Multiplier ideal sheaves in complex and algebraic geometry

Yum-Tong Siu

Department of Mathematics, Harvard University, Cambridge, MA 02138, USA (email: siu@math.harvard.edu)
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Abstract

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There are two parts in this article. The first part, which is the main part of the article, discusses the application, by the method of multiplier ideal sheaves, of analysis to complex algebraic geometry. The second part discusses the other direction which is the application of complex algebraic geometry to analysis, mainly to problems of estimates and subellipticity for the $\bar{\partial}$ operator.

1 Application of analysis to algebraic geometry

For the application of analysis to algebraic geometry, we will start out with the general technique of reducing problems in algebraic geometry to problems in $\bar{\partial}$ estimates for Stein domains spread over $\mathbb{C}^n$. The L^2 estimates of $\bar{\partial}$ correspond to the algebraic notion of multiplier ideal sheaves. This method of multiplier ideal sheaves has been successfully applied to effective problems in algebraic geometry such as problems related to the Fujita conjecture and the effective Matsusaka big theorem. It has also been applied to solve the conjecture on deformational invariance of plurigenera. There are indications that it might possibly be used to give a solution of the conjecture on the finite generation of the canonical ring. Since the application of the method of multiplier ideals to effective problems in algebraic geometry is better known, we will only very briefly mention such applications. We will explain more the application to the deformational invariance of plurigenera and discuss the techniques and ideas whose detailed implementations may lead to a solution of the conjecture of the finite generation of the canonical ring.

1.1 Algebraic geometric problems reduced to L^2 estimate for domain (spread over $\mathbb{C}^n$)

The following simple procedure, of removing certain hypersurfaces and keeping L^2 estimates, reduces problems in algebraic geometry to problems in $\bar{\partial}$ estimates for Stein domains spread over $\mathbb{C}^n$.

Let X be an n-dimensional complex manifold inside $\mathbb{P}_N$. Let S be some linear $\mathbb{P}_{N-n-1}$ inside $\mathbb{P}_N$ which is disjoint from X. We will use S as the light source for a projection. Let T be some linear $\mathbb{P}_n$ inside $\mathbb{P}_N$ which is disjoint from S. We will use T as the target for a projection. We define a projection $\pi\colon X \to T$ as follows. For $x \in X$ we define $\pi(x) \in T$ as the point of intersection of T with the linear $\mathbb{P}_{N-n}$ in $\mathbb{P}_n$ which contains S and x. Then $\pi\colon X \to T$ makes X a branched cover over T.

Let L be a holomorphic line bundle over X and s be a global (non identically zero) meromorphic section of L over X with pole-set A and zero-set B.

Let Z be some hypersurface inside T which contains the infinity hyperplane of T and contains $\pi(A \cup B)$ so that $\pi\colon X - \pi^{-1}(Z) \to T - Z$ is a local biholomorphism. After we identify $\mathbb{C}^n$ with T minus the infinity hyperplane of T, $X - \pi^{-1}(Z)$ becomes a Stein domain spread over $\mathbb{C}^n$.

Take a metric $e^{-\varphi}$ of L with φ locally bounded from above. For any open subset Ω of X and any holomorphic function f on $\Omega - \pi^{-1}(Z)$ with

$$\int_{\Omega-\pi^{-1}(Z)} |f|^2 \, e^{2\log|s|-\varphi} < \infty,$$

the section fs of L can be extended to a holomorphic section of L over Ω.

Algebraic problems concerning X and L involving sections and cohomology can be translated, through this procedure, to problems concerning functions and forms Ω involving L^2 estimates of $\bar{\partial}$ for the weight function $e^{2\log|s|-\varphi}$.

1.2   Multiplier ideal sheaves and effective problems in algebraic geometry

One important concept to facilitate the translation between algebraic geometry and analysis is that of *multiplier ideal sheaves*. For a plurisubharmonic function φ on an open subset U of $\mathbb{C}^n$ the multiplier ideal sheaf $\mathcal{I}_\varphi$ is defined as the sheaf of germs of holomorphic function-germs f on U such that $|f|^2 e^{-\varphi}$ is locally integrable[1,2]. This concept helps to translate L^2 estimates into algebraic conditions. For a holomorphic line bundle L over a compact complex manifold X with a (possibly singular) metric $e^{-\psi}$ defined by a local plurisubharmonic function ψ, the multiplier ideal sheaf $\mathcal{I}_\psi$ is a coherent ideal sheaf on X.

Later in Section 2 we will discuss another kind of multiplier ideals[3] and modules which arise from formulating, in terms of algebraic conditions, the problems of subelliptic estimates for smooth weakly pseudoconvex domains.

Multiplier ideal sheaves $\mathcal{I}_\psi$ have been used to successfully solve, or make good progress toward the solution of a number of algebraic geometric problems such as the Fujita conjecture, the effective Matsusaka big theorem, and the deformational invariance of the plurigenera. Since the use of multiplier ideal sheaves in effective problems in algebraic geometry has a somewhat longer history and is better known, our discussion of the Fujita conjecture and the effective Matsusaka big theorem will be very brief. We will fo-

cus on the deformational invariance of the plurigenera and the problem of finite generation of the canonical ring which is related to it and in a certain sense motivates it.

1.2.1 Fujita conjecture. The Fujita conjecture[4] states that, if X is a compact complex algebraic manifold of complex dimension n and L is a holomorphic line bundle on X, then $mL + K_X$ is globally free for all $m \geqslant n + 1$ and $mL + K_X$ is very ample for all $m \geqslant n + 2$.

