \mathbb{P}_n$ be a holomorphic map such that $f(\mathbb{C})$ is not contained in a hyperplane. Then*

$$\sum_{H \in F} a(H)m(f, r, H) \leq (n + 1)T(f, r) - N(W(f), r, 0) + o(\log(T(f, r))),$$

where $m(f, r, H)$ is the proximity function for the map f and the divisor H and $W(f)$ is the Wronskian of f .

Theorem (9.2) is an unpublished result of Shiffman. Stoll also has a similar result. It also follows from the work of H. & J. Weyl [W-W]. The function a is a “weight function”. To apply this theorem to get the above defect relation in the general case, one lets $F = \{S_1, \dots, S_q, H_0, H_1, H_2\}$. Let $a(D_j) = 1/k$, $a(H_i) = 1$, and the defect relation for the general case follows as in the special case.

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