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A compact Kähler surface of negative curvature not covered by the ball[1]

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The uniformization theorem for Riemann surfaces says that a simply connected Riemann surface must be the Riemann sphere, the whole complex plane, or the open unit disc. In the higher dimensional case there is no such simple trichotomy, because generic slight perturbations of the ball give rise to complex manifolds no two of which are biholomorphic [1]. However, it is conjectured that with reasonable curvature assumptions a similar trichotomy exists in the higher dimensional case. Corresponding to the case of the Riemann sphere, one has the Frankel conjecture that a compact Kähler manifold of positive sectional curvature must be biholomorphic to the complex projective space. This conjecture was proved in the dimension 2 case by Andreotti-Frankel [2] and in the dimension 3 case by Mabuchi [5]. Very recently the general case was proved independently by Mori [6] using algebraic geometry of positive characteristic and by Siu-Yau [11] using the complex-analyticity of harmonic maps. (Mori's result is stronger than the result of Siu-Yau. Mori's result assumes only that the manifold has ample tangent bundle, whereas the result of Siu-Yau assumes that the manifold has positive holomorphic bisectional curvature.)

Corresponding to the case of the complex plane, one has the conjecture that a noncompact complete Kähler manifold of positive sectional curvature must be biholomorphic to some $\mathbf{C}^n$. Or, more generally, a noncompact simply connected complete Kähler manifold with sectional curvature $K \geqq -A/r^{2+\varepsilon}$ $\Big($or even with the weaker assumption $K \geqq -k(r)$ with $k(r) \geqq 0$ and $\int_0^\infty rk(r)dr < \infty\Big)$ must be obtained from some $\mathbf{C}^n$ by some proper modifications, where A, $\varepsilon > 0$ and r is the distance from some fixed point. Siu-Yau [10] proved that a noncompact simply connected complete Kähler manifold with $0 \geqq K \geqq -A/r^{2+\varepsilon}$ must be biholomorphic to some $\mathbf{C}^n$. Greene-Wu [3] generalized their result to the case $0 \geqq K \geqq -k(r)$ with $k(r) \geqq 0$ and

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$\int_0^\infty rk(r)dr < \infty$.

Corresponding to the case of the ball, one has the conjecture that a simply connected complete Kähler manifold of negative sectional curvature must be biholomorphic to a bounded domain in $\mathbf{C}^n$. A weaker version of this conjecture says that on such a manifold there should be enough bounded holomorphic functions to separate points and give local coordinates. Up to now no one has yet succeeded in producing a single nonconstant bounded holomorphic function on such a manifold, even if one assumes that the manifold is the universal covering of a compact Kähler manifold of negative sectional curvature.

There is a second conjecture which says that the universal covering of a compact Kähler manifold of negative sectional curvature should be biholomorphic to the ball. This second conjecture is encouraged by the following results of Wong [12] and Yang [13]. Wong's result is that a bounded domain in $\mathbf{C}^n$ with smooth strongly pseudoconvex boundary whose automorphism group is noncompact must be the ball. His method of proof can also yield the result that a bounded domain in $\mathbf{C}^n$ with smooth boundary which covers a compact manifold must be the ball. Yang's result is that a bounded symmetric domain of rank >1 cannot admit a complete Kähler metric with its holomorphic bisectional curvature bounded between two negative constants. In particular, a bounded symmetric domain of rank >1 cannot cover a compact Kähler manifold with negative bisectional curvature. In this paper we give a counterexample to the second conjecture by constructing a compact Kähler surface of negative sectional curvature whose universal covering is not biholomorphic to the open 2-ball. The surface is constructed by using the result of Mostow [7], [8] on groups generated by complex reflections. The Kähler metric on the surface is constructed by delicately piecing together the Poincaré metric of the 2-ball and the Bergman metric of the domain $|z_1|^{14} + |z_2|^2 < 1$ in $\mathbf{C}^2$. The universal covering is shown to be non-biholomorphic to the ball by computing c_1^2/c_2 and verifying that it is not equal to 3. The ratio c_1^2/c_2 is 852/298 which is >2. Hence its index is positive. Besides the compact quotients of the complex 2-ball, the only known compact complex surface of positive index are the Kodaira surfaces [4]. Since the Kodaira surfaces admit nontrivial holomorphic deformation and the surface constructed in this paper is rigid, our surface is a new algebraic surface. Our surface is also the first known example of a negatively curved compact Riemannian fourfold which is not diffeomorphic to a locally symmetric manifold, because the Pontrjagin number p_1 of our surface is nonzero and the Pontrjagin number of any compact quotient of the real 4-ball, being proportional to that of the real projective 4-space which is covered by the 4-sphere cobordant to zero, is zero.

The method used to construct this surface actually can be used to construct an infinite discrete family of similar surfaces with negative sectional curvature not covered by the ball. This family is discussed in the last section of this paper. The maximum of c_1^2/c_2 for surfaces in this family is 2.9525 for the surface whose parameters are $p = 5$ and $\sigma = 6$ (for the meanings of p and σ see the last section of this paper).

All these surfaces have very strongly negative curvature tensor in the sense of [9]. Hence for them strong rigidity holds.

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1. Sectional curvature of Kähler metric

Suppose M is a complex manifold of dimension n with local holomorphic coordinate system $z^1, \cdots, z^n$. Let $2\operatorname{Re}\sum g_{\alpha\bar\beta}dz^\alpha dz^{\bar\beta}$ be a Kähler metric on M. The curvature tensor is given by

$$R_{\alpha\bar\beta\gamma\bar\delta} = \partial_{\bar\beta}\partial_\gamma g_{\alpha\bar\delta} - \sum_{\mu,\lambda} g^{\lambda\bar\mu}\partial_\alpha g_{\gamma\bar\mu}\partial_{\bar\beta}g_{\lambda\bar\delta} .$$

Let p, q be two real vectors of the complexified tangent space at a point of M. We want to calculate the sectional curvature at the plane spanned by p and q. We can write

$$p = 2\operatorname{Re}\xi ,$$
$$q = 2\operatorname{Re}\eta$$

where

$$\xi = \sum_\alpha \xi^\alpha \frac{\partial}{\partial z^\alpha} ,$$

$$\eta = \sum_\alpha \eta^\alpha \frac{\partial}{\partial z^\alpha} .$$

We use italic lower case letters h, i, j, k, etc. to run through the range $1, \cdots, n, \bar{1}, \cdots, \bar{n}$ and keep the range of the Greek lower case letters α, β, γ, δ, etc. to be $1, \cdots, n$. Recall

$$\begin{aligned} p^{\alpha} &= \xi^{\alpha} , \\ p^{\bar{\alpha}} &= \overline{\xi^{\alpha}} , \\ q^{\alpha} &= \eta^{\alpha} , \\ q^{\bar{\alpha}} &= \overline{\eta^{\alpha}} , \end{aligned}$$

and recall the following properties of the curvature tensor. The only nonzero components of R_{hijk} are of the form

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} , \quad R_{\alpha\bar{\beta}\bar{\delta}\gamma} , \quad R_{\bar{\beta}\alpha\gamma\bar{\delta}} , \quad R_{\bar{\beta}\alpha\bar{\delta}\gamma}$$

and they satisfy the symmetry relations

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= -R_{\alpha\bar{\beta}\bar{\delta}\gamma} = -R_{\bar{\beta}\alpha\gamma\bar{\delta}} = R_{\bar{\beta}\alpha\bar{\delta}\gamma} , \\ R_{\gamma\bar{\beta}\alpha\bar{\delta}} &= R_{\alpha\bar{\delta}\gamma\bar{\beta}} = R_{\alpha\bar{\beta}\gamma\bar{\delta}} , \\ R_{\beta\bar{\alpha}\delta\bar{\gamma}} &= \overline{R_{\alpha\bar{\beta}\gamma\bar{\delta}}} . \end{aligned}$$

The sectional curvature at the plane spanned by p and q is

$$\frac{\sum_{h,i,j,k} R_{hijk} p^h q^i p^j q^k}{\|p \wedge q\|^2} .$$

Let $\xi^{\bar{\alpha}} = \overline{\xi^{\alpha}}$ and $\eta^{\bar{\alpha}} = \overline{\eta^{\alpha}}$. Then

$$\begin{aligned} &\sum_{h,i,j,k} R_{hijk} p^h q^i p^j q^k \\ &\quad = \sum_{\alpha,\beta,\gamma,\delta} (R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha}\eta^{\bar{\beta}}\xi^{\gamma}\eta^{\bar{\delta}} + R_{\alpha\bar{\beta}\bar{\gamma}\delta} \xi^{\alpha}\eta^{\bar{\beta}}\xi^{\bar{\gamma}}\eta^{\delta} + R_{\bar{\alpha}\beta\gamma\bar{\delta}} \xi^{\bar{\alpha}}\eta^{\beta}\xi^{\gamma}\eta^{\bar{\delta}} + R_{\bar{\alpha}\beta\bar{\gamma}\delta} \xi^{\bar{\alpha}}\eta^{\beta}\xi^{\bar{\gamma}}\eta^{\delta}) \\ &\quad = \sum_{\alpha,\beta,\gamma,\delta} (R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha}\eta^{\bar{\beta}}\xi^{\gamma}\eta^{\bar{\delta}} - R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha}\eta^{\bar{\beta}}\eta^{\gamma}\xi^{\bar{\delta}} - R_{\alpha\bar{\beta}\gamma\bar{\delta}} \eta^{\alpha}\xi^{\bar{\beta}}\xi^{\gamma}\eta^{\bar{\delta}} + R_{\alpha\bar{\beta}\gamma\bar{\delta}} \eta^{\alpha}\xi^{\bar{\beta}}\eta^{\gamma}\xi^{\bar{\delta}}) \\ &\quad = \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (\xi^{\alpha}\eta^{\bar{\beta}} - \eta^{\alpha}\xi^{\bar{\beta}})(\xi^{\gamma}\eta^{\bar{\delta}} - \eta^{\gamma}\xi^{\bar{\delta}}) \\ &\quad = -\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (\xi^{\alpha}\eta^{\bar{\beta}} - \eta^{\alpha}\xi^{\bar{\beta}})\overline{(\xi^{\delta}\eta^{\bar{\gamma}} - \eta^{\delta}\xi^{\bar{\gamma}})} . \end{aligned}$$

Hence negative sectional curvature means

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (\xi^{\alpha}\eta^{\bar{\beta}} - \eta^{\alpha}\xi^{\bar{\beta}})\overline{(\xi^{\delta}\eta^{\bar{\gamma}} - \eta^{\gamma}\xi^{\bar{\delta}})} > 0$$

for all ξ, η with $(\mathrm{Re}\,\xi) \wedge (\mathrm{Re}\,\eta) \neq 0$.

In the case $n = 2$, we have

$$\begin{aligned} &\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (\xi^{\alpha}\eta^{\bar{\beta}} - \eta^{\alpha}\xi^{\bar{\beta}})\overline{(\xi^{\delta}\eta^{\bar{\gamma}} - \eta^{\delta}\xi^{\bar{\gamma}})} \\ &\quad = R_{1\bar{1}1\bar{1}} |\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}}|^2 \\ &\qquad + 4\,\mathrm{Re}\big(R_{1\bar{1}1\bar{2}} (\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})}\big) \\ &\qquad + 2\,R_{1\bar{1}2\bar{2}} \big(|\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}}|^2 + \mathrm{Re}\,(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})}\big) \\ &\qquad + 2\,\mathrm{Re}\big(R_{1\bar{2}1\bar{2}} (\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})}\big) \\ &\qquad + 4\,\mathrm{Re}\big(R_{2\bar{2}1\bar{2}} (\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})}\big) \\ &\qquad + R_{2\bar{2}2\bar{2}} |\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}}|^2 . \end{aligned}$$

We now compute $||p \wedge q||^2$. We will later need this only for the case $n = 2$ and $g_{1\bar{2}} = 0$. So we do our computation only in this case. (Computation in the general case is completely analogous.)

$$\begin{aligned}
p \wedge q &= \left(\xi^1 \frac{\partial}{\partial z^1} + \xi^2 \frac{\partial}{\partial z^2} + \xi^{\bar{1}} \frac{\partial}{\partial z^{\bar{1}}} + \xi^{\bar{2}} \frac{\partial}{\partial z^{\bar{2}}}\right) \\
&\qquad \wedge \left(\eta^1 \frac{\partial}{\partial z^1} + \eta^2 \frac{\partial}{\partial z^2} + \eta^{\bar{1}} \frac{\partial}{\partial z^{\bar{1}}} + \eta^{\bar{2}} \frac{\partial}{\partial z^{\bar{2}}}\right) \\
&= (\xi^1\eta^2 - \xi^2\eta^1)\frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} + (\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^{\bar{1}}} \\
&\quad + (\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}})\frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^{\bar{2}}} + (\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})\frac{\partial}{\partial z^2} \wedge \frac{\partial}{\partial z^{\bar{1}}} \\
&\quad + (\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})\frac{\partial}{\partial z^2} \wedge \frac{\partial}{\partial z^{\bar{2}}} + (\xi^{\bar{1}}\eta^{\bar{2}} - \eta^{\bar{1}}\xi^{\bar{2}})\frac{\partial}{\partial z^{\bar{1}}} \wedge \frac{\partial}{\partial z^{\bar{2}}} \, .
\end{aligned}$$

$$\begin{aligned}
||p \wedge q||^2 &= g_{1\bar{1}}^2 |\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}}|^2 + 2g_{1\bar{1}}g_{2\bar{2}}(|\xi^1\eta^2 - \eta^1\xi^2|^2 + |\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}}|^2) \\
&\quad + g_{2\bar{2}}^2 |\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}}|^2 \, .
\end{aligned}$$

2. Bergman metric of $|z_1|^{2m} + |z_2|^2 < 1$

Let D be the domain in $\mathbf{C}^2$ defined by $|z_1|^{2m} + |z_2|^2 < 1$. First we calculate its Bergman kernel function directly by using an orthonormal basis in the Hilbert space of L^2 holomorphic functions on D. The set of all polynomials is dense in this Hilbert space.

Let $z_\alpha = r_\alpha e^{i\theta_\alpha}$ $(\alpha = 1, 2)$. The Euclidean volume form dV of $\mathbf{C}^2$ is $(r_1 dr_1 d\theta_1)(r_2 dr_2 d\theta_2)$. The inner product of $z_1^k z_2^l$ and $z_1^p z_2^q$ is given by

$$\begin{aligned}
(z_1^k z_2^l, z_1^p z_2^q) &= \int_D (z_1^k z_2^l)(\overline{z_1^p z_2^q}) dV \\
&= \int_{r_1^{2m} + r_2^2 < 1} \int_{0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi} r_1^{k+p+1} r_2^{l+q+1} e^{i(k-p)\theta_1} e^{i(l-q)\theta_2} d\theta_1 d\theta_2 \, ,
\end{aligned}$$

which is zero unless $(k, l) = (p, q)$. Now consider the case $(k, l) = (p, q)$.

$$\begin{aligned}
\frac{1}{4\pi^2}(z_1^p z_2^q, z_1^p z_2^q) &= \int_{r_1^{2m} + r_2^2 < 1, r_1 \geq 0, r_2 \geq 0} r_1^{2p+1} r_2^{2q+1} dr_1 dr_2 \\
&= \int_{r_1=0}^1 r_1^{2p+1} dr_1 \int_{r_2=0}^{\sqrt{1-r_1^{2m}}} r_2^{2q+1} dr_2 \\
&= \frac{1}{2(q+1)} \int_{r_1=0}^1 r_1^{2p+1}(1 - r_1^{2m})^{q+1} dr_1 \\
&= \frac{1}{2(q+1)} \int_{u=0}^1 u^p (1 - u^m)^{q+1} du \qquad (\text{where } u = r_1^2) \\
&= \frac{m}{4(p+1)} \int_{u=0}^1 u^{p+m}(1 - u^m)^q du
\end{aligned}$$

(by using integration by parts to integrate u^p first)

$$= \frac{m(mq)}{4(p+1)(p+m+1)} \int_{u=0}^{1} u^{p+2m}(1-u^m)^{q-1}\,du$$

(by using integration by parts to integrate u^{p+2m} first)

$$= \cdots$$

$$= \frac{m(mq)\big(m(q-1)\big)\big(m(q-2)\big)\cdots(m2)(m1)}{4(p+1)(p+m+1)(p+2m+1)\cdots(p+qm+1)} \int_{u=0}^{1} u^{p+(q+1)m}\,du$$

$$= \frac{m^{q+1}q!}{4(p+1)(p+m+1)(p+2m+1)\cdots\big(p+(q+1)m+1\big)} .$$

The Bergman kernel function $\Phi(z_1, z_2)$ is given by

$$\begin{aligned}
\Phi(z_1, z_2) \\
&= \sum_{p,q=0}^{\infty} \frac{1}{m^{p+1}\pi^2} \frac{(p+1)(p+m+1)(p+2m+1)\cdots\big(p+(q+1)m+1\big)}{q!} |z_1|^{2p} |z_2|^{2q} \\
&= \sum_{p=0}^{\infty} \frac{(p+1)(p+m+1)}{m\pi^2} \\
&\quad \times \left(\sum_{q=0}^{\infty} \frac{\left(\frac{p+2m+1}{m}\right)\left(\frac{p+2m+1}{m}+1\right)\cdots\left(\frac{p+2m+1}{m}+q-1\right)}{q!} |z_2|^{2q}\right) |z_1|^{2p} \\
&= \sum_{p=0}^{\infty} \frac{(p+1)(p+m+1)}{m\pi^2} \big(1-|z_2|^2\big)^{-(p+2m+1)/m} |z_1|^{2p} \\
&= \frac{\big(1-|z_2|^2\big)^{-(2m+1)/m}}{m\pi^2} \sum_{p=0}^{\infty} (p+1)(p+m+1)\Big(\big(1-|z_2|^2\big)^{-(1/m)} |z_1|^2\Big)^p \\
&= \frac{\big(1-|z_2|^2\big)^{-(2m+1)/m}}{m\pi^2} \Big(\sum_{p=0}^{\infty} (p+1)(p+2)\Big(\big(1-|z_2|^2\big)^{-(1/m)}|z_1|^2\Big)^p \\
&\quad + \sum_{p=0}^{\infty} (m-1)(p+1)\Big(\big(1-|z_2|^2\big)^{-(1/m)}|z_1|^2\Big)^p\Big) \\
&= \frac{\big(1-|z_2|^2\big)^{-(2m+1)/m}}{m\pi^2} \Big(2\sum_{p=0}^{\infty} \frac{(-3)(-4)\cdots(-3-p+1)}{p!}\Big(-\big(1-|z_2|^2\big)^{-(1/m)}|z_1|^2\Big)^p \\
&\quad + (m-1)\sum_{p=0}^{\infty} \frac{(-2)(-3)\cdots(-2-p+1)}{p!}\Big(-\big(1-|z_2|^2\big)^{-(1/m)}|z_1|^2\Big)^p\Big) \\
&= \frac{\big(1-|z_2|^2\big)^{-(2m+1)/m}}{m\pi^2} \left(2\left(1-\frac{|z_1|^2}{\big(1-|z_2|^2\big)^{1/m}}\right)^{-3} + (m-1)\left(1-\frac{|z_1|^2}{\big(1-|z_2|^2\big)^{1/m}}\right)^{-2}\right) \\
&= \frac{1}{m\pi^2} \frac{(m+1)\big(1-|z_2|^2\big)^{1/m} - (m-1)|z_1|^2}{\big(1-|z_2|^2\big)^{2-(1/m)}\Big(\big(1-|z_2|^2\big)^{1/m} - |z_1|^2\Big)^3} .
\end{aligned}$$