For the first part of global freeness, the conjecture is proved for $n = 2$ by Reider[5], $n = 3$ by Ein-Lazarsfeld[6,7], $n = 4$ by Kawamata[8], general n with weaker $m \geqslant (1 + 2 + \cdots + n) + 1$ (which is of order n^2) instead of $m \geqslant n + 1$ by Angehrn-Siu[9], improved by Helmke[10,11] and by Heier[12] to m of order $n^{4/3}$.

By using the method of higher-order multiplier ideal sheaves the very ampleness part can be proved for $m \geqslant m_n$ with some explicit effective m_n depending on n. The detailed argument for $n = 2$ was given in ref. [13] which can be modified for the case of general dimension n. For the much simpler problem of the very ampleness of $mL + 2K_X$, a bound for m of the order 3^n is easily obtained by using the multiplier ideal sheaves $\mathcal{I}_\psi$ and the fundamental theorem of algebra[2,14−19].

1.2.2 Effective Matsusaka big theorem. For the effective Matsusaka big theorem for general dimension, the best result up to this point is the following[20] (for earlier results see refs. [21,14] and for a more precise bound in dimension 2 see ref. [22]).

Let X be a compact complex manifold of complex dimension n and L be an ample line bundle over X and B be a numerically effective line bundle over X. Then $mL - B$ is very ample for m no less than

$$C_n \left(L^{n-1}\tilde{K}_X \right)^{2^{\max(n-2,0)}} \left(1 + \frac{L^{n-1}\tilde{K}_X}{L^n} \right)^{2^{\max(n-2,0)}},$$

where

$$C_n = 2^{n-1+2^{n-1}} \left(\prod_{k=1}^n \left(k 2^{\frac{(n-k-1)(n-k)}{2}} \right)^{2^{\max(k-2,0)}} \right)$$

and $\tilde{K}_X = \left(2n\binom{3n-1}{n} + 2n + 1 \right) L + B + 2K_X$.

1.3 Background of finite generation of canonical ring

We now very briefly present the background for the problem of the finite generation of the canonical ring and its relation to the deformational invariance of plurigenera.

1.3.1 Pluricanonical bundle. Let X be a complex manifold of complex dimension n. Let K_X be the canonical line bundle so that local holomorphic sections of K_X are local holomorphic n-forms. A local holomorphic section s of $K_X^{\otimes m}$ over X is locally of the form $f \left(dz_1 \wedge \cdots \wedge dz_n \right)^m$, where f is a local holomorphic function and $z_1, \cdots, z_n$ are local coordinates of X. We will use the additive notation mK_X for $K_X^{\otimes m}$.

1.3.2 Blowup of a point. We can blow up the origin 0 of $\mathbb{C}^n$ to form $\widetilde{\mathbb{C}^n}$ which is the

topological closure of the graph of the map $\mathbb{C}^n - \{0\} \to \mathbb{P}_{n-1}$ defined by $(z_1, \cdots, z_n) \mapsto [z_1, \cdots, z_n]$. The projection $\widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ is from the natural projection of the graph onto the domain.

1.3.3 Monoidal transformation.   We can do the blow-up with a parameter space $\mathbb{C}^k$ by using the product $\mathbb{C}^n \times \mathbb{C}^k$ and blowing up $\{0\} \times \mathbb{C}^k$.

For a manifold X and a submanifold D we can blow up D to get another manifold $\tilde{X}$, because locally the pair (X, D) is the same as the pair $\left(\mathbb{C}^n \times \mathbb{C}^k, \{0\} \times \mathbb{C}^k\right)$ (if $\dim_{\mathbb{C}} X = n + k$ and $\dim_{\mathbb{C}} D = k$). This blow-up is called the *monoidal transformation* of X with nonsingular center D.

1.3.4 Resolution of singularities.   Hironaka[23] resolved singularity of a subvariety V of a compact complex manifold X by a finite number of successive monoidal transformations with nonsingular center so that the pullback of V to the final blowup manifold $\tilde{X}$ becomes a finite number of nonsingular hypersurfaces in normal crossing (i.e. they are locally like a subcollection of coordinate hyperplanes).

1.3.5 Space of pluricanonical sections unchanged in blowup and blowdown.   An important property of the pluricanonical line bundle mK_X of a compact complex manifold X is that global holomorphic pluricanonical sections (i.e. elements of $\Gamma(X, mK_X)$) remain global holomorphic pluricanonical sections in the process of blowing up and blowing down.

More generally, if we have a holomorphic map $\pi\colon Y \to X$ between two compact complex manifolds of the same dimension and a subvariety Z of codimension $\geqslant 2$ in X such that f maps $Y - f^{-1}(Z)$ biholomorphically onto $X - Z$, then every element s of $\Gamma(Y, mK_Y)$ comes from the pullback of some element s' of $\Gamma(X, mK_X)$, because the pushforward of $s|_{Y-f^{-1}(Z)}$ is a holomorphic section of the holomorphic line bundle mK_X over $X - Z$ and can be extended across the subvariety Z of codimension $\geqslant 2$ to give a holomorphic section s' of mK_X over all of X.

1.3.6 Canonical ring.   The ring (known as the canonical ring)
$$R(X, K_X) = \bigoplus_{m=0}^{\infty} \Gamma(X, mK_X)$$
is invariant under blow-ups and blow-downs.

Two compact projective algebraic complex manifolds related by blowing-ups and blowing-downs clearly have the same field of meromorphic functions (i.e. are birationally equivalent).

In order to get a representative in a birationally equivalence class which is easier to study, a most important question in algebraic geometry is the existence of good representatives called *minimal models*.

For simplicity, let us focus on complex manifolds which are of general type. A complex manifold X of complex dimension n is *of general type* if $\dim_{\mathbb{C}} \Gamma(X, mK_X) \geqslant cm^n$

for some $c > 0$ and for all m sufficiently large.

1.3.7 Conjecture on finite generation of the canonical ring.   Let X be a compact complex manifold of general type. Let

$$R(X, K_X) = \bigoplus_{m=0}^{\infty} \Gamma(X, mK_X).$$

Then the ring $R(X, K_X)$ is finitely generated.

If the canonical ring $R(X, K_X)$ is finitely generated by elements $s_1, \cdots, s_N$ with $s_j \in \Gamma(X, m_j K_X)$. Let $m_0 = \max_{1 \leqslant j \leqslant N} m_j$. We can use a basis of $\Gamma(X, (m_0!)K_X)$ to define a rational map.

The image Y may not be regular. We expect its canonical line bundle to behave somewhat like that of a manifold. More precisely, its canonical line bundle K_Y is expected to satisfy the following two conditions.

(i) K_Y (defined from the extension of the canonical line bundle of the regular part of Y) is a $\mathbb{Q}$-Cartier divisor (i.e. some positive integral power is a line bundle) and is numerically effective.

(ii) There exists a resolution of singularity $\pi\colon \tilde{Y} \to Y$ such that $K_{\tilde{Y}} = \pi^* K_Y + \sum_j a_j E_j$, where $\tilde{Y}$ is regular and $\{E_j\}_j$ is a collection of hypersurfaces of Y in normal crossing, and $a_j > 0$ and $a_j \in \mathbb{Q}$.

A compact complex variety Y of general type satisfying (i) and (ii) is called a *minimal model* (see ref. [24]). (For the purpose of comparing (ii) with the manifold case, we note that, for any proper surjective holomorphic map $\sigma\colon \tilde{Z} \to Z$ of complex manifolds of the same dimension, $K_{\tilde{Z}} = \sigma^* K_Z + \sum_j b_j F_j$, where b_j is a positive integer and each F_j is a hypersurface of Y.)

1.3.8 Minimal model conjecture (for general type).   Any compact complex manifold X of general type is birational to a minimal model.

The conjecture is known for general threefolds[25] and is still open for general dimension. Analysis offers the possibility of new tools to handle the conjecture.

Some consequences of the conjecture have already been handled with success by new tools in analysis. A prominent example is the proof of the deformational invariance of plurigenera. We will first look at the deformational invariance of plurigenera and its relation to the minimal model conjecture. Then we will return to the problem of the finite generation of the canonical ring later.

1.4 Deformational invariance of plurigenera and the two ingredients for its proof

The most general form of the conjecture on the deformational invariance of plurigenera is for the Kähler case which is stated as follows.

1.4.1 Conjecture on deformational invariance of plurigenera.   Let $\pi\colon X \to \Delta$ be a

holomorphic family of compact complex Kähler manifolds over the unit 1-disk $\Delta \subset \mathbb{C}$. Let $X_t = \pi^{-1}(t)$ for $t \in \Delta$. Then $\dim_{\mathbb{C}} \Gamma\left(X_t, mK_{X_t}\right)$ is independent of t for any positive integer m.

By the semi-continuity of $\dim_{\mathbb{C}} \Gamma\left(X_t, mK_{X_t}\right)$ as a function of t, the main problem is the extension of an element of $\Gamma\left(X_0, mK_{m_0}\right)$ to an element of $\Gamma\left(X, mK_X\right)$. Consider the short exact sequence

$$0 \to \mathcal{O}_X\left(mK_X\right) \xrightarrow{\Theta} \mathcal{O}_X\left(mK_X\right) \to \mathcal{O}_{X_0}\left(mK_{X_0}\right) \to 0,$$

where Θ is defined by multiplication by t. From its long exact cohomology sequence the vanishing of $H^1\left(X, \mathcal{O}_X\left(mK_X\right)\right)$ would give the extension.

Let us first look at the case of general type. If we have a parametrized version of the minimal model conjecture, then we have the answer[26], because pluricanonical sections are independent of blowing-ups and blowing-downs and the minimal models have numerically effective canonical line bundles for which the theorem of Kawamata-Viehweg[27,28] would still hold with the kind of mild singularities of the minimal models.

While the minimal model conjecture and the Kähler case of the deformational invariance of plurigenera are both still open (see refs. [29,30] for some partial results in the Kähler case), the algebraic case has been proved[31,32] by using a "twin-tower argument" and the following two ingredients (see ref. [31], p.664, Prop.1 and p.666, Prop.2).

1.4.2 Global generation of multiplier ideal sheaves (ingredient one). Let L be a holomorphic line bundle over an n-dimensional compact complex manifold Y with a metric which is locally of the form $e^{-\xi}$ with ξ plurisubharmonic. Let $\mathcal{I}_{\xi}$ be the multiplier ideal sheaf of the metric $e^{-\xi}$. Let A be an ample holomorphic line bundle over Y such that for every point P of Y there are a finite number of elements of $\Gamma(Y, A)$ which all vanish to order at least $n + 1$ at P and which do not simultaneously vanish outside P. Then $\Gamma(Y, \mathcal{I}_{\xi} \otimes (L + A + K_Y))$ generates $\mathcal{I}_{\xi} \otimes (L + A + K_Y)$ at every point of Y.