Let

$$g_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} \log \Phi .$$

Direct computation yields

$$
(*)\quad\begin{cases}
g_{1\bar{1}} = \dfrac{3\left(1-|z_2|^2\right)^{1/m}}{\left(\left(1-|z_2|^2\right)^{1/m}-|z_1|^2\right)^2} - \dfrac{m-1}{m+1}\dfrac{\left(1-|z_2|^2\right)^{1/m}}{\left(\left(1-|z_2|^2\right)^{1/m}-\dfrac{m-1}{m+1}|z_1|^2\right)^2} \\[2ex]
g_{1\bar{2}} = \dfrac{3\bar{z}_1 z_2\left(1-|z_2|^2\right)^{(1/m)-1}}{m\left(\left(1-|z_2|^2\right)^{1/m}-|z_1|^2\right)^2} \\[2ex]
\qquad - \dfrac{m-1}{m(m+1)}\dfrac{\bar{z}_1 z_2}{\left(1-|z_2|^2\right)^{1-(1/m)}\left(\left(1-|z_2|^2\right)^{1/m}-\dfrac{m-1}{m+1}|z_1|^2\right)^2} \\[2ex]
g_{2\bar{2}} = \dfrac{2m-1}{m}\dfrac{1}{\left(1-|z_2|^2\right)^2} + 3\dfrac{\left(1-|z_2|^2\right)^{1/m}-|z_1|^2+\dfrac{1}{m}|z_1|^2|z_2|^2}{m\left(1-|z_2|^2\right)^{2-(1/m)}\left(\left(1-|z_2|^2\right)^{1/m}-|z_1|^2\right)^2} \\[2ex]
\qquad - \dfrac{\left(1-|z_2|^2\right)^{1/m}-\dfrac{m-1}{m+1}|z_1|^2+\dfrac{m-1}{m(m+1)}|z_1|^2|z_2|^2}{m\left(1-|z_2|^2\right)^{2-(1/m)}\left(\left(1-|z_2|^2\right)^{1/m}-\dfrac{m-1}{m+1}|z_1|^2\right)^2} .
\end{cases}
$$

Moreover, at $(0, 0)$

$$
R_{1\bar{1}1\bar{1}} = \frac{4m^2+16m+4}{(m+1)^2}\,,
$$

$$
R_{1\bar{1}2\bar{2}} = \frac{2m+4}{m(m+1)}\,,
$$

$$
R_{2\bar{2}2\bar{2}} = \frac{4m+2}{m}\,.
$$

All other $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0$, since $g_{1\bar{2}} = \bar{z}_1 z_2 F_{1\bar{2}}\left(|z_1|^2, |z_2|^2\right)$ and thus $R_{\alpha\bar{\beta}1\bar{2}} \neq 0$ only if $\alpha = 2$ and $\beta = 1$. Since

$$
\left(\frac{2m+4}{m(m+1)}\right)^2 < \frac{4m+2}{m}\frac{4m^2+16m+4}{(m+1)^2}\,,
$$

it follows from the following lemma that the sectional curvature of the Bergman metric of D is negative at $(0, 0)$.

LEMMA 1. *Suppose* $2\,\mathrm{Re}\sum_{\alpha,\beta=1}^{2} g_{\alpha\bar{\beta}}\,dz^\alpha dz^{\bar{\beta}}$ *is a Kähler metric on a neighborhood of* 0 *in* $\mathbf{C}^2$ *and* $g_{1\bar{2}} = 0$ *at* 0. *Suppose all the components* $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ *of the curvature tensor are zero at the origin except* $R_{1\bar{1}1\bar{1}}$, $R_{1\bar{1}2\bar{2}}$, *and* $R_{2\bar{2}2\bar{2}}$. *Then all the sectional curvatures at the origin are negative if and only if* $R_{1\bar{1}1\bar{1}} > 0$, $R_{1\bar{1}2\bar{2}} > 0$, $R_{2\bar{2}2\bar{2}} > 0$, *and* $(R_{1\bar{1}2\bar{2}})^2 < R_{1\bar{1}1\bar{1}}R_{2\bar{2}2\bar{2}}$.

Proof. In this case

$$
\begin{aligned}
&\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^\alpha\eta^{\bar{\beta}} - \eta^\alpha\xi^{\bar{\beta}})\overline{(\xi^\delta\eta^{\bar{\gamma}} - \eta^\delta\xi^{\bar{\gamma}})} \\
&\quad = R_{1\bar{1}1\bar{1}}|\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}}|^2 + 2R_{1\bar{1}2\bar{2}}\left(|\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}}|^2 + (\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})}\right) \\
&\qquad + R_{2\bar{2}2\bar{2}}|\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}}|^2\,.
\end{aligned}
$$

The sectional curvature is negative if and only if this expression is $\geqq$ some positive multiple of

$$|\xi^1\eta^{\bar 1} - \eta^1\xi^{\bar 1}|^2 + |\xi^1\eta^2 - \eta^1\xi^2|^2 + |\xi^1\eta^{\bar 2} - \eta^1\xi^{\bar 2}|^2 + |\xi^2\eta^{\bar 2} - \eta^2\xi^{\bar 2}|^2 .$$

Sufficiency follows from

$$\begin{aligned}|2R_{1\bar 12\bar 2}(\xi^1\eta^{\bar 1} - \eta^1\xi^{\bar 1})\overline{(\xi^2\eta^{\bar 2} - \eta^2\xi^{\bar 2})}| &\leqq |2(R_{1\bar 11\bar 1}R_{2\bar 22\bar 2})^{1/2}(\xi^1\eta^{\bar 1} - \eta^1\xi^{\bar 1})\overline{(\xi^2\eta^{\bar 2} - \eta^2\xi^{\bar 2})}| \\ &\leqq R_{1\bar 11\bar 1}|\xi^1\eta^{\bar 1} - \eta^1\xi^{\bar 1}|^2 + R_{2\bar 22\bar 2}|\xi^2\eta^{\bar 2} - \eta^2\xi^{\bar 2}|^2\end{aligned}$$

and from

$$\begin{aligned}|\xi^1\eta^2 - \eta^1\xi^2|^2 &= |\xi^1\eta^{\bar 2} - \eta^1\xi^{\bar 2}|^2 - (\xi^1\eta^{\bar 1} - \eta^1\xi^{\bar 1})\overline{(\xi^2\eta^{\bar 2} - \eta^2\xi^{\bar 2})} \\ &\leqq |\xi^1\eta^{\bar 2} - \eta^1\xi^{\bar 2}|^2 + \frac{1}{2}|\xi^1\eta^{\bar 1} - \eta^1\xi^{\bar 1}|^2 + \frac{1}{2}|\xi^2\eta^{\bar 2} - \eta^2\xi^{\bar 2}|^2 .\end{aligned}$$

Necessity follows from considering the following special values of ξ^α and η^α.

i) $\xi^2 = \eta^2 = 0$ yields $R_{1\bar 11\bar 1} > 0$,

ii) $\xi^1 = \eta^1 = 0$ yields $R_{2\bar 22\bar 2} > 0$,

iii) $\xi^1 = \eta^2 = 0$ yields $R_{1\bar 12\bar 2} > 0$,

iv) $\xi^1 = a\sqrt{-1}$, $\xi^2 = -\sqrt{-1}$, $\eta^1 = a$, $\eta^2 = 1$ for $a > 0$ yields

$$R_{1\bar 11\bar 1}4a^4 - 2R_{1\bar 12\bar 2}4a^2 + R_{2\bar 22\bar 2}4 > 0 .$$

Hence $(R_{1\bar 12\bar 2})^2 < R_{1\bar 11\bar 1}R_{2\bar 22\bar 2}$.

3. Curvature of the sum of metrics

As in Section 2 let $D \subset \mathbf{C}^2$ be defined by $|z_1|^{2m} + |z_2|^2 < 1$. Let B be the open unit 2-ball with coordinates w_1, w_2. Let $\sigma: D \to B$ be defined by

$$\sigma(z_1, z_2) = (z_1^m, z_2) .$$

Let $2\mathrm{Re}\sum g_{\alpha\bar\beta}dz^\alpha dz^{\bar\beta}$ be the Bergman metric of D from Section 2. Let $2\mathrm{Re}\sum h_{\alpha\bar\beta}dw^\alpha dw^{\bar\beta}$ be the Poincaré metric of B.

LEMMA 2. *There exists an open neighborhood U of 0 in D such that for $\lambda \geqq 0$ the metric which is the sum of the Bergman metric of D and λ times the pullback of the Poincaré metric of B (i.e., the metric $2\mathrm{Re}(\sum g_{\alpha\bar\beta}dz^\alpha d\bar z^\beta) + \lambda\sigma^*(\sum h_{\alpha\bar\beta}dw^\alpha dw^{\bar\beta})$ on D) has strictly negative sectional curvature on U.*

Proof. First we observe that for $|b| < 1$,

$$\begin{cases} w_1' = \dfrac{w_1\sqrt{1 - |b|^2}}{\bar b w_2 + 1} \\[2ex] w_2' = \dfrac{w_2 + b}{\bar b w_2 + 1} \end{cases}$$

is a biholomorphism of the ball sending $B \cap \{w_2 = -b\}$ to $B \cap \{w_2 = 0\}$. (The fact that it is a biholomorphism follows from the simply identity

$$|\bar{a}z+1|^2\left(1-\left|\frac{z+a}{\bar{a}z+1}\right|^2\right)=(1-|z|^2)(1-|a|^2)$$

for $a, z \in \mathbf{C}$ with $|a|<1$, $|z|<1$.) Moreover, this biholomorphism of B is covered by the following automorphism of D:

$$\begin{cases} z_1' = \dfrac{z_1(1-|b|^2)^{1/2m}}{(\bar{b}z_2+1)^{1/m}} \\ z_2' = \dfrac{z_2+b}{\bar{b}z_2+1}\ , \end{cases}$$

where $(\bar{b}z_2+1)^{1/m}$ is any branch. Hence instead of proving the existence of U, it suffices to show that for $\alpha \in \mathbf{C}$ sufficiently small the sectional curvature of the sum metric is negative at $(z_1, z_2)=(\alpha, 0)$. We will do our computation of the sectional curvature at $(z_1, z_2)=(\alpha, 0)$ with $\alpha \neq 0$ and will use the coordinate system (w_1, w_2) coming from B instead of using the coordinate system (z_1, z_2). Let $a=\alpha^m$. Then the σ-image of $(z_1, z_2)=(\alpha, 0)$ is $(w_1, w_2)=(a, 0)$. Let

$$\sum \varphi_{\alpha\bar{\beta}}dw^\alpha dw^{\bar{\beta}} = \sum g_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}}\ .$$

Then

$$\begin{cases} \varphi_{1\bar{1}} = g_{1\bar{1}}\dfrac{1}{m^2}w_1^{(1/m)-1}\bar{w}_1^{(1/m)-1} \\ \varphi_{1\bar{2}} = g_{1\bar{2}}\dfrac{1}{m}w_1^{(1/m)-1} \\ \varphi_{2\bar{2}} = g_{2\bar{2}}\ , \end{cases}$$

where $g_{\alpha\bar{\beta}}$ is given by (*) of Section 2 and the m^{th} roots (and all other m^{th} roots from this point on) are taken in a way compatible with the point $(\alpha, 0)$.

Consider the following biholomorphism of B

$$\begin{cases} w_1 = \dfrac{\zeta_1+a}{\bar{a}\zeta_1+1} \\ w_2 = \dfrac{\zeta_2\sqrt{1-|a|^2}}{\bar{a}\zeta_1+1} \end{cases}$$

which sends $(\zeta_1, \zeta_2)=(0, 0)$ to $(w_1, w_2)=(a, 0)$. We are going to calculate the sectional curvature of the sum metric using the coordinates (ζ_1, ζ_2). Let

$$\sum h_{\alpha\bar{\beta}}dw^\alpha dw^{\bar{\beta}} = \sum H_{\alpha\bar{\beta}}d\zeta^\alpha d\zeta^{\bar{\beta}}\ ,$$
$$\sum \varphi_{\alpha,\bar{\beta}}dw^\alpha dw^{\bar{\beta}} = \sum \psi_{\alpha\bar{\beta}}d\zeta^\alpha d\zeta^{\bar{\beta}}\ .$$

By ∂_α and $\partial_{\bar{\alpha}}$ we mean respectively $\partial/\partial\zeta_\alpha$ and $\partial/\partial\bar{\zeta}_\alpha$. Since $2\operatorname{Re}\sum h_{\alpha\bar{\beta}}dw_\alpha d\bar{w}_\beta$ is the Poincaré metric of B, it follows that

$$(*)\qquad \begin{cases} H_{1\bar 2} = 0 \\ dH_{\alpha\bar\beta} = 0 \end{cases} \qquad \text{at} \quad (\zeta_1, \zeta_2) = (0, 0)\ .$$

We claim that

$$(\dagger)\qquad \begin{cases} \psi_{1\bar 2} = 0 \\ \partial_2\psi_{1\bar 1} = 0 \\ \partial_1\psi_{1\bar 2} = 0\ , \quad \partial_{\bar 2}\psi_{1\bar 2} = 0 \\ \partial_{\bar 2}\psi_{2\bar 2} = 0 \end{cases} \qquad \text{at} \quad (\zeta_1, \zeta_2) = (0, 0)\ .$$

To prove the claim, we express $\psi_{\alpha\bar\beta}$ in terms of $\varphi_{\alpha\bar\beta}$.

$$\begin{cases} \psi_{1\bar 1} = \sum_{\alpha,\beta}\varphi_{\alpha\bar\beta}\dfrac{\partial w_\alpha}{\partial\zeta_1}\overline{\dfrac{\partial w_\beta}{\partial\zeta_1}} \\ \qquad = \varphi_{1\bar 1}\left|\dfrac{1-|a|^2}{(\bar a\zeta_1+1)^2}\right|^2 + \varphi_{1\bar 2}\dfrac{1-|a|^2}{(\bar a\zeta_1+1)^2}\left(\overline{\dfrac{-\bar a\zeta_2\sqrt{1-|a|^2}}{(\bar a\zeta_1+1)^2}}\right) \\ \qquad\quad + \varphi_{2\bar 1}\left(\dfrac{-\bar a\zeta_2\sqrt{1-|a|^2}}{(\bar a\zeta_1+1)^2}\right)\left(\overline{\dfrac{1-|a|^2}{(\bar a\zeta_1+1)^2}}\right) + \varphi_{2\bar 2}\left|\dfrac{-\bar a\zeta_2\sqrt{1-|a|^2}}{(\bar a\zeta_1+1)^2}\right|^2 \\ \psi_{1\bar 2} = \sum_{\alpha,\beta}\varphi_{\alpha\bar\beta}\dfrac{\partial w_\alpha}{\partial\zeta_1}\overline{\dfrac{\partial w_\beta}{\partial\zeta_2}} \\ \qquad = \varphi_{1\bar 2}\dfrac{1-|a|^2}{(\bar a\zeta_1+1)^2}\left(\overline{\dfrac{\sqrt{1-|a|^2}}{\bar a\zeta_1+1}}\right) + \varphi_{2\bar 2}\left(\dfrac{-\bar a\zeta_2\sqrt{1-|a|^2}}{(\bar a\zeta_1+1)^2}\right)\left(\overline{\dfrac{\sqrt{1-|a|^2}}{\bar a\zeta_1+1}}\right) \\ \psi_{2\bar 2} = \sum_{\alpha\,\beta}\varphi_{\alpha\bar\beta}\dfrac{\partial w_\alpha}{\partial\zeta_2}\overline{\dfrac{\partial w_\beta}{\partial\zeta_2}} = \varphi_{2\bar 2}\dfrac{1-|a|^2}{|\bar a\zeta_1+1|^2}\ . \end{cases}$$