1.4.3 Extension theorem of Ohsawa-Takegoshi type (ingredient two). Let Y be a complex manifold of complex dimension n. Let w be a bounded holomorphic function on Y with nonsingular zero-set Z so that dw is nonzero at every point of Z. Let L be a holomorphic line bundle over Y with a (possibly singular) metric $e^{-\kappa}$ whose curvature current is semipositive. Assume that Y is projective algebraic (or, more generally, assume that there exists a hypersurface V in Y such that $V \cap Z$ is a subvariety of codimension at least 1 in Z and $Y - V$ is the union of a sequence of Stein subdomains Ω_{ν} of smooth boundary and Ω_{ν} is relatively compact in $\Omega_{\nu+1}$). If f is an L-valued holomorphic $(n-1)$-form on Z with

$$\int_Z |f|^2\, e^{-\kappa} < \infty,$$

then $f\, dw$ can be extended to an L-valued holomorphic n-form F on Y such that

$$\int_Y |F|^2\, e^{-\kappa} \leqslant 8\pi\, e\, \sqrt{2 + \frac{1}{e}}\, \left(\sup_Y |w|^2\right) \int_Z |f|^2\, e^{-\kappa}.$$

For the first ingredient we actually need its effective version which is as follows.

1.4.4 Theorem (effective version of global generation of multiplier ideal sheaves). Let L be a holomorphic line bundle over an n-dimensional compact complex manifold Y with a metric which is locally of the form $e^{-\xi}$ with ξ plurisubharmonic. Assume that for every point P_0 of Y one has a coordinate chart $\tilde{U}_{P_0} = \{|z^{(P_0)}| < 2\}$ of Y with coordinates

$$z^{(P_0)} = \left(z_1^{(P_0)}, \cdots, z_n^{(P_0)} \right)$$

centered at P_0 such that the set U_{P_0} of points of $\tilde{U}_{P_0}$, where $\left|z^{(P_0)}\right| < 1$ is relatively compact in $\tilde{U}_{P_0}$. Let ω_0 be a Kähler form of Y. Let C_Y be a positive number such that the supremum norm of $\bar{\partial}z_j^{(P_0)}$ with respect to ω_0 is $\leqslant C_Y$ on U_{P_0} for $1 \leqslant j \leqslant n$. Let $0 < r_1 < r_2 \leqslant 1$. Let A be an ample line bundle over Y with a smooth metric h_A of positive curvature. Assume that, for every point P_0 of Y, there exists a singular metric h_{A,P_0} of A, whose curvature current dominates $c_A \omega_0$ for some positive constant c_A, such that

$$\frac{h_A}{\left|z^{(P_0)}\right|^{2(n+1)}} \leqslant h_{A,P_0}$$

on U_{P_0} and

$$\sup_{r_1 \leqslant |z^{(P_0)}| \leqslant r_2} \frac{h_{A,P_0}(z^{(P_0)})}{h_A(z^{(P_0)})} \leqslant C_{r_1,r_2}$$

and

$$\sup_Y \frac{h_A}{h_{A,P_0}} \leqslant C^{\sharp}$$

for some constants C_{r_1,r_2} and $C^{\sharp} \geqslant 1$ independent of P_0. Let

$$C^{\flat} = 2n\left(\frac{1}{r_1^{2(n+1)}} + 1 + C^{\sharp} \frac{1}{c_A} C_{r_1,r_2} \left(\frac{2\, r_2\, C_Y}{r_2^2 - r_1^2} \right)^2 \right).$$

Let $0 < r < 1$ and let

$$\hat{U}_{P_0,r} = U_{P_0} \cap \left\{ \left|z^{(P_0)}\right| < \frac{r}{n\sqrt{C^{\flat}}} \right\}.$$

Let N be the complex dimension of the subspace of all elements

$$s \in \Gamma\left(Y, L + K_Y + A\right)$$

such that

$$\int_Y |s|^2\, e^{-\xi}\, h_A < \infty.$$

Then there exist

$$s_1, \cdots, s_N \in \Gamma\left(Y, L + K_Y + A\right)$$

with

$$\int_Y |s_k|^2\, e^{-\xi} h_A \leqslant 1$$

$(1 \leqslant k \leqslant N)$ such that, for any $P_0 \in Y$ and for any holomorphic section s of $L + K_Y + A$ over U_{P_0} with

$$\int_{U_{P_0}} |s|^2 e^{-\xi} h_A = C_s < \infty,$$

one can find holomorphic functions $b_{P_0,k}$ on $\hat{U}_{P_0,r}$ such that

$$s = \sum_{k=1}^{N} b_{P_0,k} \, s_k$$

on $\hat{U}_{P_0,r}$ and

$$\sup_{\hat{U}_{P_0,r}} \sum_{k=1}^{N} |b_{P_0,k}|^2 \leqslant C^\natural \, C_s,$$

where $C^\natural = C^\flat/(1-r)^2$.

Theorem (1.4.4) was proved in ref. [32] (p.234, Th. 2.1) for the case where the line bundle L is mK_Y. The proof there works for the case of a general line bundle L. In its application here and in ref. [32] also, only the case of $L = mK_Y$ is used.

1.5   Twin tower argument for invariance of plurigenera

The detailed proof of the deformational invariance of the plurigenera for the projective algebraic case was presented in refs. [32,33]. The proofs given there are for more general settings. In order to more transparently present the essence of the argument, we give here the proof just for the original conjecture without including any setting which is more general.

We now start the proof of the deformational invariance of the plurigenera for the general case of projective algebraic manifolds not necessarily of general type.

Let m_0 be a positive integer. Take $s^{(m_0)} \in \Gamma\left(X_0, m_0 K_X\right)$. For the proof of the deformational invariance of the plurigenera we have to extend $s^{(m_0)}$ to an element of $\Gamma\left(X, m_0 K_X\right)$. Let A be a positive line bundle over X which is sufficiently positive for the purpose of the global generation of multiplier ideal sheaves in the sense of (1.4.2) and (1.4.4). Let h_A be a smooth metric for A with positive curvature form on X. Let s_A be a global holomorphic section of A over X whose restriction to X_0 is not identically zero.

1.5.1   Use of most singular metric still giving finite norm of given initial section.   Let $\psi = \frac{1}{m_0} \log \left|s^{(m_0)}\right|^2$. Fix arbitrarily $l \in \mathbb{N}$. Let $m_1 = l m_0$. For $1 \leqslant p < m_1$, let $\varphi_{p-1} = (p-1)\psi$ and let N_p be the complex dimension of the subspace of all elements $s \in \Gamma\left(X_0, p K_{X_0} + A\right)$ such that

$$\int_{X_0} |s|^2 \, e^{-\varphi_{p-1}} \, h_A < \infty.$$

Let $\tilde{N} = \sup_{1 \leqslant p < m_1} N_p$. Then $\tilde{N}$ is bounded independently of l. This supremum bound being independent of l is a key point in this proof. (The metric $e^{-\psi}$ is chosen to be as singular as possible so as to make N_p as small as possible to guarantee the finiteness of $\tilde{N}$, and yet the initial section $s^{(m_0)}$ still has finite L^2 norm with respect to the appropriate power of $e^{-\psi}$.)

1.5.2   Local trivializations of line bundles.   Let $\{U_\lambda\}_{1 \leqslant \lambda \leqslant \Lambda}$ be a finite covering of X_0 by Stein open subsets such that

(i) for some U_λ^* which contains U_λ as a relatively compact subset, the assumptions and conclusions of Theorem (1.4.4) are satisfied with each U_λ being some $\hat{U}_{P_0,r}$ from Theorem (1.4.4) when L with metric $e^{-\xi}$ on Y is replaced by $p\,K_{X_0}$ with metric $e^{-p\,\psi}$ on X_0 for $0 \leqslant p \leqslant m_1 - 1$, and

(ii) for $1 \leqslant \lambda \leqslant \Lambda$, there exists nowhere zero
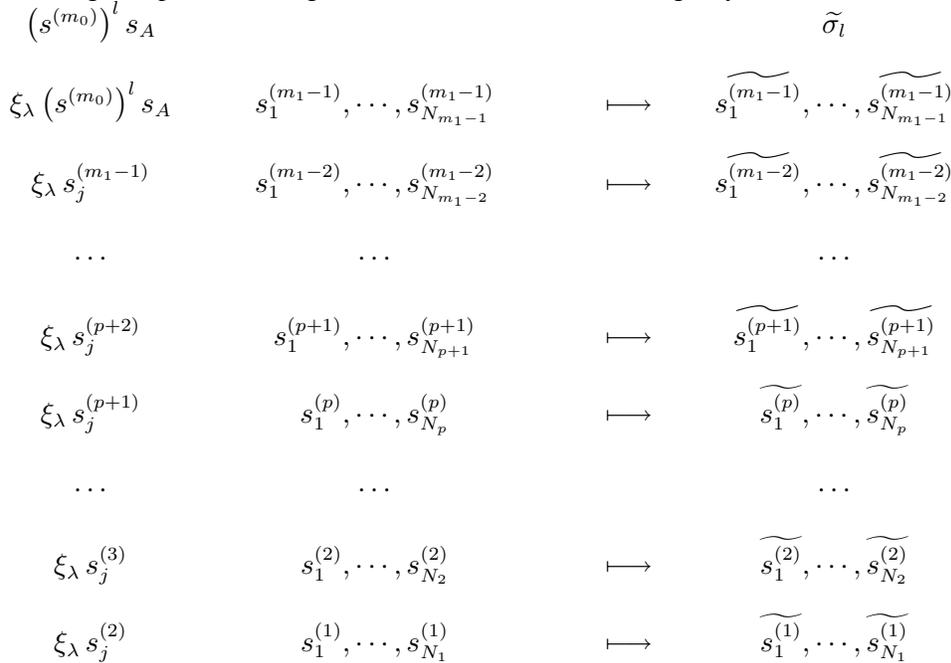
$$\xi_\lambda \in \Gamma\left(U_\lambda^{**}, -K_{X_0}\right)$$

for some open subset U_λ^{**} of X_0 which contains U_λ^* as a relatively compact subset.

1.5.3 Diagram of "twin tower argument." First we schematically explain the "twin tower argument" and then we give the details about estimates and convergence. For the "twin tower argument" we start with $\left(s^{(m_0)}\right) s_A$ for a large integer l at the upper right-hand corner of the following picture and goes down the tower left of the column

$$\longmapsto$$
$$\longmapsto$$
$$\cdots$$
$$\longmapsto$$
$$\longmapsto$$
$$\cdots$$
$$\longmapsto$$
$$\longmapsto$$
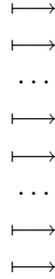
and then goes up the tower right of the column in the following way.

$\left(s^{(m_0)}\right)^l s_A$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\widetilde{\sigma}_l$

$\xi_\lambda \left(s^{(m_0)}\right)^l s_A \qquad s_1^{(m_1-1)}, \cdots, s_{N_{m_1-1}}^{(m_1-1)} \qquad \longmapsto \qquad \widetilde{s_1^{(m_1-1)}}, \cdots, \widetilde{s_{N_{m_1-1}}^{(m_1-1)}}$

$\xi_\lambda\, s_j^{(m_1-1)} \qquad s_1^{(m_1-2)}, \cdots, s_{N_{m_1-2}}^{(m_1-2)} \qquad \longmapsto \qquad \widetilde{s_1^{(m_1-2)}}, \cdots, \widetilde{s_{N_{m_1-2}}^{(m_1-2)}}$