Since $g_{1\bar 2} = 0$ when $z_2 = 0$, it follows that $\psi_{1\bar 2} = 0$ at $(\zeta_1, \zeta_2) = (0, 0)$. Observe that from (*) of Section 2 we have

$$\begin{cases} g_{1\bar 1}(z_1, z_2) = F_{1\bar 1}(|z_1|^2, |z_2|^2) \text{ with } F_{1\bar 1}(0, 0) > 0 \\ g_{1\bar 2}(z_1, z_2) = \bar z_1 z_2 F_{1\bar 2}(|z_1|^2, |z_2|^2) \\ g_{2\bar 2}(z_1, z_2) = F_{2\bar 2}(|z_1|^2, |z_2|^2) \end{cases}$$

for some smooth functions, $F_{1\bar 1}$, $F_{1\bar 2}$, $F_{2\bar 2}$ of two real variables. Hence by (#) of Section 3,

$$\begin{cases} \varphi_{1\bar 1}(w_1, w_2) = |w_1|^{(2/m)-2}\Phi_{1\bar 1}(|w_1|^{2/m}, |w_2|^2) \text{ with } \Phi_{1\bar 1}(0, 0) > 0 \\ \varphi_{1\bar 2}(w_1, w_2) = \dfrac{w_2}{w_1}\Phi_{1\bar 2}(|w_1|^{2/m}, |w_2|^2) \\ \varphi_{2\bar 2}(w_1, w_2) = \Phi_{2\bar 2}(|w_1|^{2/m}, |w_2|^2) \end{cases}$$

for some smooth functions $\Phi_{1\bar 1}$, $\Phi_{1\bar 2}$, $\Phi_{2\bar 2}$ of two real variables. It follows that

$$\begin{cases} \psi_{1\bar 1} = |\zeta_1 + a|^{(2/m)-2}\psi_{1\bar 1}(\zeta_1, |\zeta_1 + a^{\,2/m}|, |\zeta_2|^2) \text{ with } \psi_{1\bar 1}(0, 0, 0) > 0 \\ \psi_{1\bar 2} = \dfrac{\zeta_2}{\zeta_1 + a}\psi_{1\bar 2}(\zeta_1, |\zeta_1 + a|^{2/m}, |\zeta_2|^2) \\ \psi_{2\bar 2} = \psi_{2\bar 2}(\zeta_1, |\zeta_1 + a|^{2/m}, |\zeta_2|^2) \end{cases}$$

for some smooth functions $\psi_{1\bar{1}}$, $\psi_{1\bar{2}}$, $\psi_{2\bar{2}}$ of a complex variable and two real variables. It is now obvious that all the equations in (†) of Section 3 are satisfied. Let

$$G_{\alpha\bar{\beta}} = \psi_{\alpha\bar{\beta}} + \lambda H_{\alpha\bar{\beta}}$$

where λ is a positive number. Let $R^{G}_{\alpha\bar{\beta}\gamma\bar{\delta}}$, $R^{\psi}_{\alpha\bar{\beta}\gamma\bar{\delta}}$, $R^{H}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ be the curvature tensors of respectively the metrics $2\operatorname{Re}\sum G_{\alpha\bar{\beta}}d\zeta_\alpha d\bar{\zeta}_\beta$, $2\operatorname{Re}\sum \psi_{\alpha\bar{\beta}}d\zeta_\alpha d\bar{\zeta}_\beta$, and $2\operatorname{Re}\sum H_{\alpha\bar{\beta}}d\zeta_\alpha d\bar{\zeta}_\beta$. From (*) and (†) of Section 3 it follows that at $(\zeta_1, \zeta_2) = (0, 0)$

$$\begin{cases} R^{G}_{1\bar{1}1\bar{1}} = R^{\psi}_{1\bar{1}1\bar{1}} + \lambda R^{H}_{1\bar{1}1\bar{1}} + \left(\dfrac{1}{\psi_{1\bar{1}}} - \dfrac{1}{\psi_{1\bar{1}} + \lambda H_{1\bar{1}}}\right)|\partial_1\psi_{1\bar{1}}|^2 \\ R^{G}_{1\bar{1}2\bar{2}} = R^{\psi}_{1\bar{1}2\bar{1}} + \lambda R^{H}_{1\bar{1}1\bar{2}} + \left(\dfrac{1}{\psi_{2\bar{2}}} - \dfrac{1}{\psi_{2\bar{2}} + \lambda H_{1\bar{1}}}\right)|\partial_1\psi_{2\bar{2}}|^2 \\ R^{G}_{\alpha\bar{\beta}\gamma\bar{\delta}} = R^{\psi}_{\alpha\bar{\beta}\gamma\bar{\delta}} + \lambda R^{H}_{\alpha\bar{\beta}\gamma\bar{\delta}} \text{ for } (\alpha, \bar{\beta}, \gamma, \bar{\delta}) = (1, \bar{1}, 1, \bar{2}), (1, \bar{2}, 1, \bar{2}), \\ \qquad (1, \bar{2}, 2, \bar{2}) \text{ and } (2, \bar{2}, 2, \bar{2})\,. \end{cases}$$

Hence

$$\begin{aligned} &\textstyle\sum_{\alpha,\beta,\gamma,\delta} R^{G}_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^\alpha\eta^{\bar{\beta}} - \eta^\alpha\xi^{\bar{\beta}})\overline{(\xi^\delta\eta^{\bar{\gamma}} - \eta^\delta\xi^{\bar{\gamma}})} \\ &\quad = \textstyle\sum_{\alpha,\beta,\gamma,\delta} R^{\psi}_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^\alpha\eta^{\bar{\beta}} - \eta^\alpha\xi^{\bar{\beta}})\overline{(\xi^\delta\eta^{\bar{\gamma}} - \eta^\delta\xi^{\bar{\gamma}})} \\ &\qquad + \lambda\textstyle\sum_{\alpha,\beta,\gamma,\delta} R^{H}_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^\alpha\eta^{\bar{\beta}} - \eta^\alpha\xi^{\bar{\beta}})\overline{(\xi^\delta\eta^{\bar{\gamma}} - \eta^\delta\xi^{\bar{\gamma}})} \\ &\qquad + \left(\frac{1}{\psi_{1\bar{1}}} - \frac{1}{\psi_{1\bar{1}} + \lambda H_{1\bar{1}}}\right)|\partial_1\psi_{1\bar{1}}|^2|\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}}|^2 \\ &\qquad + 2\left(\frac{1}{\psi_{2\bar{2}}} - \frac{1}{\psi_{2\bar{2}} + \lambda H_{2\bar{2}}}\right)|\partial_1\psi_{2\bar{2}}|^2|\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}}|^2 \\ &\qquad + 2\left(\frac{1}{\psi_{2\bar{2}}} - \frac{1}{\psi_{2\bar{2}} + \lambda H_{2\bar{2}}}\right)|\partial_1\psi_{2\bar{2}}|^2\operatorname{Re}(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})}\,. \end{aligned}$$

We want to show that the sectional curvature of the metric $2\operatorname{Re}\sum G_{\alpha\bar{\beta}}d\zeta_\alpha d\bar{\zeta}_\beta$ is $\leqq -c$ at $(\zeta_1, \zeta_2) = (0, 0)$ for $|a| < \varepsilon$ and $\lambda \geqq 0$, where c, ε are suitable positive numbers. Only the last term on the right-hand side is an undesirable term and we have to dominate it by other terms. The absolute value of $\partial_1\psi_{2\bar{2}}$ at $(\zeta_1, \zeta_2) = (0, 0)$ is $\leqq C_1 a^{(2/m)-1}$ where C_1 is a positive constant independent of a when $|a| < \varepsilon$ and ε is a fixed positive number <1. Hence the absolute value of

$$2\left(\frac{1}{\psi_{2\bar{2}}} - \frac{1}{\psi_{2\bar{2}} + \lambda H_{2\bar{2}}}\right)|\partial_1\psi_{2\bar{2}}|^2\operatorname{Re}(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{2} - \eta^2\xi^{\bar{2}})}$$

is

$$\leqq C_2|a|^{(4/m)-2}|\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}}||\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}}|\,,$$

where C_2 is again a positive constant independent of a and λ, because

$$\frac{1}{\psi_{2\bar{2}}} - \frac{1}{\psi_{2\bar{2}} + \lambda H_{2\bar{2}}} \leqq \frac{1}{\psi_{2\bar{2}}}\,.$$

Since the sectional curvature of the Bergman metric of D is negative at the center of D, it follows that for $|a| \leqq \varepsilon$ and ε sufficiently small, there exists a positive number s_1 such that

$$\begin{aligned}\sum\nolimits_{\alpha,\beta,\gamma,\delta} & R^{\psi}_{\alpha\bar\beta\gamma\bar\delta}(\xi^\alpha\eta^{\bar\beta} - \eta^\alpha\xi^{\bar\beta})\overline{(\xi^\delta\eta^{\bar\gamma} - \eta^\delta\xi^{\bar\gamma})} \\ & \geqq s_1\psi_{1\bar1}^2|\xi^1\eta^{\bar1} - \eta^1\xi^{\bar1}|^2 + s_1\psi_{2\bar2}^2|\xi^2\eta^{\bar2} - \eta^2\xi^{\bar2}|^2 \\ & \geqq s_2|a|^{(4/m)-4}|\xi^1\eta^{\bar1} - \eta^1\xi^{\bar1}|^2 + s_2|\xi^2\eta^{\bar2} - \eta^2\xi^{\bar2}|^2\end{aligned}$$

where s_2 is a positive constant independent of a. Since

$$\begin{aligned} & C_2|a|^{(4/m)-2}|\xi^1\eta^{\bar1} - \eta^1\xi^{\bar1}||\xi^2\eta^{\bar2} - \eta^2\xi^{\bar2}| \\ & \quad \leqq \frac{1}{2}C_2|a|^{2/m}\left(|a|^{(4/m)-4}|\xi^1\eta^{\bar1} - \eta^1\xi^{\bar1}|^2 + |\xi^2\eta^{\bar2} - \eta^2\xi^{\bar2}|^2\right),\end{aligned}$$

it follows that

$$\begin{aligned} & \frac{1}{2}\sum\nolimits_{\alpha,\beta,\gamma,\delta} R^{\psi}_{\alpha\bar\beta\gamma\bar\delta}(\xi^\alpha\eta^{\bar\beta} - \eta^\alpha\xi^{\bar\beta})\overline{(\xi^\delta\eta^{\bar\gamma} - \eta^\delta\xi^{\bar\gamma})} \\ & \quad \geqq C_2|a|^{(4/m)-2}|\xi^1\eta^{\bar1} - \eta^1\xi^{\bar1}||\xi^2\eta^{\bar2} - \eta^2\xi^{\bar2}| \\ & \quad \geqq \left|2\left(\frac{1}{\psi_{2\bar2}} - \frac{1}{\psi_{2\bar2} + \lambda H_{2\bar2}}\right)|\partial_1\psi_{2\bar2}|^2 \operatorname{Re}(\xi^1\eta^{\bar1} - \eta^1\xi^{\bar1})\overline{(\xi^2\eta^{\bar2} - \eta^2\xi^{\bar2})}\right|\end{aligned}$$

when $|a| \leqq (s_2C_2^{-1})^{m/2}$. Hence for a sufficiently small and for all $\lambda \geqq 0$

$$\begin{aligned}\sum\nolimits_{\alpha,\beta,\gamma,\delta} & R^{G}_{\alpha\bar\beta\gamma\bar\delta}(\xi^\alpha\eta^{\bar\beta} - \eta^\alpha\xi^{\bar\beta})\overline{(\xi^\delta\eta^{\bar\gamma} - \eta^\delta\xi^{\bar\gamma})} \\ & \geqq \frac{1}{2}\sum\nolimits_{\alpha,\beta,\gamma,\delta} R^{\psi}_{\alpha\bar\beta\gamma\bar\delta}(\xi^\alpha\eta^{\bar\beta} - \eta^\alpha\xi^{\bar\beta})\overline{(\xi^\delta\eta^{\bar\gamma} - \eta^\delta\xi^{\bar\gamma})}.\end{aligned}$$

Now the lemma follows from the fact that the sectional curvature of the Bergman metric of D is negative at the center of D.

The sum of the two metrics actually satisfies a curvature condition stronger than the negativity of the sectional curvature on a neighborhood of 0 for all $\lambda \geqq 0$. First we introduce this curvature condition. Define

$$A_{\alpha\bar\beta,\gamma\bar\delta} = R_{\alpha\bar\beta\delta\bar\gamma}.$$

Then

$$\overline{A_{\alpha\bar\beta,\gamma\bar\delta}} = \overline{R_{\alpha\bar\beta\delta\bar\gamma}} = R_{\gamma\bar\delta\beta\bar\alpha} = A_{\gamma\bar\delta,\alpha\bar\beta}.$$

Hence, as a matrix with row index $(\alpha, \bar\beta)$ and column index $(\gamma, \bar\delta)$, $A_{\alpha\bar\beta,\gamma\bar\delta}$ is a hermitian matrix. We say that the curvature tensor is *very strongly negative* if this hermitian matrix is strictly positive definite. In other words,

$$\sum\nolimits_{\alpha,\beta,\gamma,\delta} A_{\alpha\bar\beta,\gamma\bar\delta}\theta_{\alpha\bar\beta}\overline{\theta_{\gamma\bar\delta}} \geqq c\sum\nolimits_{\alpha,\beta}|\theta_{\alpha\bar\beta}|^2$$

for some $c > 0$; that is,

$$\sum\nolimits_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar\beta\gamma\bar\delta}\theta_{\alpha\bar\beta}\overline{\theta_{\delta\bar\gamma}} \geqq c\sum\nolimits_{\alpha,\beta}|\theta_{\alpha\bar\beta}|^2.$$

We recall that the negativity of the sectional curvature means that

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^{\alpha}\eta^{\bar{\beta}} - \eta^{\alpha}\xi^{\bar{\beta}})\overline{(\xi^{\delta}\eta^{\bar{\gamma}} - \eta^{\delta}\xi^{\bar{\delta}})} \geqq c\sum_{\alpha,\beta}|\xi^{\alpha}\eta^{\bar{\beta}} - \eta^{\alpha}\xi^{\bar{\beta}}|^2 ,$$

for some $c > 0$. Clearly the negativity of the sectional curvature follows from the very strong negativity of the curvature tensor, because one can set

$$\theta_{\alpha\bar{\beta}} = \xi^{\alpha}\eta^{\bar{\beta}} - \eta^{\alpha}\xi^{\bar{\beta}} .$$

To prove that the sum of the two metrics has a very strongly negative curvature tensor, one observes that

$$\begin{aligned}\sum_{\alpha,\beta,\gamma,\delta} R^{G}_{\alpha\bar{\beta}\gamma\bar{\delta}}\theta_{\alpha\bar{\beta}}\overline{\theta_{\delta\bar{\gamma}}} &= \sum_{\alpha,\beta,\gamma,\delta} R^{\psi}_{\alpha\bar{\beta}\gamma\bar{\delta}}\theta_{\alpha\bar{\beta}}\overline{\theta_{\delta\bar{\gamma}}} + \lambda\sum_{\alpha,\beta,\gamma,\delta} R^{H}_{\alpha\bar{\beta}\gamma\bar{\delta}}\theta_{\alpha\bar{\beta}}\overline{\theta_{\delta\bar{\gamma}}} \\ &\quad + \left(\frac{1}{\psi_{1\bar{1}}} - \frac{1}{\psi_{1\bar{1}} + \lambda H_{1\bar{1}}}\right)|\partial_1\psi_{1\bar{1}}|^2|\theta_{1\bar{1}}|^2 \\ &\quad + \left(\frac{1}{\psi_{2\bar{2}}} - \frac{1}{\psi_{2\bar{2}} + \lambda H_{2\bar{2}}}\right)|\partial_1\psi_{2\bar{2}}|^2\left(|\theta_{1\bar{2}}|^2 + |\theta_{2\bar{1}}|^2\right) \\ &\quad + 2\left(\frac{1}{\psi_{2\bar{2}}} - \frac{1}{\psi_{2\bar{2}} + \lambda H_{2\bar{2}}}\right)\operatorname{Re}(\theta_{1\bar{1}}\overline{\theta_{2\bar{2}}}) .\end{aligned}$$

Moreover, one observes that the Bergman metric of D has a very strongly negative curvature tensor at 0, because the very strong negativity of the curvature tensor is equivalent to the positive definiteness of the matrix

$$(A_{\alpha\bar{\beta},\gamma\bar{\delta}}) = \begin{pmatrix} R_{1\bar{1}1\bar{1}} & R_{1\bar{1}2\bar{1}} & R_{1\bar{1}1\bar{2}} & R_{1\bar{1}2\bar{2}} \\ R_{1\bar{2}1\bar{1}} & R_{1\bar{2}2\bar{1}} & R_{1\bar{2}1\bar{2}} & R_{1\bar{2}2\bar{2}} \\ R_{2\bar{1}1\bar{1}} & R_{2\bar{1}2\bar{1}} & R_{2\bar{1}1\bar{2}} & R_{2\bar{1}2\bar{2}} \\ R_{2\bar{2}1\bar{1}} & R_{2\bar{2}2\bar{1}} & R_{2\bar{2}1\bar{2}} & R_{2\bar{2}2\bar{2}} \end{pmatrix}$$

which becomes in this case

$$\begin{pmatrix} R_{1\bar{1}1\bar{1}} & 0 & 0 & R_{1\bar{1}2\bar{2}} \\ 0 & R_{1\bar{1}2\bar{2}} & 0 & 0 \\ 0 & 0 & R_{1\bar{1}2\bar{2}} & 0 \\ R_{1\bar{1}2\bar{2}} & 0 & 0 & R_{2\bar{2}2\bar{2}} \end{pmatrix}$$

which is positive definite because

$$R_{1\bar{1}1\bar{1}} > 0 ,$$

$$\det\begin{pmatrix} R_{1\bar{1}1\bar{1}} & 0 \\ 0 & R_{1\bar{1}2\bar{2}} \end{pmatrix} = R_{1\bar{1}1\bar{1}}R_{1\bar{1}2\bar{2}} > 0 ,$$

$$\det\begin{pmatrix} R_{1\bar{1}1\bar{1}} & 0 & 0 \\ 0 & R_{1\bar{1}2\bar{2}} & 0 \\ 0 & 0 & R_{1\bar{1}2\bar{2}} \end{pmatrix} = R_{1\bar{1}1\bar{1}}(R_{1\bar{1}2\bar{2}})^2 > 0 ,$$

$$\det\begin{pmatrix} R_{1\bar{1}1\bar{1}} & 0 & 0 & R_{1\bar{1}2\bar{2}} \\ 0 & R_{1\bar{1}2\bar{2}} & 0 & 0 \\ 0 & 0 & R_{1\bar{1}2\bar{2}} & 0 \\ R_{1\bar{1}2\bar{2}} & 0 & 0 & R_{2\bar{2}2\bar{2}} \end{pmatrix} = (R_{1\bar{1}2\bar{2}})^2\bigl(R_{1\bar{1}1\bar{1}}R_{2\bar{2}2\bar{2}} - (R_{1\bar{1}2\bar{2}})^2\bigr) > 0 \ .$$

Since the ball is the special case of D with $m = 1$, it follows that the curvature tensor of the invariant metric of the ball is very strongly negative. Borrowing earlier notations, we have, when $|a| \leqq (s_2C_2^{-1})^{m/2}$,

$$\begin{aligned}
&\left|2\Bigl(\frac{1}{\psi_{2\bar{2}}} - \frac{1}{\psi_{2\bar{2}} + \lambda H_{2\bar{2}}}\Bigr)|\partial_1\psi_{2\bar{2}}|^2 \operatorname{Re}\theta_{1\bar{1}}\overline{\theta_{2\bar{2}}}\right| \\
&\leqq C_2|a|^{(4/m)-2}|\theta_{1\bar{1}}||\theta_{2\bar{2}}| \\
&\leqq \frac{1}{2}C_2|a|^{2/m}\bigl(|a|^{(4/m)-4}|\theta_{1\bar{1}}|^2 + |\theta_{2\bar{2}}|^2\bigr) \\
&\leqq \frac{1}{2}s_2\bigl(|a|^{(4/m)-4}|\theta_{1\bar{1}}|^2 + |\theta_{2\bar{2}}|^2\bigr) \\
&\leqq \frac{1}{2}s_1\bigl(\psi_{1\bar{1}}^2|\theta_{1\bar{1}}|^2 + \psi_{2\bar{2}}^2|\theta_{2\bar{2}}|^2\bigr) \\
&\leqq \frac{1}{2}R^{\psi}_{\alpha\bar{\beta}\gamma\bar{\delta}}\theta_{\alpha\bar{\beta}}\overline{\theta_{\delta\bar{\gamma}}} \ .
\end{aligned}$$

Hence

$$\sum\nolimits_{\alpha,\beta,\gamma,\delta} R^{G}_{\alpha\bar{\beta}\gamma\bar{\delta}}\theta_{\alpha\bar{\beta}}\overline{\theta_{\delta\bar{\gamma}}} \geqq \frac{1}{2}\sum\nolimits_{\alpha,\beta,\gamma,\delta} R^{\psi}_{\alpha\bar{\beta}\gamma\bar{\delta}}\theta_{\alpha\bar{\beta}}\overline{\theta_{\delta\bar{\gamma}}} \ .$$

The significance of the very strong negativity of the curvature tensor is that Siu [9] proved that any compact Kähler manifold having the same homotopy type as a compact Kähler manifold of dimension $\geqq 2$ with a very strongly negative curvature tensor must be either biholomorphic or conjugate biholomorphic to it. In particular, strong rigidity holds for compact Kähler manifolds of dimension $\geqq 2$ with a very strongly negative curvature tensor.