$\cdots \qquad\qquad\qquad \cdots \qquad\qquad\qquad\qquad\qquad \cdots$

$\xi_\lambda\, s_j^{(p+2)} \qquad s_1^{(p+1)}, \cdots, s_{N_{p+1}}^{(p+1)} \qquad \longmapsto \qquad \widetilde{s_1^{(p+1)}}, \cdots, \widetilde{s_{N_{p+1}}^{(p+1)}}$

$\xi_\lambda\, s_j^{(p+1)} \qquad s_1^{(p)}, \cdots, s_{N_p}^{(p)} \qquad \longmapsto \qquad \widetilde{s_1^{(p)}}, \cdots, \widetilde{s_{N_p}^{(p)}}$

$\cdots \qquad\qquad\qquad \cdots \qquad\qquad\qquad\qquad\qquad \cdots$

$\xi_\lambda\, s_j^{(3)} \qquad s_1^{(2)}, \cdots, s_{N_2}^{(2)} \qquad \longmapsto \qquad \widetilde{s_1^{(2)}}, \cdots, \widetilde{s_{N_2}^{(2)}}$

$\xi_\lambda\, s_j^{(2)} \qquad s_1^{(1)}, \cdots, s_{N_1}^{(1)} \qquad \longmapsto \qquad \widetilde{s_1^{(1)}}, \cdots, \widetilde{s_{N_1}^{(1)}}$

1.5.4 Going down the left tower. We start out from the top of the left tower with

$$\left(s^{(m_0)}\right)^l s_A \in \Gamma\left(X_0, m_1 K_{X_0} + A\right).$$

We descend one level down the left tower by locally multiplying it by ξ_λ to form $\xi_\lambda(s^{(m_0)})^l s_A$. By Theorem (1.4.4) (and because each U_λ equals some $\hat{U}_{P_0,r}$), we can write

$$\xi_\lambda \left(s^{(m_0)}\right)^l s_A = \sum_{k=1}^{N_{m_1-1}} b_k^{(m_1-1,\lambda)} s_k^{(m_1-1)} \tag{1.5.4.1}$$

on U_λ, where $b_k^{(m_1-1,\lambda)}$ are holomorphic functions on U_λ and where

$$s_1^{(m_1-1)}, \cdots, s_{N_p}^{(m_1-1)} \in \Gamma\left(X_0, (m_1-1)K_{X_0} + A\right)$$

with

$$\int_{X_0} \left|s_k^{(m_1-1)}\right|^2 e^{-(m_1-2)\psi} h_A \leqslant 1.$$

We now inductively go down the left tower one level at a time. For $1 \leqslant p \leqslant m_1 - 2$, at the $(p+1)$-st level we have

$$s_j^{(p+1)} \in \Gamma\left(X_0, (p+1)K_{X_0} + A\right)$$

and we descend one level down the left tower to the p-th level by locally multiplying it by ξ_λ to form $\xi_\lambda s_j^{(p+1)}$. Again, by Theorem (1.4.4) (and because each U_λ equals some $\hat{U}_{P_0,r}$), we can write

$$\xi_\lambda s_j^{(p+1)} = \sum_{k=1}^{N_p} b_{j,k}^{(p,\lambda)} s_k^{(p)} \tag{1.5.4.2$_p$}$$

on U_λ, where $b_{j,k}^{(p,\lambda)}$ are holomorphic functions on U_λ and where

$$s_1^{(p)}, \cdots, s_{N_p}^{(p)} \in \Gamma\left(X_0, p\,K_{X_0} + A\right)$$

with

$$\int_{X_0} \left|s_k^{(p)}\right|^2 e^{-(p-1)\psi} h_A \leqslant 1.$$

When we get to the bottom of the left tower, the value of p becomes 1.

Theorem (1.4.4) gives us the estimates:

$$\sup_{U_\lambda} \sum_{k=1}^{N_{m_1-1}} \left|b_k^{(m_1-1,\lambda)}\right|^2 \leqslant C^\natural,$$

$$\sup_{U_\lambda} \sum_{k=1}^{N_p} \left|b_{j,k}^{(p,\lambda)}\right|^2 \leqslant C^\natural$$

for $1 \leqslant p \leqslant m_1 - 2$. From these estimates and $(1.5.4.1)$ and $(1.5.4.2)_p$ we obtain the following estimates:

$$\int_{X_0} \frac{\left|\left(s^{(m_0)}\right)^l s_A\right|^2}{\max_{1\leqslant j\leqslant N_{m_1-1}} \left|\widetilde{s_j^{(m_1-1)}}\right|^2} \leqslant C^\natural, \tag{1.5.4.3}$$

$$\int_{X_0} \frac{\left| s_j^{(p+1)} \right|^2}{\max\limits_{1 \leqslant j \leqslant N_p} \left| \widetilde{s_j^{(p)}} \right|^2} \leqslant C^{\natural} \qquad (1.5.4.4)_p$$

for $1 \leqslant p \leqslant m_1 - 2$.

1.5.5 Going up the right tower. Now at the bottom level we move from the left tower to the right tower and then move up the right tower one level at a time, by using the extension theorem of Ohsawa-Takegoshi type (1.4.3).

Let $C^{\sharp} = 8\pi\, e \sqrt{2 + \frac{1}{e}}$. At the bottom level of $p = 1$, because of

$$\int_{X_0} \left| s_j^{(1)} \right|^2 h_A \leqslant 1,$$

we can extend

$$s_j^{(1)} \in \Gamma\left(X_0, K_{X_0} + A\right)$$

to

$$\widetilde{s_j^{(1)}} \in \Gamma\left(X, K_X + A\right)$$

with

$$\int_X \left| \widetilde{s_j^{(1)}} \right|^2 h_A \leqslant C^{\sharp}.$$

In the picture of the "twin tower" the line

$$s_1^{(1)}, \cdots, s_{N_1}^{(1)} \longmapsto \widetilde{s_1^{(1)}}, \cdots, \widetilde{s_{N_1}^{(1)}}$$

signifies the extension of $s_j^{(1)}$ to $\widetilde{s_j^{(1)}}$.

Inductively we are going to move up one level at a time on the right tower. Suppose we have already moved up to the p-th level for $1 \leqslant p \leqslant m_1 - 2$ so that we have the extension of

$$s_j^{(p)} \in \Gamma\left(X_0, pK_{X_0} + A\right)$$

to

$$\widetilde{s_j^{(p)}} \in \Gamma\left(X, pK_X + A\right).$$

We use

$$\frac{1}{\max\limits_{1 \leqslant j \leqslant N_p} \left| \widetilde{s_j^{(p)}} \right|^2}$$

as the metric for $pK_X + A$ on X and apply the extension theorem of Ohsawa-Takegoshi type (1.4.3). From the estimate $(1.5.4.4)_p$ we can extend

$$s_j^{(p+1)} \in \Gamma\left(X_0, (p+1)K_{X_0} + A\right)$$

to

$$\widetilde{s_j^{(p+1)}} \in \Gamma\left(X, (p+1)K_X + A\right)$$

with

$$\int_X \frac{\left|\widetilde{s_j^{(p+1)}}\right|^2}{\max\limits_{1\leqslant k\leqslant N_p}\left|\widetilde{s_k^{(p)}}\right|^2} \leqslant C^\natural C^\sharp. \qquad (1.5.5.1)_p$$

In the picture of the "twin tower" the line

$$s_1^{(p+1)},\cdots,s_{N_{p+1}}^{(p+1)} \longmapsto \widetilde{s_1^{(p+1)}},\cdots,\widetilde{s_{N_{p+1}}^{(p+1)}}$$

signifies the extension of $s_j^{(p+1)}$ to $\widetilde{s_j^{(p+1)}}$.

When we get to the second highest level on the right tower signified by the line

$$s_1^{(m_1-1)},\cdots,s_{N_{m_1-1}}^{(m_1-1)} \longmapsto \widetilde{s_1^{(m_1-1)}},\cdots,\widetilde{s_{N_{m_1-1}}^{(m_1-1)}},$$

we can use

$$\frac{1}{\max\limits_{1\leqslant j\leqslant N_{m_1-1}}\left|\widetilde{s_j^{(m_1-1)}}\right|^2}$$

as the metric for $(m_1 - 1)K_X + A$ on X and apply the extension theorem of Ohsawa-Takegoshi type (1.4.3). From the estimate (1.5.4.3) we can extend

$$\left(s^{(m_0)}\right)^l s_A \in \Gamma\left(X_0, m_1 K_{X_0} + A\right)$$

to

$$\widetilde{\sigma}_l \in \Gamma\left(X, m_1 K_X + A\right)$$

with

$$\int_X \frac{|\widetilde{\sigma}_l|^2}{\max\limits_{1\leqslant k\leqslant N_{m_1-1}}\left|\widetilde{s_k^{(m_1-1)}}\right|^2} \leqslant C^\natural C^\sharp. \qquad (1.5.5.2)$$

1.5.6   Limit metric and its convergence.   We are going to use

$$\frac{1}{\limsup\limits_{l\to\infty}|\widetilde{\sigma}_l|^{\frac{2}{lm_0}}}$$

to define a (possibly singular) metric $e^{-\chi}$ for K_X with χ plurisubharmonic so that from

$$\int_{X_0}\left|s^{(m_0)}\right|^2 e^{-(m_0-1)\chi} < \infty,$$

we get an extension $\widetilde{s^{(m_0)}} \in \Gamma\left(X, m_0 K_X\right)$ of $s^{(m_0)}$.

1.6   Convergence argument with the most singular metric

We need the convergence of

$$\limsup\limits_{l\to\infty}|\widetilde{\sigma}_l|^{\frac{2}{lm_0}},$$

which has to come from the L^2 estimates $(1.5.5.1)_p$ and $(1.5.5.2)$. Since the L^2 estimate of each factor in a product (of at least two factors) would not be able to yield any estimate of the product, we are forced to use the concavity of the logarithmic function to convert

quotients to differences. For this process it is essential that the dimension bound $\tilde{N}$ is independent of l. Finally we get pointwise estimates from integral estimates by using the subharmonicity of the logarithm of the absolute value of a holomorphic function. The details of the estimates are as follows.

1.6.1   Key point of using the most singular allowable metric and uniform bound of dimensions of spaces of sections.   For nonnegative valued functions $g_1, \cdots, g_N$, in general the coefficient N on the right-hand side of the inequality

$$\int \max \left(g_1, \cdots, g_N\right) \leqslant N \max \left(\int g_1, \cdots, \int g_N\right)$$

cannot be lowered, because the supports of $g_1, \cdots, g_N$ may all be disjoint or close to being disjoint. So from $(1.5.5.1)_p$ we get

$$\int_X \frac{\max\limits_{1 \leqslant j \leqslant N_{p+1}} \left|\widetilde{s_j^{(p+1)}}\right|^2}{\max\limits_{1 \leqslant k \leqslant N_p} \left|\widetilde{s_k^{(p)}}\right|^2} \leqslant N_{p+1} \, C^\natural \, C^\sharp,$$

where the factors N_{p+1} on the right-hand side cannot be lowered in general. Since $\tilde{N} = \sup_{1 \leqslant p < m_1} N_p$, it follows that

$$\int_X \frac{\max\limits_{1 \leqslant j \leqslant N_{p+1}} \left|\widetilde{s_j^{(p+1)}}\right|^2}{\max\limits_{1 \leqslant k \leqslant N_p} \left|\widetilde{s_k^{(p)}}\right|^2} \leqslant \tilde{N} \, C^\natural \, C^\sharp. \qquad (1.6.1.1)_p$$

The main point is that the right-hand side $\tilde{N} \, C^\natural \, C^\sharp$ is now independent of p, which is essential for the estimate in passing to the final limit metric.