4. Construction of the surface

The main idea of the construction is to construct first a subgroup Γ of $\operatorname{Aut} B$ generated by three complex reflections. Γ is not discrete, but is almost discrete in the sense that there is a complex surface Y and a holomorphic map σ from Y onto B such that Γ lifts to a discrete subgroup $\widetilde{\Gamma}$ of $\operatorname{Aut} Y$, and the only kind of singularity of σ is simple winding singularity along an infinite number of disjoint complex curves whose images are complex lines in B. The compact surface is $Y/\widetilde{\Gamma}_0$ for some subgroup $\widetilde{\Gamma}_0$ of $\widetilde{\Gamma}$ of finite index chosen for the sole purpose of making $Y/\widetilde{\Gamma}_0$ nonsingular.

The investigation of subgroups of $\operatorname{Aut} B$ generated by three complex reflections of order p for $p = 3, 4, 5$ was extensively carried out by Mostow [7], [8]. We will describe below the construction of the subgroup we need, but will refer to [8] for detailed proofs of the properties of this subgroup. We will need only the case $p = 5$.

Let $a = 1/(2\sin(\pi/5))$ and φ be a complex number of modulus 1 whose argument will be given later. Let e_1, e_2, e_3 be a basis of $\mathbf{C}^3$ over $\mathbf{C}$. Consider the hermitian form $\langle\cdot, \cdot\rangle$ defined with respect to the basis e_1, e_2, e_3 by the matrix

$$\begin{pmatrix} 1 & -a\varphi & -a\bar{\varphi} \\ -a\bar{\varphi} & 1 & -a\varphi \\ -a\varphi & -a\bar{\varphi} & 1 \end{pmatrix}$$

whose determinant is $\Delta = 1 - 3a^2 - a^3(\varphi^3 + \varphi^{-3})$. We consider only values of $\arg\varphi$ for which Δ is negative. Let V be the set of all $x \in \mathbf{C}^3$ with $\langle x, x\rangle < 0$. The quotient of V by the $\mathbf{C}^*$ action is biholomorphic to B.

Let $e_j^\perp$ be the set of all $x \in \mathbf{C}^3$ with $\langle x, e_j\rangle = 0$. Define the complex reflection R'_j in C^3 of order 5 about the complex plane $e_j^\perp$ as follows.

$$R'_j(x) = x + (e^{2\pi\sqrt{-1}/5} - 1)\langle x, e_j\rangle e_j \ .$$

Since $\langle\cdot, \cdot\rangle$ is invariant under R'_j and R'_j commutes with the $\mathbf{C}^*$ action, it follows that R'_j defines an element R_j of $\operatorname{Aut} B$.

Let Γ_{ij} be the subgroup of $\operatorname{Aut} B$ generated by R_i and R_j. The only relations of R_i and R_j are $R_i^5 = R_j^5 = 1$ and $R_iR_jR_i = R_jR_iR_j$. The group Γ_{ij} has order 600. Let p_{ij} be the image of $e_i^\perp \cap e_j^\perp$ in B. Then p_{ij} is the only fixed point of Γ_{ij}. We give B the invariant metric. Then p_{12}, p_{23}, p_{13} form the vertices of an equilateral triangle. Let p_0 be the center of the circumscribed ball of this equilateral triangle. /e

For $\gamma \in \operatorname{Aut} B$ let

$$\hat{\gamma} = \{x \in B \mid d(x, p_0) = d(x, \gamma^{-1}p_0)\} \ ,$$
$$X(\gamma) = \{x \in B \mid d(x, p_0) \leqq d(x, \gamma^{-1}p_0)\}$$

where $d(\cdot, \cdot)$ is the invariant distance in B. For any subset D_* of $\operatorname{Aut} B$ let

$$X(D_*) = \textstyle\bigcap_{\gamma \in D_*} X(\gamma) \ ,$$
$$\tilde{\gamma} = \hat{\gamma} \cap X(D_*) \ .$$

For $i \neq j$ let D_{ij} be the subset of Γ_{ij} consisting of the ten elements

$$D_{ij} = \{R_i^{\pm1}, R_j^{\pm1}, (R_iR_j)^{\pm1}, (R_jR_i)^{\pm1}, (R_iR_jR_i)^{\pm1}\} \ .$$

Then $X(D_{ij})$ is a fundamental domain of Γ_{ij}.

Let Γ denote the subgroup of $\operatorname{Aut} B$ generated by R_1, R_2, R_3. Let

$$D_* = \{R_i^{\pm 1}, (R_iR_j)^{\pm 1}, (R_iR_jR_i)^{\pm 1}, i \neq j, i, j = 1, 2, 3\} .$$

That is, Γ is generated by Γ_{12}, Γ_{23}, Γ_{13} and $D_* = D_{12} \cup D_{23} \cup D_{13}$. Let $\Omega = X(D_*)$. Γ is discrete for certain values of $\arg \varphi$. In these cases Ω is a fundamental domain for Ω modulo the finite subgroup of Γ which stabilizes Ω. In general, Γ is not discrete. For any φ such that $|\arg \varphi^3| < \pi/2 - \pi/5$, Ω has twenty-four 3-faces given by

$$\widetilde{R}_i^{\pm 1}, (\widetilde{R_iR_j})^{\pm 1}, (\widetilde{R_iR_jR_i})^{\pm 1}; i \neq j, i, j = 1, 2, 3$$

(note that $R_iR_jR_i = R_jR_iR_j$). Each of the twenty-four 3-faces of Ω is oriented so that together with the inner normal to Ω, it carries the orientation of the ambient oriented space B. Inasmuch as each $\gamma \in \Gamma$ is an orientation-preserving homeomorphism of B which carries $\tilde{\gamma}$ into $\tilde{\gamma}^{-1}$ and an inner normal to an outer normal, we see that for each 3-face $\tilde{\gamma}$ of Ω, the map $\gamma: \tilde{\gamma} \to \tilde{\gamma}^{-1}$ reverses the assigned orientation of the 3-face $\tilde{\gamma}$. Next, we orient each 2-face of each $\tilde{\gamma}$ by the inner-normal rule, using the assigned orientation of the ambient orientable 3-manifold $\partial\Omega$. Since $\partial\Omega$ is an oriented manifold without boundary, each 2-face appears in exactly two 3-faces and is assigned two opposite orientations. We adopt the convention: whenever we represent a 2-face as the intersection of two 3-faces $F' \cap F''$, it carries the orientation induced by the first 3-face F'.

The vertices of Ω are p_{ij}, s_{ij}, $\tilde{s}_{ij}$, t_{ij} ($i \neq j$, $i, j = 1, 2, 3$) defined by

$$\begin{aligned} s_{ij} = s_{ji} &= \widetilde{R}_i \cap \widetilde{R}_j \cap \widetilde{R_iR_j} \cap \widetilde{R_jR_i} \cap \widetilde{R_iR_jR_i} \cap \widetilde{R}_k^{-1} \cap (\widetilde{R_kR_i})^{-1} \cap (\widetilde{R_kR_j})^{-1} , \\ \tilde{s}_{ij} = \tilde{s}_{ji} &= \widetilde{R}_i^{-1} \cap \widetilde{R}_j^{-1} \cap (\widetilde{R_iR_j})^{-1} \cap (\widetilde{R_jR_i})^{-1} \cap \widetilde{R_iR_jR_i} \cap \widetilde{R}_k \cap \widetilde{R_iR_k} \cap \widetilde{R_jR_k} , \\ t_{ij} &= \widetilde{R}_k \cap \widetilde{R}_k^{-1} \cap \widetilde{R_kR_j} \cap (\widetilde{R_iR_k})^{-1} \cap R_k\widetilde{R_jR_k} \cap (R_k\widetilde{R_iR_k})^{-1} , \end{aligned}$$

where $\{i, j, k\} = \{1, 2, 3\}$. For $i \neq j$ there are two 1-faces joining p_{ij} and s_{ij}. To distinguish them we select a point s_{ij}^* on one and a point s_{ji}^* on the other. The actions of R_j on some of the vertices are

$$\begin{aligned} R_js_{ij} &= \tilde{s}_{ik} , \\ R_k\tilde{s}_{ij} &= s_{ij} , \\ R_iR_jt_{kj} &= t_{ik} \end{aligned}$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Later we will use the fact that for $\varphi = 1$, in homogeneous coordinates of $\mathbf{P}_2$,

$$\begin{aligned} s_{23} &= [1, e^{-(\pi\sqrt{-1}/5)}\sqrt{-1}, e^{-(\pi\sqrt{-1}/5)}\sqrt{-1}] , \\ \tilde{s}_{21} &= [-e^{\pi\sqrt{-1}/5}\sqrt{-1}, -e^{\pi\sqrt{-1}/5}\sqrt{-1}, 1] . \end{aligned}$$

When $|\arg \varphi^3| < \pi/2 - \pi/5$, the only 2-faces not passing through any of the vertices p_{ij} are the ones with vertices $t_{ik}, s_{jk}, \tilde{s}_{ij}$. We denote it by Δ_{ijk}. It lies in exactly two 3-faces: $\widetilde{R_jR_k}$ and $(\widetilde{R_iR_j})^{-1}$. Moreover, it lies in a complex line pointwise fixed under $(R_iR_jR_k)^2$. $(R_iR_jR_k)^2$ is a complex reflection about this line. Its action can best be described by looking at a complex line perpendicular to Δ_{ijk}. There are six 2-faces Δ_{ijk} and the cyclic permutation $1 \to 2 \to 3$ yields the isometries $\Delta_{123} \to \Delta_{231} \to \Delta_{321}$ and $\Delta_{321} \to \Delta_{213} \to \Delta_{132}$. So we consider only Δ_{312} and Δ_{213}. The geodesic triangle $p_{12}\, t_{32}\, p_{31}$ intersects Δ_{312} at t_{32} and is perpendicular to Δ_{312}. The angle at t_{32} of the triangle $p_{12}\, t_{32}\, p_{31}$ is $\pi/2 - \pi/5 + \arg \varphi^3$. $(R_3R_1R_2)^2$ rotates the triangle $p_{12}\, t_{32}\, p_{31}$ on its plane about the vertex t_{32} by an angle equal to $3\pi/5 + \pi/2 - \arg \varphi^9$. The geodesic triangle $p_{12}\, t_{23}\, p_{31}$ intersects Δ_{213} at t_{23} and is perpendicular to Δ_{213}. The angle at t_{23} of the triangle $p_{12}\, t_{23}\, p_{31}$ is $\pi/2 - \pi/5 - \arg \varphi^3$. $(R_2R_1R_3)^2$ rotates the triangle $p_{12}\, t_{23}\, p_{31}$ on its plane about the vertex t_{23} by an angle equal to $3\pi/5 + \pi/2 + \arg \varphi^9$.

For later use we list also here the magnitudes of certain angles. In the geodesic triangles Δ_{312} and Δ_{213},

$$\measuredangle\, s_{12} = \measuredangle\, \tilde{s}_{31} = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{5} - \arg \varphi^3\right),$$

$$\measuredangle\, \tilde{s}_{12} = \measuredangle\, s_{31} = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{5} + \arg \varphi^3\right).$$

The angle at p_{21} of the triangle $t_{23}\, p_{21}\, t_{32}$ and the angle at p_{31} of the triangle $t_{23}\, p_{31}\, t_{32}$ are both $\pi/10$.

In [8] it is proved that Γ is discrete and Ω is a fundamental domain for Γ modulo the stabilizer in Γ of Ω if and only if for all distinct i, j, k the interior of the triangle $p_{jk}\, t_{ik}\, p_{ij}$ intersects its image under $(R_iR_jR_k)^{2\nu}$ only when $(R_iR_jR_k)^{2\nu} = 1$. In other words, a necessary and sufficient condition is that the following two conditions are satisfied.

Let $\nu_+ = \operatorname{order}(R_3R_1R_2)^2$ and $\nu_- = \operatorname{order}(R_2R_1R_3)^2$.

i) 2π is an integral multiple of $(\pi/2 - \pi/5 + \arg \varphi^3)\nu_+$,

ii) 2π is an integral multiple of $(\pi/2 - \pi/5 - \arg \varphi^3)\nu_-$.

The reason for this criterion of discreteness is the following. Clearly Γ is discrete and Ω is its fundamental domain modulo the stabilizer in Γ of Ω if for $\gamma \in \Gamma$ the interior of Ω intersects its image under γ only when γ belongs to the stabilizer. The elements $(R_iR_jR_k)^{2n}$ such that $(\text{Interior } \Omega) \cap (R_iR_jR_k)^{2n}\Omega = \varnothing$ are the obstructions to the fulfillment of these requirements. The above criterion stipulates the removal of these obstructions.

Now we choose $\arg \varphi^3 = \pi/20$,

$$\begin{cases} < p_{12}t_{32}p_{31} = \dfrac{\pi}{2} - \dfrac{\pi}{5} + \dfrac{\pi}{20} = \dfrac{7\pi}{20} . \\ \text{Angle of rotation of } (R_3R_1R_2)^2 = \dfrac{3\pi}{5} + \dfrac{\pi}{2} - 3\arg\varphi^3 = \dfrac{19\pi}{20} . \end{cases}$$

$$\begin{cases} < p_{12}t_{23}p_{31} = \dfrac{\pi}{2} - \dfrac{\pi}{5} - \dfrac{\pi}{20} = \dfrac{\pi}{4} . \\ \text{Angle of rotation of } (R_2R_1R_3)^2 = \dfrac{3\pi}{5} + \dfrac{\pi}{2} + 3\arg\varphi^3 = \dfrac{5}{4}\pi . \end{cases}$$

In this example, condition ii) is satisfied, but condition i) is not satisfied. It is proved in [8] that Γ is not discrete. Nevertheless it is not far from acting discretely. The obstruction to Ω being a fundamental domain near Δ_{312} can be described as follows. In C we have a closed sector S of angle $7\pi/20$ with the origin as vertex and we have the action of a cyclic group G whose generator g is the rotation of angle $19\pi/20$. S is not a fundamental domain for G. However, we can take a 7-sheeted analytic cover $\alpha: \tilde{\mathbf{C}} \to \mathbf{C}$ with the origin as the branching locus. Then $\tilde{\mathbf{C}}$ is the union of 40 sets $\tilde{S}_\nu$ $(1 \leqq \nu \leqq 40)$ so that α maps each $\tilde{S}_\nu$ bijectively onto a sector in C of angle $7\pi/20$ with the origin as vertex and $\alpha(\tilde{S}_1) = S$. The generator g is the same as the rotation of angle $-(21\pi/20)$. The cyclic group action of G on C can be lifted to a cyclic group action of $\tilde{G}$ on $\tilde{\mathbf{C}}$ whose generator $\tilde{g}$ is the rotation of angle $-(21\pi/20)$ with the angle measured from C. $\tilde{S}$ is a fundamental domain for $\tilde{G}$. Observe that $\tilde{\mathbf{C}}$ is the same as the quotient $S \times G/\sim$ where $\sim$ is defined as follows: $(s, g) \sim (s', g')$ if and only if $gs = g's'$ and $s, s' \in \partial S$. This description fits precisely the situation at hand with C being the complex line containing Δ_{312}, the sector being defined by the two geodesics $t_{32}\,p_{12}$ and $t_{32}\,p_{31}$, and g being the restriction of $(R_3R_1R_2)^2$ to the complex line containing Δ_{312}. Besides the obstruction to discreteness of Γ due to $(R_3R_1R_2)^2$, there are similar obstructions due to $(R_1R_2R_3)^2$ and $(R_2R_3R_1)^2$. Hence to remove all the obstructions to discreteness, it takes more than construction of a 7-sheeted analytic cover of B with branching locus along the complex line containing Δ_{312}. One has to construct also 7-sheeted analytic covers with branching locus along the complex lines containing respectively Δ_{123} and Δ_{231}. Moreover, the set of 3 complex lines containing Δ_{312}, Δ_{123}, Δ_{231} is far from invariant under the action Γ. This necessitates the construction of analytic covers with branching locus along their images under Γ. There is a natural way of constructing a Γ-space lying over B with Ω as a fundamental domain modulo $\operatorname{Aut}\Omega$; on such a space Γ will necessarily be discrete.

We observe first that $\operatorname{Aut}\Omega$ is induced by the 3 cyclic permutations

$1 \to 2 \to 3$ of the coordinates of $\mathbf{C}^3$. It is shown in [8] that for $\arg \varphi^3 = \pi/20$, these 3 elements are in Γ. Define $Y = \Gamma \times \Omega/\sim$, where the relation $\sim$ is generated by the conditions: $(\gamma, x) \sim (\gamma', x')$ if

$$(1) \qquad \gamma x = \gamma' x' \quad \text{with} \quad x, x' \in \partial\Omega \text{ and } \gamma^{-1}\gamma' \in D_*$$

or

$$(2) \qquad \gamma^{-1}\gamma' \in \operatorname{Aut}\Omega .$$

Define $\sigma\colon Y \to B$ as the map induced by $\Gamma \times \Omega \to B$ which sends (γ, x) to γx. It is proved in [8] that Y can be given the structure of a complex manifold such that σ is holomorphic and Γ acts holomorphically on M.

Let E be the set of all points of Y where the Jacobian determinant of σ is zero. E is a disjoint union of a countable number of nonsingular complex curves E_i. Each E_i is biholomorphic to $\sigma(E_i)$ under σ. The set of all $\sigma(E_i)$ is precisely the set of all images of the 3 complex lines containing Δ_{123}, Δ_{231}, Δ_{312} under elements of Γ. Note that, though the E_i's are mutually disjoint and are locally finite, the $\sigma(E_i)$'s are not mutually disjoint and their union is dense in B. It is true though that the 3 complex lines in B containing Δ_{123}, Δ_{231} and Δ_{312} respectively are disjoint. For every i, there exist an open neighborhood Q_i of E_i in Y and an open neighborhood U_i of $\sigma(E_i)$ in B such that we have the following commutative diagram

$$\begin{array}{ccc} Q_i & \xrightarrow{\theta} & \pi^{-1}(\tau(U_i)) \subset D \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi} \\ U_i & \xrightarrow{\tau} & \tau(U_i) \end{array}$$

where $\sigma(Q_i) = U_i$, θ is biholomorphic, $\tau \in \operatorname{Aut} B$ sends $\sigma(E_i)$ to the complex line $\{w_1 = 0\} \cap B$ with B realized as $\{(w_1, w_2) \in \mathbf{C}^2 \mid |w_1|^2 + |w_2|^2 < 1\}$, $D = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^{14} + |z_2|^2 < 1\}$, and $\pi(w_1, w_2) = (w_1^7, w_2)$.

For $\gamma_0 \in \Gamma$, the map $\Gamma \times \Omega \to \Gamma \times \Omega$ sending (γ, x) to $(\gamma_0\gamma, x)$ induces an element γ_0^* of Aut Y. γ_0^* covers γ_0 in the sense that $\sigma\gamma_0^* = \gamma_0\sigma$. The group $\Gamma^* \subset \operatorname{Aut} Y$ of all such γ_0^* is discrete. The quotient Y/Γ^* is compact, but in general has singularities. We can remove the singularities by using, instead of Γ^*, a normal subgroup Γ_0^* of finite index in Γ which contains no element of finite order except the identity. For example, Γ can be regarded as a subgroup of $\mathrm{PSL}(3, \mathbf{Z}(e^{\pi\sqrt{-1}/20}))$ and it suffices to take Γ_0 as the subgroup of all elements $\equiv 1$ modulo some prime divisor of 3 in $\mathbf{Z}(e^{\pi\sqrt{-1}/20})$, and take as Γ_0^* the subgroup of Γ^* corresponding to Γ_0. Let $M = Y/\Gamma_0^*$. Then M is a compact nonsingular complex surface and is the surface we want to construct.

5. Construction of the Kähler metric

Let $\kappa\colon Y \to M$ be the quotient map. $\kappa(E)$ is the disjoint union of a finite number of nonsingular complex curves C_ν $(1 \leqq \nu \leqq l)$ in M. Let $\{E_i^{(\nu)}\}$ be the set of all E_i such that $\kappa(E_i) = C_\nu$. Each $E_i^{(\nu)}$ is a universal covering of C_ν. Let $\Gamma_i^{(\nu)} \subset \Gamma_0^*$ be the stabilizer of $E_i^{(\nu)}$. Then $C_\nu = E_i^{(\nu)}/\Gamma_i^{(\nu)}$.

Let W_ν be an open neighborhood of C_ν in M such that the W_ν's are mutually disjoint. Take a C^∞ function $0 \leqq \rho_\nu \leqq 1$ on W_ν with compact support such that $\rho_\nu \equiv 1$ on an open neighborhood of C_ν. For $1 \leqq \nu \leqq l$, fix some $E_{i_\nu}^{(\nu)}$. We have an open neighborhood $Q^{(\nu)}$ of $E_{i_\nu}^{(\nu)}$ and an open neighborhood $U^{(\nu)}$ of $\sigma(E_{i_\nu}^{(\nu)})$ such that $\sigma(Q^{(\nu)}) = U^{(\nu)}$ and

$$\begin{array}{ccc} Q^{(\nu)} & \xrightarrow{\theta_\nu} & \pi^{-1}\big(\tau_\nu^{-1}(U^{(\nu)})\big) \subset D \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ U^{(\nu)} & \xrightarrow{\tau_\nu} & \tau_\nu(U^{(\nu)}) \subset B \end{array}$$

is commutative, where θ_ν is biholomorphic, $\tau_\nu \in \operatorname{Aut} B$ sends $\sigma(E_{i_\nu}^{(\nu)})$ to the complex line $\{w_1 = 0\} \cap B$ with B realized as $\{(w_1, w_2) \in \mathbf{C}^2 \,|\, |w_1|^2 + |w_2|^2 < 1\}$, $D = \{(z_1, z_2) \in \mathbf{C}^2 \,|\, |z_1|^{14} + |z_2|^2 < 1\}$, and $\pi(w_1, w_2) = (w_1^7, w_2)$. By replacing $Q^{(\nu)}$ by a possibly smaller neighborhood, we can assume that $Q^{(\nu)}$ is a component of $\kappa^{-1}(W_\nu)$ and is mapped bijectively onto W_ν by κ. $Q^{(\nu)}$ is stabilized by $\Gamma_{i_\nu}^{(\nu)}$.

Let Φ be the Bergman kernel form on D. Let $\Phi_{i_\nu}^{(\nu)} = \theta_\nu^*\Phi$. So $\Phi_{i_\nu}^{(\nu)}$ is a $(2, 2)$-form on $Q^{(\nu)}$ and is invariant under $\Gamma_i^{(\nu)}$. Define $\Phi_i^{(\nu)} = \gamma^*\Phi_{i_\nu}^{(\nu)}$ and $Q_i^{(\nu)} = \gamma^{-1}(Q^{(\nu)})$, where $\gamma \in \Gamma_0^*$ maps $E_i^{(\nu)}$ to $E_{i_\nu}^{(\nu)}$. These definitions are independent of the choice of γ. The $Q_i^{(\nu)}$'s are mutually disjoint. Let $\widetilde{Q} = \cup_{i,\nu} Q_i^{(\nu)}$ and let $\widetilde{\Phi}$ be the $(2, 2)$-form on $\widetilde{Q}$ which agrees with $\Phi_i^{(\nu)}$ on $Q_i^{(\nu)}$. Then both $\widetilde{Q}$ and $\widetilde{\Phi}$ are Γ_0^*-invariant.

Let ρ be the function on $\widetilde{Q}$ which agrees with $\rho_\nu \circ \kappa$ on $Q_\nu^{(i)}$. Then ρ has compact support on $\widetilde{Q}$ mod Γ_0^*, is identically 1 on an open neighborhood of E, and is Γ_0^*-invariant. Let Ψ be the Bergman kernel form on B and let $\widetilde{\Psi} = \sigma^*\Psi$. Define $\Theta = \rho\widetilde{\Phi} + (1 - \rho)\widetilde{\Psi}$. At every point of E the $(2, 2)$-form $\widetilde{\Phi}$ is a positive multiple of the volume form. At every point of Y the $(2, 2)$-form $\widetilde{\Psi}$ is a nonnegative multiple of the volume form. At every point of $Y - E$ the $(2, 2)$-form $\widetilde{\Psi}$ is a positive multiple of the volume form. It follows that at every point of Y the $(2, 2)$-form Θ is a positive multiple of the volume form. Θ is clearly Γ_0^*-invariant. Consider the Kähler metric $\partial\bar{\partial} \log \Theta$ on Y (i.e., $\partial\bar{\partial}$ of the log of the coefficient of Θ). This Kähler metric is Γ_0^*-invariant and it agrees with $\partial\bar{\partial} \log \widetilde{\Phi}$ on some open neighborhood of E. It follows from the result of Section 3 that on some open neighborhood of E the Kähler metric $\partial\bar{\partial} \log \widetilde{\Phi} + \lambda\sigma^*(\partial\bar{\partial} \log \Psi)$ has negative sectional curvature (and actually satisfies a stronger negative curvature condition as defined in § 3) for all $\lambda \geqq 0$. Take a compact subset A of Y with $\kappa(A) = M$. Then for λ sufficiently large, the Kähler metric $\partial\bar{\partial} \log \tilde{\Phi} + \lambda\sigma^*(\partial\bar{\partial} \log \Psi)$ has negative sectional curvature (actually satisfies a stronger negative curvature condition) at every point of A. As a consequence this Γ_0^*-invariant Kähler metric for sufficiently large λ induces a Kähler metric on M which has negative sectional curvature (and actually satisfies a stronger negative curvature condition).

6. Computation of $c_2(M)$

We will show that the universal covering of M is not biholomorphic to the open 2-ball B by calculating c_2 and c_1^2 for M and verifying that $c_1^2 \neq 3c_2$. We first calculate $c_2(M)$ which is simply the Euler characteristic of M. We do it by choosing a suitable cellular decomposition of M. For this purpose we have to understand how Γ identifies the ν-faces of Ω for $\nu = 0, 1, 2, 3$. For detailed proofs of this identification, we again refer to [8]. Let $p =$ the order of R_1, $N =$ the order of the group Γ_{12}, $r =$ the order of $(R_1R_2R_3)^2$ and $s =$ the order of $(R_3R_2R_1)^2$. In our case $p = 5$, $N = 600$, $r = 40$, and $s = 8$.

i) The 3 cyclic permutations $1 \to 2 \to 3$ for the coordinates of $\mathbf{C}^3$ define 3 elements of Γ and these 3 elements form precisely the stabilizer in Γ of Ω.

ii) Under Γ every 3-face of Ω is identified with one of the four 3-faces $\tilde{R}_1$, $\widetilde{R_1R_2}$, $\widetilde{R_2R_1}$, $\widetilde{R_1R_2R_1}$. There is no identification among these four 3-faces. The stabilizer in Γ of any one of them is trivial.

iii) Under Γ every 2-face of Ω is identified with one of the nine 2-faces

$$\tilde{R}_1 \cap (\widetilde{R_1R_2}),\ \tilde{R}_2 \cap (\widetilde{R_1R_2}),\ \tilde{R}_1 \cap (\widetilde{R_1R_2R_1}),$$
$$\tilde{R}_2 \cap (\widetilde{R_1R_2R_1}),\ \widetilde{R_1R_2} \cap (\widetilde{R_1R_2R_1}),\ (\widetilde{R_2R_1}) \cap (\widetilde{R_1R_2}),$$

the complex line cell in Ω defined by $e_1^\perp$ (which we will simply call $e_1^\perp$), Δ_{123}, Δ_{231}. There is no identification among these nine 2-faces. The stabilizers in Γ of the first six 2-faces are all trivial. The stabilizer in Γ of $e_1^\perp$ is of order p. The stabilizer in Γ of Δ_{123} is of order r. The stabilizer in Γ of Δ_{321} is of order s.

iv) Under Γ every 1-face of Ω is identified with one of the following eight 1-faces among which there is no identification. $p_{23}t_{13}$ (order of stabilizer $= p$), $p_{23}t_{31}$ (order of stabilizer $= p$), $p_{12}s_{12}$ via s_{12}^* (order of stabilizer $=$ 1), $p_{12}s_{12}$ via s_{21}^* (order of stabilizer $= 1$), $s_{23}\tilde{s}_{12}$ (order of stabilizer $= 2r$), $s_{21}\tilde{s}_{32}$ (order of stabilizer $= 2s$), $s_{23}t_{13}$ (order of stabilizer $= r$), $s_{21}t_{23}$ (order of stabilizer $= s$).

v) Under Γ every 0-face of Ω can be indentified with one of the following four 0-faces among which there is no identification: p_{12} (order of stabilizer $= N$), t_{13} (order of stabilizer $= pr$), t_{31} (order of stabilizer $= ps$), s_{12} (order of stabilizer $= rs$).

Ω can be lifted to a domain $\widetilde{\Omega}$ in Y. We construct a cellular decomposition L of Y by using $\widetilde{\Omega}$ and its ν-faces ($\nu = 0, 1, 2, 3$) corresponding to the ν-faces of Ω and by using all their images under Γ^*. This cellular decomposition L induces a cellular decomposition K of M. (Note that since the action of Γ_0^* on Y is fixed point free, it follows that no element of Γ_0^* other than the identity element stabilizes any cell of L.) Let k be the index of Γ_0^* in Γ^*. Then the number of distinct ν-cells in K which each ν-face on our list above gives rise to is equal to k divided by the order of the stabilizer in Γ^* of that ν-face. Let ε_ν be the number of ν-cells in K. Then

$$\begin{cases} \varepsilon_4 = \dfrac{k}{3} \\ \varepsilon_3 = 4k \\ \varepsilon_2 = k\left(6 + \dfrac{1}{p} + \dfrac{1}{r} + \dfrac{1}{s}\right) \\ \varepsilon_1 = k\left(\dfrac{2}{p} + 2 + \dfrac{1}{2r} + \dfrac{1}{2s} + \dfrac{1}{r} + \dfrac{1}{s}\right) \\ \varepsilon_0 = k\left(\dfrac{1}{N} + \dfrac{1}{pr} + \dfrac{1}{ps} + \dfrac{1}{rs}\right). \end{cases}$$

Hence the Euler characteristic is

$$\begin{aligned} &k\left(\frac{1}{3} - 4 + \left(6 + \frac{1}{p} + \frac{1}{r} + \frac{1}{s}\right) - \left(\frac{2}{p} + 2 + \frac{1}{2r} + \frac{1}{2s} + \frac{1}{r} + \frac{1}{s}\right)\right. \\ &\qquad \left. + \left(\frac{1}{N} + \frac{1}{pr} + \frac{1}{ps} + \frac{1}{rs}\right)\right) \\ &\quad = k\left(\frac{1}{3} + \frac{1}{N} + \frac{1}{pr} + \frac{1}{ps} + \frac{1}{rs} - \frac{1}{p} - \frac{1}{2r} - \frac{1}{2s}\right) \\ &\quad = k\left(\frac{1}{3} + \frac{1}{600} + \frac{1}{200} + \frac{1}{40} + \frac{1}{320} - \frac{1}{5} - \frac{1}{80} - \frac{1}{16}\right) \\ &\quad = \frac{149k}{1600}. \end{aligned}$$

This method of calculating the Euler characteristic works also for other values of $\arg \varphi^3$.

7. Noneuclidean volume of a fundamental domain

The calculation of c_1^2 for M is much more complicated than that of c_2. At one point the noneuclidean volume of Ω is needed in the calculation. So we have to calculate first the noneuclidean volume of Ω.

Let the open 2-ball B be realized by $\{w = (w_1, w_2) \in \mathbf{C}^2 \,|\, |w_1|^2 + |w_2|^2 < 1\}$. Let $\tilde{\Xi}(w) = \log\big(1/(1 - |w_1|^2 - |w_2|^2)^2\big)$. Then the invariant Kähler metric of B with holomorphic sectional curvature -1 is given by $2\,\mathrm{Re}\big(\sum(\partial^2\tilde{\Xi}/\partial w_\alpha \partial\bar{w}_\beta) dw_\alpha d\bar{w}_\beta\big)$. We have the following transformation law for $\tilde{\Xi}$:

$$\tilde{\Xi}(\gamma w) + \frac{2}{3}\log|J_\gamma(w)|^2 = \tilde{\Xi}(w)$$

for any automorphism $w \mapsto \gamma w$ of B, where J_γ is the Jacobian determinant for γ (calculated with respect to the coordinates $w = (w_1, w_2)$).

We introduce a new coordinate system for B. We realize B as the set of all points $[x_1, x_2, x_3] \in \mathbf{P}_3$ such that

$$(x_1, x_2, x_3)\begin{pmatrix} 1 & -a\varphi & -a\bar{\varphi} \\ -a\bar{\varphi} & 1 & -a\varphi \\ -a\varphi & -a\bar{\varphi} & 1 \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} < 0\,.$$

Recall that $a = 1/(2\sin(\pi/5))$ and φ is a complex number of modulus 1 so that the determinant of the matrix $\Delta = 1 - 3a^2 - a^3(\varphi^3 + \bar{\varphi}^3)$ is negative. Introduce the coordinate system $z = (z_1, z_2)$ on B defined by

$$\begin{cases} z_1 = \dfrac{x_1}{x_3} \\[2ex] z_2 = \dfrac{x_2}{x_3}\,. \end{cases}$$

This is indeed a global coordinate system on B, because the positive definiteness of the matrix

$$\begin{pmatrix} 1 & -a\varphi \\ -a\bar{\varphi} & 1 \end{pmatrix}$$

implies that B is disjoint from $\{x_3 = 0\}$. Let $w = f(z)$ be the function relating the two coordinate systems w and z on B. Define

$$\Xi(z) = \tilde{\Xi}\big(f(z)\big) + \frac{2}{3}\log|J_f(z)|^2\,.$$

Clearly the invariant metric of B of holomorphic sectional curvature -1 is given by

$$2\,\mathrm{Re}\Big(\sum \frac{\partial^2\Xi}{\partial z_\alpha \partial\bar{z}_\beta} dz_\alpha d\bar{z}_\beta\Big)\,.$$

We claim that Ξ obeys the following transformation law for any automorphism $z \mapsto \theta z$ of B.

$$\Xi(\theta z) + \frac{2}{3}\log|J_0(z)|^2 = \Xi(z) .$$

The verification is straightforward. In the rest of this section, we will use only the coordinate system z and all the Jacobian determinants of automorphisms of B will be calculated with respect to this coordinate system.

The twenty-four 3-faces of Ω occur in pairs. We denote them by $F_j^\pm$ $(j = 1, \cdots, 12)$. More precisely we agree that

$$\{F_j^+\}_{j=1,\cdots,12} = \{\tilde{R}_i, \widetilde{R_iR_j}, \widetilde{R_iR_jR_i}\}_{i,j=1,2,3,i\neq j} ,$$

$$\{F_j^-\}_{j=1,\cdots,12} = \{\tilde{R}_i^{-1}, (\widetilde{R_iR_j})^{-1}, (\widetilde{R_iR_jR_i})^{-1}\}_{i,j=1,2,3,i\neq j} .$$

(Recall that $R_iR_jR_i = R_jR_iR_j$.) Ω is oriented. Its orientation is described in Section 4. R_i maps $\tilde{R}_i$ to $\tilde{R}_i^{-1}$, R_iR_j maps $\widetilde{R_iR_j}$ to $(\widetilde{R_iR_j})^{-1}$, and $R_iR_jR_i$ maps $\widetilde{R_iR_jR_i}$ to $(\widetilde{R_iR_jR_i})^{-1}$. These maps precisely reverse the orientations of the 3-faces. We use $f_j\colon F_j^+ \to F_j^-$ to denote these automorphisms. When we take orientations into account, f_j actually maps $+F_j^+$ to $-F_j^-$.

$$\begin{aligned}\pm\operatorname{vol}(\Omega) &= \int_\Omega \frac{(\sqrt{-1}\,\partial\bar\partial\Xi)^2}{2} = -\frac{1}{2}\int_{\partial\Omega}\bar\partial\Xi \wedge \partial\bar\partial\Xi \\ &= -\frac{1}{2}\sum_{j=1}^{12}\left(\int_{F_j^+}\bar\partial\Xi \wedge \partial\bar\partial\Xi + \int_{F_j^-}\bar\partial\Xi \wedge \partial\bar\partial\Xi\right) \\ &= -\frac{1}{2}\sum_{j=1}^{12}\int_{F_j^+}(\bar\partial\Xi \wedge \partial\bar\partial\Xi) - (\bar\partial\Xi \wedge \partial\bar\partial\Xi)\circ f_j .\end{aligned}$$

We use $\pm\operatorname{vol}(\Omega)$, because we do not know whether the orientation we have chosen for $\partial\Omega$ in Section 4 is compatible with the orientation induced by the volume form. Since $\Xi \circ f_j = \Xi - 2/3\log|J_{f_j}|^2$, it follows that

$$(\partial\bar\partial\Xi)\circ f_j = \partial\bar\partial\Xi ,$$

$$(\bar\partial\Xi)\circ f_j = \bar\partial\Xi - \frac{2}{3}\overline{d\log J_{f_j}} .$$

Hence

$$\begin{aligned}&\int_{F_j^+}(\bar\partial\Xi \wedge \partial\bar\partial\Xi) - (\bar\partial\Xi \wedge \partial\bar\partial\Xi)\circ f_j \\ &= \frac{2}{3}\int_{F_j^+}\overline{d\log J_{f_j}} \wedge \partial\bar\partial\Xi \\ &= \frac{2}{3}\int_{\partial F_j^+}\log\bar J_{f_j}\partial\bar\partial\Xi\end{aligned}$$

after we choose a branch for $\log J_{f_j}$ on B, and

$$\begin{aligned}\pm \operatorname{vol}(\Omega) &= -\frac{1}{3}\sum_{j=1}^{12}\int_{\partial F_j^+} \log \bar{J}_{f_j}\partial\bar{\partial}\Xi \\ &= \frac{\sqrt{-1}}{3}\sum_{j=1}^{12}\int_{\partial F_j^+} (\log \bar{J}_{f_j})\omega ,\end{aligned}$$

where $\omega = \sqrt{-1}\partial\bar{\partial}\Xi$ is the Kähler form for the invariant metric on B with holomorphic sectional curvature -1.

As is well-known, $J_{\gamma_1\gamma_2} = (J_{\gamma_1}\circ\gamma_2)J_{\gamma_2}$. From this we get for $x \in e_j^\perp$,

$$1 = J_{R_j^5}(x) = (J_{R_j}(x))^5 .$$

We define the branch of $\log J_{f_j}$ in the following way. R_j is a complex reflection fixing $e_j^\perp$ pointwise. The order of R_j is 5. We define the branches:

$$\log J_{R_j} = \frac{2\pi\sqrt{-1}}{5} + 2\pi n\sqrt{-1}$$

at $e_j^\perp$ for $j = 1, 2, 3$, where n is an indeterminate integer;

$$\log J_{R_iR_j} = (\log J_{R_i})\circ R_j + \log J_{R_j} ;$$

$$\log J_{R_iR_jR_i} = (\log J_{R_i})\circ(R_jR_i) + (\log J_{R_j})\circ R_i + \log J_{R_i} .$$

Note that the last two are defined from the first one according to the chain rule. Moreover,

$$\log J_{R_iR_jR_i}(p_{ij}) = 2\pi\sqrt{-1}\left(\frac{3}{5} + 3n\right) = \log J_{R_jR_iR_j}(p_{ij}) .$$

Since $R_iR_jR_i = R_jR_iR_j$, it follows that $\log J_{R_iR_jR_i} = \log J_{R_jR_iR_j}$ and is well-defined.

To compute $\sum_{j=1}^{12}\int_{\partial F_j^+}(\log \bar{J}_{f_j})\omega$, we first observe that

$$\partial F_j^+ = (\bigcup_{i=1\ i\neq j}^{12} F_j^+ \cap F_i^+) \cup (\bigcup_{i=1}^{12} F_j^+ \cap F_i^-)$$

(where of course some of the $F_j^+ \cap F_i^+$ and $F_j^+ \cap F_i^-$ are empty). Every 2-face of Ω belongs precisely to two 3-faces and therefore is of the form $F_j^+ \cap F_i^+$, $F_j^+ \cap F_i^-$, or $F_j^- \cap F_i^-$. We recall that whenever we represent a 2-face as the intersection of two 3-faces $F' \cap F''$, it carries the orientation induced by the first 3-face F'. So $F' \cap F'' = -F'' \cap F'$. Using this orientation convention we have

$$\begin{aligned}&\sum_{j=1}^{12}\int_{\partial F_j^+}(\log \bar{J}_{f_j})\omega \\ &= \sum_{j=1}^{12}\left(\sum_{i=1\ i\neq j}^{12}\int_{F_j^+\cap F_i^+}(\log \bar{J}_{f_j})\omega + \sum_{i=1}^{12}\int_{F_j^+\cap F_i^-}(\log \hat{J}_{f_j})\omega\right) \\ &= \sum_\nu \int_{S_\nu} \bar{\eta}_\nu\omega\end{aligned}$$

where $\{S_\nu\}$ is the set of all 2-faces of Ω and

i) when S_ν is of the form $F_j^+ \cap F_i^+$, $\eta_\nu = \log J_{f_j} - \log J_{f_i}$;

ii) when S_ν is of the form $F_j^+ \cap F_i^-$, $\eta_\nu = \log J_{f_j}$;

iii) when S_ν is of the form $F_j^- \cap F_i^-$, $\eta_\nu = 0$.

So in order to compute $\sum_{j=1}^{12} \int_{\partial F_j^+} (\log \bar{J}_{f_j})\omega$, we have to look at the set of all 2-faces of Ω. There are the following 3 types of 2-faces of Ω.

i) The eighteen 2-faces passing through one of the three points p_{12}, p_{23}, p_{13}.

ii) The three 2-faces $\tilde{R}_1 \cap \tilde{R}_1^{-1}$, $\tilde{R}_2 \cap \tilde{R}_2^{-1}$, $\tilde{R}_3 \cap \tilde{R}_3^{-1}$ (i.e., $e_1^\perp$, $e_2^\perp$, $e_3^\perp$).

iii) The six 2-faces Δ_{ijk} with $\{i, j, k\} = \{1, 2, 3\}$ $\big(\Delta_{ijk}$ = the triangle $t_{ik}s_{jk}\tilde{s}_{ij} = \widetilde{R_jR_k} \cap (\widetilde{R_iR_j})^{-1}\big)$.

First, consider the eighteen 2-faces passing through some p_{ij}. They can be grouped in six groups of three each. We give them below in the following six diagrams.

$$\begin{tikzcd}
& \widetilde{(R_2R_1)} \cap \widetilde{(R_1R_2R_1)} = (p_{12}, s_{12}, t_{32}) \arrow[dr, "R_2R_1"] & \\
(p_{12}, \tilde{s}_{21}, t_{23}) = \widetilde{(R_1R_2R_1)}^{-1} \cap \tilde{R}_1^{-1} \arrow[ur, "(R_1R_2R_1)^{-1}"] & & \tilde{R}_1 \cap \widetilde{(R_2R_1)}^{-1} = (p_{12}, s_{31}, t_{23}) \arrow[ll, "R_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
& \tilde{R}_1 \cap \widetilde{(R_2R_1)} = (p_{21}, \tilde{s}_{32}, s_{12}) \arrow[dr, "R_1"] & \\
(p_{21}, \tilde{s}_{21}, s_{31}) = \widetilde{(R_2R_1)}^{-1} \cap \tilde{R}_2^{-1} \arrow[ur, "(R_2R_1)^{-1}"] & & \tilde{R}_2 \cap \tilde{R}_1^{-1} = (p_{21}, s_{23}, \tilde{s}_{13}) \arrow[ll, "R_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
& \tilde{R}_2 \cap \widetilde{(R_1R_2)} = (p_{21}, s_{12}, \tilde{s}_{13}) \arrow[dr, "R_2"] & \\
(p_{21}, s_{23}, \tilde{s}_{21}) = \widetilde{(R_1R_2)}^{-1} \cap \tilde{R}_1^{-1} \arrow[ur, "(R_1R_2)^{-1}"] & & \tilde{R}_1 \cap \tilde{R}_2^{-1} = (p_{21}, \tilde{s}_{32}, s_{31}) \arrow[ll, "R_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \tilde{R}_1 \cap \widetilde{R_1R_2R_1} = (p_{12}, t_{32}, s_{12}) \arrow[dr, "R_1"] & \\
(p_{12}, t_{13}, \tilde{s}_{21}) = (\widetilde{R_1R_2R_1})^{-1} \cap (\widetilde{R_1R_2})^{-1} \arrow[ur, "(R_1R_2R_1)^{-1}"] & & \widetilde{R_1R_2} \cap \tilde{R}_1^{-1} = (p_{12}, t_{32}, \tilde{s}_{13}) \arrow[ll, "R_1R_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & \widetilde{R_1R_2} \cap \widetilde{R_1R_2R_1} = (p_{12}, s_{12}, t_{32}) \arrow[dr, "R_1R_2"] & \\
(p_{12}, \tilde{s}_{21}, t_{13}) = (\widetilde{R_1R_2R_1})^{-1} \cap \tilde{R}_2^{-1} \arrow[ur, "(R_1R_2R_1)^{-1}"] & & R_2 \cap (\widetilde{R_1R_2})^{-1} = (p_{12}, s_{23}, t_{13}) \arrow[ll, "R_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & \tilde{R}_2 \cap \widetilde{R_1R_2R_1} = (p_{21}, t_{31}, s_{12}) \arrow[dr, "R_2"] & \\
(p_{21}, t_{23}, \tilde{s}_{21}) = (\widetilde{R_1R_2R_1})^{-1} \cap (\widetilde{R_2R_1})^{-1} \arrow[ur, "(R_2R_1R_2)^{-1}"] & & \widetilde{R_2R_1} \cap \tilde{R}_2^{-1} = (p_{21}, t_{31}, \tilde{s}_{32}) \arrow[ll, "R_2R_1"']
\end{tikzcd}$$

In these diagrams the arrows give elements of Γ mapping a 2-face to another 2-face. These maps preserve orientations when the 2-faces are given the orientations in the diagrams. A triple of points in the diagrams denotes the triangle with the three points as vertices. These triples of points are ordered and they give also the orientations of the triangles. The equality signs mean not just equality as sets but also equality in orientation. We have

$$\sum\nolimits_{S_\nu \text{ in one of the above groups}} \int_{S_\nu} \eta_\nu \omega = 0 .$$

We prove this only for the first group. The proofs for the other five groups are completely analogous.

$$\begin{aligned}
&\sum\nolimits_{S_\nu \text{ in the first group}} \int_{S_\nu} \eta_\nu \omega \\
&= \int_{\widetilde{R_2R_1} \cap \widetilde{R_1R_2R_1}} (\log \bar{J}_{R_2R_1} - \log \bar{J}_{R_1R_2R_1})\omega + \int_{\tilde{R}_1 \cap (\widetilde{R_2R_1})^{-1}} (\log \bar{J}_{R_1})\omega \\
&= \int_{\widetilde{R_2R_1} \cap \widetilde{R_1R_2R_1}} (\log \bar{J}_{R_2R_1} - \log \bar{J}_{R_1R_2R_1})\omega + \int_{\widetilde{R_2R_1} \cap \widetilde{R_1R_2R_1}} (\log \bar{J}_{R_1}) \circ (R_2R_1)\omega \\
&= \int_{\widetilde{R_2R_1} \cap \widetilde{R_1R_2R_1}} (\log \bar{J}_{R_2R_1} - \log \bar{J}_{R_1R_2R_1} + (\log \bar{J}_{R_1}) \circ (R_2R_1))\omega \\
&= 0
\end{aligned}$$

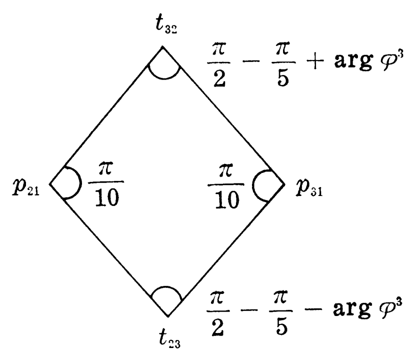
from the definition of the branch of $\log \bar{J}_{f_j}$. Thus we have

$$\begin{aligned}
\pm \operatorname{vol}(\Omega) = &\frac{\sqrt{-1}}{3} \sum\nolimits_{i=1}^{3} \int_{\tilde{R}_i \cap \tilde{R}_i^{-1}} (\log \bar{J}_{R_i})\,\omega \\
&+ \frac{\sqrt{-1}}{3} \sum\nolimits_{\{i,j,k\}=\{1,2,3\}} \int_{\Delta_{ijk}} (\log \bar{J}_{R_jR_k})\omega\ .
\end{aligned}$$

Now

$$\int_{\tilde{R}_i \cap \tilde{R}_i^{-1}} (\log \bar{J}_{R_i})\,\omega = \left(-\frac{2\pi}{5}\sqrt{-1} - 2\pi n\sqrt{-1}\right)(\text{signed area of } \tilde{R}_i \cap \tilde{R}_i^{-1})\ .$$

$\tilde{R}_1 \cap \tilde{R}_1^{-1}$ is the quadrilateral $t_{23}p_{31}t_{32}p_{21}$.



The area of $\tilde{R}_1 \cap \tilde{R}_1^{-1}$ is therefore

$$2\pi - \left(\left(\frac{\pi}{2} - \frac{\pi}{5} + \arg \varphi^3\right) + \left(\frac{\pi}{2} - \frac{\pi}{5} - \arg \varphi^3\right) + \frac{\pi}{5}\right) = \frac{6\pi}{5}\ .$$

Recall that the cyclic permutations $1 \to 2 \to 3$ of the coordinates of $\mathbf{C}^3$ define elements of Γ. Hence some element of Γ maps $\tilde{R}_1 \cap \tilde{R}_1^{-1}$ to $\tilde{R}_2 \cap \tilde{R}_2^{-1}$ with orientation preserved. It follows that

$$(\dagger) \qquad \frac{\sqrt{-1}}{3} \sum\nolimits_{i=1}^{3} \int_{\tilde{R}_i \cap \tilde{R}_i^{-1}} (\log \bar{J}_{R_i})\,\omega = \pm \frac{1}{3}\left(\frac{6\pi}{5} + 6\pi n\right)\frac{6\pi}{5}\ .$$

We have the $\pm$ sign on the right-hand side, because we do not know whether the signed area of $\tilde{R}_i \cap \tilde{R}_i^{-1}$ is $+$ or $-$ the area of $\tilde{R}_i \cap \tilde{R}_i^{-1}$.

The six Δ_{ijk} can be divided into two groups. We give them in the following two diagrams.

$$\begin{tikzcd}[column sep=small]
 & \Delta_{123} = \widetilde{R_2R_3} \cap \widetilde{(R_1R_2)}^{-1} = (t_{13}, s_{23}, \tilde{s}_{21}) \arrow[dr, "R_2R_3"] & \\
\Delta_{312} = \widetilde{R_1R_2} \cap \widetilde{(R_3R_1)}^{-1} = (t_{32}, s_{12}, \tilde{s}_{13}) \arrow[ur, "R_1R_2"] & & \Delta_{231} = \widetilde{R_3R_1} \cap \widetilde{(R_2R_3)}^{-1} = (t_{21}, s_{31}, \tilde{s}_{32}) \arrow[ll, bend left, "R_3R_1"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \Delta_{132} = \widetilde{R_3R_2} \cap \widetilde{(R_1R_3)}^{-1} = (t_{12}, s_{32}, \tilde{s}_{13}) \arrow[dr, "R_3R_2"] & \\
\Delta_{213} = \widetilde{R_1R_3} \cap \widetilde{(R_2R_1)}^{-1} = (t_{23}, s_{13}, \tilde{s}_{21}) \arrow[ur, "R_1R_3"] & & \Delta_{321} = \widetilde{R_2R_1} \cap \widetilde{(R_3R_2)}^{-1} = (t_{31}, s_{21}, \tilde{s}_{32})\ . \arrow[ll, bend left, "R_2R_1"]
\end{tikzcd}$$

These two diagrams are to be interpreted in the same way as the preceding six diagrams.

From the first diagram it follows that

$$\begin{aligned}
&\int_{\Delta_{123}} (\log \bar{J}_{R_2R_3})\omega + \int_{\Delta_{231}} (\log \bar{J}_{R_3R_1})\omega + \int_{\Delta_{321}} (\log \bar{J}_{R_1R_2})\omega \\
&\quad = \int_{\Delta_{123}} (\log \bar{J}_{R_2R_3})\omega + \int_{\Delta_{123}} \big((\log \bar{J}_{R_3R_1}) \circ (R_2R_3)\big)\omega \\
&\qquad + \int_{\Delta_{123}} \big((\log \bar{J}_{R_1R_2}) \circ (R_3R_1R_2R_3)\big)\omega \\
&\quad = \int_{\Delta_{123}} (\log \bar{J}_{(R_1R_2R_3)^2})\omega
\end{aligned}$$

where the branch $\log \bar{J}_{(R_1R_2R_3)^2}$ is defined as

$$\log \bar{J}_{R_2R_3} + (\log \bar{J}_{R_3R_1}) \circ (R_2R_3) + (\log \bar{J}_{R_1R_2}) \circ (R_3R_1R_2R_3)\ ,$$

that is, according to the chain rule.

Likewise from the second diagram it follows that

$$\int_{\Delta_{132}}(\log \bar{J}_{R_3R_2})\omega + \int_{\Delta_{321}}(\log \bar{J}_{R_2R_1})\omega + \int_{\Delta_{213}}(\log \bar{J}_{R_1R_3})\omega = \int_{\Delta_{321}}(\log \bar{J}_{(R_3R_2R_1)^2})\omega$$

where the branch $\log \bar{J}_{(R_3R_2R_1)^2}$ is defined according to the chain rule. We know that

$$J_{(R_1R_2R_3)^2} = \sqrt{-1}\,e^{(3\pi/5)\sqrt{-1}}\bar{\varphi}^9 \quad \text{at} \quad \Delta_{123}\,,$$
$$J_{(R_3R_2R_1)^2} = \sqrt{-1}\,e^{(3\pi/5)\sqrt{-1}}\varphi^9 \quad \text{at} \quad \Delta_{321}\,.$$

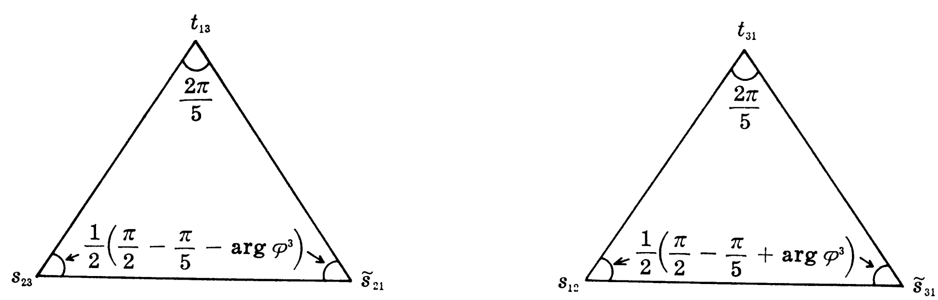
Hence

$$\log \bar{J}_{(R_1R_2R_3)^2} = \sqrt{-1}\left(-\frac{3\pi}{5} - \frac{\pi}{2} + 3\arg\varphi^3 - 12\pi n + 2\pi q\right) \qquad \text{at } \Delta_{123}\,,$$

$$\log \bar{J}_{(R_3R_2R_1)^2} = \sqrt{-1}\left(-\frac{3\pi}{5} - \frac{\pi}{2} - 3\arg\varphi^3 - 12\pi n + 2\pi q'\right) \qquad \text{at } \Delta_{321}\,,$$

where q and q' and integers independent of n, because the branches $\log \bar{J}_{(R_1R_2R_3)^2}$ and $\log \bar{J}_{(R_3R_2R_1)^2}$ are defined from the branches $\log \bar{J}_{R_1}$, $\log \bar{J}_{R_3}$, $\log \bar{J}_{R_3}$ by the chain rule. We will determine q and q' later.

$$\frac{\sqrt{-1}}{3}\sum_{\{i,j,k\}=\{1,2,3\}}\int_{\Delta_{ijk}}(\log \bar{J}_{R_jR_k})\omega$$
$$= -\frac{1}{3}\Bigg(\left(-\frac{3\pi}{5} - \frac{\pi}{2} + 3\arg\varphi^3 - 12\pi n + 2\pi q\right)(\text{signed area of } \Delta_{123})$$
$$+\left(-\frac{3\pi}{5} - \frac{\pi}{2} - 3\arg\varphi^3 - 12\pi n + 2\pi q'\right)(\text{signed area of } \Delta_{321})\Bigg)\,.$$



Δ_{123} Δ_{321}

The area of Δ_{123} is

$$\pi - \left(\frac{2\pi}{5} + \left(\frac{\pi}{2} - \frac{\pi}{5} - \arg\varphi^3\right)\right) = \frac{3\pi}{10} + \arg\varphi^3\,.$$

The area of Δ_{321} is

$$\pi - \left(\frac{2\pi}{5} + \left(\frac{\pi}{2} - \frac{\pi}{5} + \arg\varphi^3\right)\right) = \frac{3\pi}{10} - \arg\varphi^3\,.$$

Hence

$$\frac{\sqrt{-1}}{3}\sum_{\{i,j,k\}=\{1,2,3\}}\int_{\Delta_{ijk}}(\log \bar{J}_{R_jR_k})\omega$$
$$= -\frac{1}{3}\left(\pm\left(-\frac{11\pi}{10} + 3\arg\varphi^3 - 12\pi n + 2\pi q\right)\left(\frac{3\pi}{10} + \arg\varphi^3\right)\right.$$
$$\left.\pm\left(-\frac{11\pi}{10} - 3\arg\varphi^3 - 12\pi n + 2\pi q'\right)\left(\frac{3\pi}{10} - \arg\varphi^3\right)\right).$$

Combining this with (†), we obtain

$$\pm\operatorname{vol}(\Omega) = \pm\frac{1}{3}\left(\frac{6\pi}{5} + 6\pi n\right)\frac{6\pi}{5}$$
$$-\frac{1}{3}\left(\pm\left(-\frac{11\pi}{10} + 3\arg\varphi^3 - 12\pi n + 2\pi q\right)\left(\frac{3\pi}{10} + \arg\varphi^3\right)\right.$$
$$\left.\pm\left(-\frac{11\pi}{10} - 3\arg\varphi^3 - 12\pi n + 2\pi q'\right)\left(\frac{3\pi}{10} - \arg\varphi^3\right)\right).$$

The left-hand side is independent of n. The right-hand side should also be independent of n. Since $|\arg\varphi^3| < \pi/2 - \pi/5$, it follows from

$$0 = \pm\frac{6\pi}{5}n \pm 2\left(\frac{3\pi}{10} + \arg\varphi^3\right)n \pm 2\left(\frac{3\pi}{10} - \arg\varphi^3\right)n$$

that the sign for the first term should be opposite to the signs of the last two terms. Hence

$$(\sharp)\qquad \pm\operatorname{vol}(\Omega) = \frac{1}{3}\left(\frac{6\pi}{5}\right)^2 + \frac{1}{3}\left(-\frac{11\pi}{10} + 3\arg\varphi^3 + 2\pi q\right)\left(\frac{3\pi}{10} + \arg\varphi^3\right)$$
$$+\frac{1}{3}\left(-\frac{11\pi}{10} - 3\arg\varphi^3 + 2\pi q'\right)\left(\frac{3\pi}{10} - \arg\varphi^3\right)$$
$$= \frac{13\pi^2}{50} + 2(\arg\varphi^3)^2 + \frac{2\pi q}{3}\left(\frac{3\pi}{10} + \arg\varphi^3\right)$$
$$+\frac{2\pi q'}{3}\left(\frac{3\pi}{10} - \arg\varphi^3\right).$$

The integer n is introduced only to determine the signs of the signed areas and we have no further use for it. From this point on, we choose $n = 0$.

Now we compute the integers q and q'. By continuity considerations, it suffices to compute q and q' for the special case $\arg\varphi^3 = 0$. q and q' will then have the same values for a general $\arg\varphi^3$. Assume $\varphi = 1$. Let $\eta = e^{\pi\sqrt{-1}/5}$. Then in terms of the homogeneous coordinates $[x_1, x_2, x_3]$,

$$s_{23} = [1, \bar{\eta}\sqrt{-1}, \bar{\eta}\sqrt{-1}],$$
$$\tilde{s}_{21} = [-\eta\sqrt{-1}, \eta\sqrt{-1}, 1].$$

We first calculate q. Since s_{23} belongs to Δ_{123} and since the branch

$\log J_{(R_1R_2R_3)^2}$ is defined from the branches $\log J_{R_j}$ by the chain rule, it suffices to calculate the following values:

$$
\begin{array}{lll}
\log J_{R_3} & \text{at} & s_{23} \\
\log J_{R_2} & \text{at} & R_3 s_{23} = \widetilde{s}_{13} \\
\log J_{R_1} & \text{at} & R_2 \widetilde{s}_{13} = s_{31} \\
\log J_{R_3} & \text{at} & R_1 s_{31} = \widetilde{s}_{21} \\
\log J_{R_2} & \text{at} & R_3 \widetilde{s}_{21} = s_{12} \\
\log J_{R_1} & \text{at} & R_2 s_{12} = \widetilde{s}_{32}\ .
\end{array}
$$

The value of $\log J_{(R_1R_2R_3)^2}$ at s_{23} will then be the sum of these six values. Recall that $a = 1/(2\sin(\pi/5))$. In the homogeneous coordinate system $[x_1, x_2, x_3]$,

$$
\begin{cases}
R_1 x_1 = \eta^2 x_1 - a(\eta^2 - 1)x_2 - a(\eta^2 - 1)x_3 \\
R_1 x_2 = x_2 \\
R_1 x_3 = x_3
\end{cases}
$$

$$
\begin{cases}
R_2 x_1 = x_1 \\
R_2 x_2 = -a(\eta^2 - 1)x_1 + \eta^2 x_2 - a(\eta^2 - 1)x_3 \\
R_2 x_3 = x_3
\end{cases}
$$

$$
\begin{cases}
R_3 x_1 = x_1 \\
R_3 x_2 = x_2 \\
R_3 x_3 = -a(\eta^2 - 1)x_1 - a(\eta^2 - 1)x_2 + \eta^2 x_3\ .
\end{cases}
$$

Hence in the coordinate system $z = (z_1, z_2)$,

$$
\begin{cases}
R_1 z_1 = \eta^2 z_1 - a(\eta^2 - 1)z_2 - a(\eta^2 - 1) \\
R_1 z_2 = z_2
\end{cases}
$$

$$
\begin{cases}
R_2 z_1 = z_1 \\
R_2 z_2 = -a(\eta^2 - 1)z_1 + \eta^2 z_2 - a(\eta^2 - 1)
\end{cases}
$$

$$
\begin{cases}
R_3 z_1 = \dfrac{z_1}{l(z_1, z_2)} \\
R_3 z_2 = \dfrac{z_2}{l(z_1, z_2)}\ ,
\end{cases}
$$

where $l(z_1, z_2) = -a(\eta^2 - 1)z_1 - a(\eta^2 - 1)z_2 + \eta^2$.

$$
\begin{cases}
J_{R_1} \equiv \eta^2 \\
J_{R_2} \equiv \eta^2 \\
J_{R_3} \equiv \dfrac{\eta_2}{l(z_1, z_2)^3}\ .
\end{cases}
$$

It follows that the branches $\log J_{R_1}$, $\log J_{R_2}$ will be identically $\log \eta^2$ on the whole ball B. As multi-valued functions

$$\log J_{R_3} = \log \frac{\eta^2}{l(z_1, z_2)^3} = \log \frac{\eta^2}{\left(-a(\eta^2-1)\right)^3} - 3\log\left(z_1 + z_2 - \frac{\eta^2}{a(\eta^2-1)}\right)$$

$$= \frac{13}{10}\pi\sqrt{-1} - 3\log(z_1 + z_2 + \eta\sqrt{-1}) \text{ because } a(\eta^2 - 1) = \eta\sqrt{-1}.$$

We have to obtain at s_{23} and $\tilde{s}_{21}$ the values of the branch of $\log J_{R_3}$ which assumes the value $\log \eta^2$ at every point of $e_3^{\perp}$. Now $e_3^{\perp}$ is defined by

$$(x_1, x_2, x_3)\begin{pmatrix} 1 & -a & -a \\ -a & 1 & -a \\ -a & -a & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 ;$$

that is,

$$-a(x_1 + x_2) + x_3 = 0$$

or equivalently

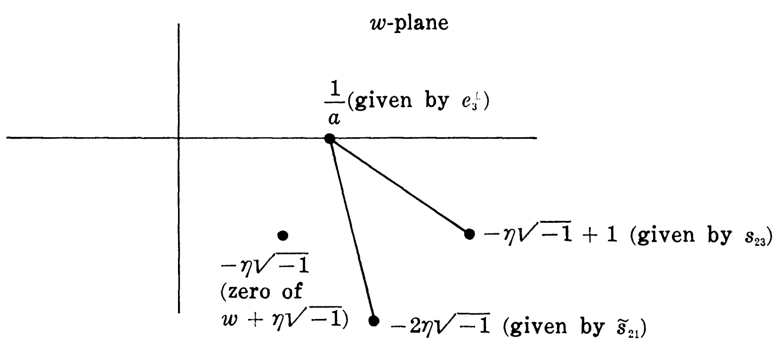
$$z_1 + z_2 = \frac{1}{a} .$$

$$\text{At } s_{23}, \quad z_1 + z_2 = -\eta\sqrt{-1} + 1 .$$
$$\text{At } \tilde{s}_{21}, \quad z_1 + z_2 = -2\eta\sqrt{-1} .$$

A branch of $\log J_{R_3}$ corresponds to a branch of $\log(z_1 + z_2 + \eta\sqrt{-1})$. Our task is now reduced to calculating at $z_1 + z_2 = -\eta\sqrt{-1} + 1$ and $-2\eta\sqrt{-1}$ the values of the branch of $\log(z_1 + z_2 + \eta\sqrt{-1})$ on B which assumes at $z_1 + z_2 = 1/a$ the value

$$\frac{\sqrt{-1}}{3}\left(\frac{13\pi}{10} - \frac{2\pi}{5}\right) = \frac{3\pi}{10}\sqrt{-1} .$$

Let $w = z_1 + z_2$. In B we join a point on $e_3^{\perp}$ to s_{23} by a straight line segment and join a point on $e_3^{\perp}$ to $\tilde{s}_{21}$ by a straight line segment. Their images in the w-plane are straight line segments $[1/a, -\eta\sqrt{-1} + 1]$ and $[1/a, -2\eta\sqrt{-1}]$.



It is clear from the picture that the branch of $\log(z_1 + z_2 + \eta\sqrt{-1}) = \log(w + \eta\sqrt{-1})$ on B, which assumes at $w = 1/a$ the value $(3\pi/10)\sqrt{-1}$, should assume 0 at $w = -\eta\sqrt{-1} + 1$ and assume $-(3\pi/10)\sqrt{-1}$ at $w = -2\eta\sqrt{-1}$. Hence

$$\log J_{R_3}(s_{23}) = \frac{13}{10}\pi\sqrt{-1}$$

$$\log J_{R_3}(\tilde{s}_{21}) = \frac{13}{10}\pi\sqrt{-1} - 3\left(-\frac{3\pi}{10}\sqrt{-1}\right) = \frac{11\pi}{5}\sqrt{-1}$$

and

$$\log J_{(R_1R_2R_3)^2}(s_{23}) = 4\frac{2\pi\sqrt{-1}}{5} + \frac{13\pi\sqrt{-1}}{10} + \frac{11\pi\sqrt{-1}}{5} = \frac{51\pi\sqrt{-1}}{10}.$$

Comparing this with

$$\log \bar{J}_{(R_1R_2R_3)^2} = \sqrt{-1}\left(-\frac{3\pi}{5} - \frac{\pi}{2} + 2\pi q\right) \quad \text{at } \Delta_{123}$$

for the case $\arg\varphi = 0$, we conclude that $q = -2$.

In the case $\arg\varphi = 0$, the roles of the coordinates x_1, x_2 can be interchanged. Hence we conclude also that $q' = -2$.

Since $\operatorname{vol}(\Omega) > 0$ for $\arg\varphi = 0$, the sign in front of $\operatorname{vol}\Omega$ in equation (♯) should be $+$. Finally we have the formula

$$\operatorname{vol}(\Omega) = \frac{27\pi}{50} - 2(\arg\varphi^3)^2.$$

In the case $\arg\varphi^3 = \pi/20$, $\operatorname{vol}(\Omega) = 107\pi^2/200$.

8. Chern class of the normal bundle of a curve

In our computation of c_1^2 for M we have to calculate the Chern class of the normal bundle of a curve. We carry out below the calculation of such Chern classes.

LEMMA 3. *Let the open 2-ball B be realized as $\{(w_1, w_2) \in \mathbf{C}^2 \mid |w_1|^2 + |w_2|^2 < 1\}$. Then any element of* Aut B *which stabilizes $\{w_1 = 0\}$ must be of the form*

$$(w_1, w_2) \longmapsto \left(e^{\sqrt{-1}\alpha}\frac{w_1\sqrt{1-|b|^2}}{\bar{b}w_2 + 1}, e^{\sqrt{-1}\alpha'}\frac{w_2 + b}{\bar{b}w_2 + 1}\right)$$

with $b \in \mathbf{C}$, $|b| < 1$, and $\alpha, \alpha' \in \mathbf{R}$.

Proof. Any element of Aut B is of the form

$$(w_1, w_2) \longmapsto \left(\frac{a_{11}w_1 + a_{12}w_2 + a_{13}}{a_{31}w_1 + a_{32}w_2 + a_{33}}, \frac{a_{21}w_1 + a_{22}w_2 + a_{23}}{a_{31}w_1 + a_{32}w_2 + a_{33}}\right)$$

where $(a_{ij})_{1\leq i,j\leq 3}$ belongs to $SU(2,1)$. If this element of $\operatorname{Aut} B$ stabilizes $\{w_1 = 0\}$, then we have $a_{12} = a_{13} = 0$. It follows that $a_{21} = a_{31} = 0$, $|a_{11}| = 1$, and $\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$ belongs to $U(1,1)$. The lemma follows from the fact that every element of $U(1,1)$ is of the form

$$\begin{pmatrix} \dfrac{e^{\sqrt{-1}\beta}}{\sqrt{1-|b|^2}} & \dfrac{be^{\sqrt{-1}\beta}}{\sqrt{1-|b|^2}} \\ \dfrac{e^{\sqrt{-1}\beta'}\bar{b}}{\sqrt{1-|b|^2}} & \dfrac{e^{\sqrt{-1}\beta'}}{\sqrt{1-|b|^2}} \end{pmatrix}$$

for some $b \in \mathbf{C}$ with $|b| < 1$ and some $\beta, \beta' \in \mathbf{R}$.

Suppose H is a discrete subgroup of $\operatorname{Aut} B$ such that H is fixed point free and every element of H stabilizes $\{w_1 = 0\}$. Let $S = B/H$ and let C be the image of $\{w_1 = 0\}$ in S. We assume that C is compact. Let N be the normal bundle of C in M.

LEMMA 4. *$c_1(N) = (1/2)c_1(T_C)$, where T_C is the tangent bundle of C.*

Proof. Let $L = B \cap \{w_1 = 0\}$. Then L is the universal cover of C and $C = L/H$. Let $\pi\colon L \to C$ be the quotient map.

Let N^* be the dual bundle of N. The sheaf of germs of holomorphic sections of N^* over C is isomorphic to the analytic restriction to C of the ideal-sheaf of C. In other words, $N^* = [C]_C^{-1}$.

Take a connected and simply connected open subset U of C. Let $\pi^{-1}(U) = \bigcup_i U_i$. A holomorphic section of N^* over U corresponds to $\{f_i\}$, where $f_i(w_2)$ is a holomorphic function on U_i such that, if $\gamma \in H$ maps U_i to U_j, then γ maps $f_i(w_2)w_1$ to $f_j(w_2)w_1$. That is, if $\gamma(\zeta_1, \zeta_2) = (w_1, w_2)$ with

$$(*)\qquad \begin{cases} w_1 = \dfrac{e^{\sqrt{-1}\alpha}\zeta_1\sqrt{1-|b|^2}}{\bar{b}\zeta_2 + 1} \\ w_2 = \dfrac{e^{\sqrt{-1}\alpha'}(\zeta_2 + b)}{\bar{b}\zeta_2 + 1}, \end{cases}$$

then

$$f_i(\zeta_2)\zeta_1 = f_j\left(\frac{e^{\sqrt{-1}\alpha'}(\zeta_2 + b)}{\bar{b}\zeta_2 + 1}\right)\frac{e^{\sqrt{-1}\alpha}\sqrt{1-|b|^2}}{\bar{b}\zeta_2 + 1}\zeta_1,$$

or equivalently

$$f_i(\zeta_2) = f_j\left(\frac{e^{\sqrt{-1}\alpha'}(\zeta_2 + b)}{\bar{b}\zeta_2 + 1}\right)\frac{e^{\sqrt{-1}\alpha}\sqrt{1-|b|^2}}{\bar{b}\zeta_2 + 1}.$$

Now we look at the tangent bundle of C. A holomorphic section of T_C over U corresponds to $\{g_i\}$, where $g_i(w_2)$ is a holomorphic function on U_i such that, if $\gamma \in H$ maps U_i to U_j, then γ sends $g_i(w_2)\partial/\partial w_2$ to $g_j(w_2)\partial/\partial w_2$.

That is, if $\gamma(\zeta_1, \zeta_2) = (w_1, w_2)$ is given by (*), then

$$g_i(\zeta_2)\frac{\partial}{\partial\zeta_2} = g_j\left(\frac{e^{\sqrt{-1}\alpha'}(\zeta_2 + b)}{\bar{b}\zeta_2 + 1}\right)\frac{\partial}{\partial w_2}$$

$$= g_j\left(\frac{e^{\sqrt{-1}\alpha'}(\zeta_2 + b)}{\bar{b}\zeta_2 + 1}\right)\frac{1}{\frac{\partial w_2}{\partial\zeta_2}}\frac{\partial}{\partial\zeta_2},$$

or equivalently,

$$g_i(\zeta_2) = g_j\left(\frac{e^{\sqrt{-1}\alpha'}(\zeta_2 + b)}{\bar{b}\zeta_2 + 1}\right)\frac{1}{\frac{e^{\sqrt{-1}\alpha'}(1 - |b|^2)}{(\bar{b}\zeta_2 + 1)^2}}.$$

Let us look at the transition functions of N and T_C. Cover C by an open covering $\{W_\mu\}$ such that both W_μ and $W_\mu \cap W_\nu$ are connected and simply connected. Select a component $\widetilde{W}_\mu$ of $\pi^{-1}(W_\mu)$. When W_μ intersects W_ν, $\widetilde{W}_\mu \cap \pi^{-1}(W_\mu \cap W_\nu)$ and $\widetilde{W}_\nu \cap \pi^{-1}(W_\mu \cap W_\nu)$ are components of $\pi^{-1}(W_\mu \cap W_\nu)$. There exists a unique $\gamma_{\mu\nu} \in H$ mapping $\widetilde{W}_\mu \cap \pi^{-1}(W_\mu \cap W_\nu)$ to $\widetilde{W}_\nu \cap \pi^{-1}(W_\mu \cap W_\nu)$. By identifying holomorphic sections of N^* (respectively T_C) over open subsets of W_μ with holomorphic functions on open subsets of $\widetilde{W}_\mu$ and by applying the preceding arguments to $U = W_\mu \cap W_\nu$, we conclude that if $\gamma_{\mu\nu}$ sends (w_1, w_2) to

$$\left(\frac{e^{\sqrt{-1}\alpha_{\mu\nu}}w_1\sqrt{1 - |b_{\mu\nu}|^2}}{\bar{b}_{\mu\nu}w_2 + 1}, \frac{e^{\sqrt{-1}\alpha'_{\mu\nu}}(w_2 + b_{\mu\nu})}{\bar{b}_{\mu\nu}w_2 + 1}\right),$$

then the transition function from W_ν to W_μ for N^* is

$$\frac{e^{\sqrt{-1}\alpha_{\mu\nu}}\sqrt{1 - |b_{\mu\nu}|^2}}{\bar{b}_{\mu\nu}w_2 + 1}$$

and the transition function from W_ν to W_μ for T_C is

$$\frac{1}{\frac{e^{\sqrt{-1}\alpha'_{\mu\nu}}(1 - |b_{\mu\nu}|^2)}{(\bar{b}_{\mu\nu}w_2 + 1)^2}}.$$

Hence $T_C N^{-2}$ is a flat bundle over C, i.e., the transition functions of $T_C N^{-2}$ can be chosen to be locally constant. Since the Chern class of a flat bundle is zero, it follows that $c_1(N) = (1/2)c_1(T_C)$.

9. Computation of $c_1^2(M)$

Denote by K_B (respectively K_Y, K_M) the canonical line bundle of B (respectively Y, M). Let h be the Hermitian metric for K_B defined by the reciprocal of the Bergman function of B. Let θ be the curvature form of h. Then θ equals $3/(4\pi)$ times the Kähler form of the invariant metric of B (by the invariant metric we mean the metric of B of holomorphic sectional curvature -1). The metric h for K_B defines a metric $\widetilde{h}$ for the line bundle $K_Y - 6[E]$ on Y, because the nonsingular curve E in Y is the singular set of $\sigma\colon Y \to B$ and the vanishing order of the Jacobian determinant of σ along E is 6. Since h is invariant under Γ, $\widetilde{h}$ is invariant under Γ^* and $\widetilde{h}$ defines a metric $\hat{h}$ for the line bundle $K_M - 6\sum_{\nu=1}^{l}[C_\nu]$ on M. The curvature form of $\widetilde{h}$ is equal to $\sigma^*\theta$ and the curvature form of $\hat{h}$ is therefore equal to the form $\hat{\theta}$ induced by $\sigma^*\theta$ on M.

$$c_1^2(K_M - 6\textstyle\sum_{\nu=1}^{l}[C_\nu]) = \displaystyle\int_M \hat{\theta}^2 ,$$

which is equal to $\int_Q (\sigma^*\theta)^2$ for any open subset Q which is mapped one - one onto a set of M whose complement is of measure zero. Since the index of Γ_0^* in Γ^* is k and since the stabilizer in Γ of Ω is of order 3, it follows that

$$\int_Q (\sigma^*\theta)^2 = \frac{k}{3}\int_Q \theta^2 = \frac{2k}{3}\left(\frac{3}{4\pi}\right)^2 \mathrm{vol}(\Omega) .$$

On the other hand,

$$\begin{aligned} &c_1^2(K_M - 6\textstyle\sum_{\nu=1}^{l}[C_\nu]) \\ &\quad = c_1^2(K_M) - 12(K_M - 6\textstyle\sum_{\nu=1}^{l}[C_\nu])\cdot\sum_{\nu=1}^{l}[C_\nu] - 36(\sum_{\nu=1}^{l}[C_\nu])^2 \\ &\quad = c_1^2(K_M) - 12\textstyle\sum_{\nu=1}^{l}\displaystyle\int_{C_\nu}\hat{\theta} - 36(\textstyle\sum_{\nu=1}^{l}[C_\nu])^2 . \end{aligned}$$

Since the curves C_ν are mutually disjoint and the self-intersection number of C_ν equals the Chern number of its normal bundle N_ν in M, it follows that

$$c_1^2(K_M - 6\textstyle\sum_{\nu=1}^{l}[C_\nu]) = c_1^2(K_M) - 12\sum_{\nu=1}^{l}\displaystyle\int_{C_\nu}\hat{\theta} - 36\textstyle\sum_{\nu=1}^{l}c_1(N_\nu) .$$

First we calculate $c_1(N_\nu)$. Recall that $E_i^{(\nu)}$ is a universal covering of C_ν and $\sigma(E_i^{(\nu)})$ is a complex line in B which is the image under an element of Γ of the complex line L containing Δ_{123}. After changing the position of Ω by an element of Γ we can assume without loss of generality that $\sigma(E_i^{(\nu)})$ is the complex line L containing Δ_{123}. Let $(\Gamma_0)_L$ (respectively Γ_L) be the stabilizer in Γ_0 (respectively Γ) of L. The stabilizer $(\Gamma_0^*)_{E_i^{(\nu)}}$ in Γ_0^* of $E_i^{(\nu)}$ corresponds to $(\Gamma_0)_L$ in the correspondence $\Gamma^* \leftrightarrow \Gamma$. Since the quotient space of L by $(\Gamma_0)_L$ is biholomorphic to C_ν, Γ_0 is a fixed point free discrete subgroup of $\mathrm{Aut}\,B$. Let $T = Y/(\Gamma_0^*)_{E_i^{(\nu)}}$ and $S = B/(\Gamma_0)_L$. Let F (respectively A) be the image of $E_i^{(\nu)}$ (respectively L) in T (respectively S). The map $\sigma\colon Y \to B$ induces a map $\tilde{\sigma}\colon T \to S$. $\tilde{\sigma}$ maps F biholomorphically onto A. There exists some open neighborhood U of F in T such that F is the singular set of $\sigma | U$ whose Jacobian determinant vanishes to order 6 along F and U is biholomorphic to an open neighborhood of C_ν in M under some map which sends F to C_ν. Hence N_ν is isomorphic to the normal bundle N_F of F in T and the normal bundle N_A of A in S is isomorphic to the tensor product of 6 copies of N_F. It follows that

$$c_1(N_\nu) = c_1(N_F) = \frac{1}{6} c_1(N_A) .$$

By Section 8, $c_1(N_A) = (1/2)c_1(T_A)$, where T_A is the tangent bundle of A. Hence $c_1(N_\nu) = (1/12)c_1(T_A)$.

Let n be the index of $(\Gamma_0)_L$ in Γ_L. Since Δ_{123} is a fundamental domain in L for Γ_L modulo the stabilizer in Γ of Δ_{123} which is of order 40, it follows that the area of a fundamental domain W in L for $(\Gamma_0)_L$ (calculated with respect to the invariant metric of B) is $n/40$ times the area of Δ_{123}, i.e., $\big(7\pi/(40\times 20)\big)n$. On the other hand, we know that $c_1(T_A)$ equals $-1/(2\pi)$ times the area of A (calculated with respect to the metric of holomorphic sectional curvature -1). This area of A is simply the area of W. Hence

$$c_1(N_\nu) = -\frac{7n}{24 \times 40 \times 20} .$$

Recall that k is the index of Γ_0 in Γ. From the diagram

$$\begin{tikzcd}[column sep=small] & \Gamma \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_0 \arrow[dr, dash] & & \Gamma_L \arrow[dl, dash] \\ & (\Gamma_0)_L = \Gamma_0 \cap \Gamma_L & \end{tikzcd}$$

it follows that the index of $\Gamma_0/(\Gamma_0)_L$ in Γ/Γ_L is k/n. It is easily seen that this index equals l (which is the number of C_ν's). Hence

$$\sum_{\nu=1}^{l} c_1(N_\nu) = -\frac{7k}{24 \times 40 \times 20} .$$

Finally we have to calculate $\int_{C_\nu} \hat{\theta}$. Clearly we have

$$\int_{C_\nu} \hat{\theta} = \int_W \theta = \frac{n}{40}\int_{\Delta_{123}} \theta = \frac{n}{40} \cdot \frac{3}{4\pi}(\text{area of } \Delta_{123}) = \frac{3 \times 7n}{4 \times 40 \times 20}$$

and

$$\sum_{\nu=1}^{l} \int_{C_\nu} \hat{\theta} = \frac{3 \times 7k}{4 \times 40 \times 20} .$$

Combining all these calculations we have

$$c_1^2(K_M) = \frac{k}{3}\left(\frac{3}{4\pi}\right)^2 \frac{214\pi^2}{200} + 12\frac{3 \times 7k}{4 \times 40 \times 20} - 36 \times \frac{7k}{24 \times 40 \times 20} = \frac{852k}{3200}$$

and

$$\frac{c_1^2(M)}{c_2(M)} = \frac{852}{298} .$$

10. An infinite discrete family of similar surfaces

The surface we have constructed above with $p = 5$ (where p is the order of the three complex reflections) and $\arg \varphi^3 = \pi/20$ is only one member of an infinite discrete family of negatively compact Kähler surfaces not covered by the ball which can be constructed in a similar way with $p = 3, 4, 5$ and a discrete set of suitable values of $\arg \varphi^3$. We discuss below the set of suitable values of $\arg \varphi^3$ and compute c_1^2/c_2 for the surface constructed with such a value of $\arg \varphi^3$. Two surfaces from the family with different values of c_1^2/c_2 are not biholomorphic.

We let p take on the values 3, 4, 5, and σ_p be the least positive integer which is $\geqq 1/(1/2 - 1/p)$. The discrete family of surfaces is to be parametrized by a subset of the set of all pairs of numbers of the form (p, σ), where $p = 3, 4, 5$ and σ is an integer $\geqq \sigma_p$. For a given pair (p, σ) define

$$\arg \varphi^3 = \pi\left(\frac{1}{2} - \frac{1}{p} - \frac{2}{\sigma}\right) .$$

With these values p and $\arg \varphi^3$, we can construct a compact Kähler surface with negative sectional curvature by following the construction given in Sections 4 and 5. In this general case,

$$\begin{cases} \angle\, p_{12}t_{32}p_{31} = \dfrac{\pi}{2} - \dfrac{\pi}{p} + \arg \varphi^3 \\ \text{Angle of rotation of } (R_3R_1R_2)^2 = \dfrac{3\pi}{p} + \dfrac{\pi}{2} - 3\arg \varphi^3 , \end{cases}$$

$$\begin{cases} \angle\, p_{12}t_{23}p_{31} = \dfrac{\pi}{2} - \dfrac{\pi}{p} - \arg \varphi^3 \\ \text{Angle of rotation of } (R_2R_1R_3)^2 = \dfrac{3\pi}{p} + \dfrac{\pi}{2} + 3\arg \varphi^3 . \end{cases}$$

Le ρ be the order of $\bar{\eta} i \varphi^3$ with $\eta = e^{\pi\sqrt{-1}/p}$. For the surface we constructed in Sections 4 and 5 with $p = 5$ and $\arg \varphi^3 = \pi/20$, the group $\operatorname{Aut} \Omega$ is of order 3 and consists of the three elements defined by the 3 cyclic permutations $1 \to 2 \to 3$ of the coordinates of $\mathbf{C}^3$. In the general case, the order of $\operatorname{Aut} \Omega$ is 1 or 3 according as both ρ and σ are divisible by 3 or not. Let m be the branching order of the map $Y \to B$ along E. Then

$$\rho\Big(\frac{\pi}{2} - \frac{\pi}{p} + \arg \varphi^3\Big) = 2m\pi\,.$$

The computations of Sections 6 and 7 yield in this general case

$$\frac{c_1^2}{c_2} = \frac{\frac{3}{8}\Big(\frac{3}{2} - \frac{6}{p} + \frac{6}{p^2} - 2t^2\Big) + \frac{5(m-1)}{4\rho}\Big(\frac{1}{2} - \frac{1}{p} + t\Big)}{\frac{1}{3} - \frac{1}{p} + \frac{1}{N} - \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(\frac{1}{\rho} + \frac{1}{\sigma}\Big) + \frac{1}{\rho\sigma}}\,,$$

where $N = 24(p/(6-p))^2$ and $t = 1/2 - 1/p - 2/\sigma$. This ratio c_1^2/c_2 is 3 if and only if $m = 1$. The only pairs (p, σ) that can give $m = 1$ are the following:

$$\begin{cases} p = 3,\ \sigma = 6, 7, 8, 9, 10, 12, 15, 18, 24, 42, \infty \\ p = 4,\ \sigma = 4, 5, 6, 8, 12, 20, \infty \\ p = 5,\ \sigma = 4, 5, 10, 12\,. \end{cases}$$

Our discrete family of surfaces is parametrized by (p, σ) with $\sigma \geqq \sigma_p$ and $m \neq 1$. Two surfaces are biholomorphic (in both the cases $m = 1$ and $m \neq 1$) if and only if their parameters (p, σ) give the same $|t|$.

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