1.6.2   Preliminary notations for local trivializations of line bundles.   Let $0 < r_0 < r_1 < r_2 < 1$. Choose a finite number of coordinate charts $\tilde{W}_\lambda$ in X with coordinates

$$\left(z^{(\lambda)}, t\right) = \left(z_1^{(\lambda)}, \cdots, z_n^{(\lambda)}, t\right)$$

for $1 \leqslant \lambda \leqslant \hat{\Lambda}$ such that

(i) each

$$W_\lambda'' := \left\{\left|z_1^{(\lambda)}\right| < r_2, \cdots, \left|z_n^{(\lambda)}\right| < r_2, |t| < r_2\right\}$$

is relatively compact in $\tilde{W}_\lambda$ for $1 \leqslant \lambda \leqslant \hat{\Lambda}$,

(ii) $X \cap \{|t| < r_2\} = \bigcup_{\lambda=1}^{\hat{\Lambda}} W_\lambda''$,

(iii) $X \cap \{|t| < r_0\} = \bigcup_{\lambda=1}^{\hat{\Lambda}} W_\lambda$, where
$$W_\lambda = \left\{\left|z_1^{(\lambda)}\right| < r_0, \cdots, \left|z_n^{(\lambda)}\right| < r_0, |t| < r_0\right\}$$

for $1 \leqslant \lambda \leqslant \hat{\Lambda}$, and

(iv) there exist nowhere zero

$$\hat{\tau}_{\lambda,A} \in \Gamma\left(W_\lambda'', A\right), \quad \hat{\xi}_\lambda \in \Gamma\left(W_\lambda'', -K_X\right)$$

for $1 \leqslant \lambda \leqslant \hat{\Lambda}$.

Let $dV_{z^{(\lambda)},t}$ be the Euclidean volume form in the coordinates system $\left(z^{(\lambda)}, t \right)$. Let

$$W'_\lambda = \left\{ \left| z_1^{(\lambda)} \right| < r_1, \cdots, \left| z_n^{(\lambda)} \right| < r_1, |t| < r_1 \right\}.$$

1.6.3   Concavity of logarithm for conversion from integrals of quotients to differences of integrals.   From the concavity of the logarithmic function we conclude that

$$\frac{1}{(\pi r_1^2)^{n+1}} \int_{W'_\lambda} \left(\log \max_{1 \leqslant j \leqslant N_{p+1}} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{p+1}\, \tilde{s}_j^{(p+1)} \right|^2 \right) dV_{z^{(\lambda)},t}$$

$$- \frac{1}{(\pi r_1^2)^{n+1}} \int_{W'_\lambda} \left(\log \max_{1 \leqslant k \leqslant N_p} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{p}\, \tilde{s}_k^{(p)} \right|^2 \right) dV_{z^{(\lambda)},t}$$

$$= \frac{1}{(\pi r_1^2)^{n+1}} \int_{W'_\lambda} \log \frac{\max_{1 \leqslant j \leqslant N_{p+1}} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{p+1}\, \tilde{s}_j^{(p+1)} \right|^2}{\max_{1 \leqslant k \leqslant N_p} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{p}\, \tilde{s}_k^{(p)} \right|^2} dV_{z^{(\lambda)},t}$$

$$\leqslant \log \left(\frac{1}{(\pi r_1^2)^{n+1}} \int_{W'_\lambda} \frac{\max_{1 \leqslant j \leqslant N_{p+1}} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{p+1}\, \tilde{s}_j^{(p+1)} \right|^2}{\max_{1 \leqslant k \leqslant N_p} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{p}\, \tilde{s}_k^{(p)} \right|^2} dV_{z^{(\lambda)},t} \right)$$

$$\leqslant \log \left(\sup_{W'_\lambda} \left(\frac{1}{(\pi r_1^2)^{n+1}} \left| \hat{\xi}_\lambda \right|^2 dV_{z^{(\lambda)},t} \right) \int_{W'_\lambda} \frac{\max_{1 \leqslant j \leqslant N_{p+1}} \left| \tilde{s}_j^{(p+1)} \right|^2}{\max_{1 \leqslant k \leqslant N_p} \left| \tilde{s}_k^{(p)} \right|^2} \right)$$

$$\leqslant \log \left(\tilde{N}\, C^\natural\, C^\sharp \sup_{W'_\lambda} \left(\frac{1}{(\pi r_1^2)^{n+1}} \left| \hat{\xi}_\lambda \right|^2 dV_{z^{(\lambda)},t} \right) \right)$$

for $1 \leqslant p \leqslant m_1 - 2$, where $(1.6.1.1)_p$ is used for the last inequality. Likewise

$$\frac{1}{(\pi r_1^2)^{n+1}} \int_{W'_\lambda} \left(\log \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{m_1}\, \tilde{\sigma}_l \right|^2 \right) dV_{z^{(\lambda)},t}$$

$$- \frac{1}{(\pi r_1^2)^{n+1}} \int_{W'_\lambda} \left(\log \max_{1 \leqslant k \leqslant N_{m_1-1}} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{m_1-1}\, \tilde{s}_k^{(m_1-1)} \right|^2 \right) dV_{z^{(\lambda)},t}$$

$$\leqslant \log \left(C^\natural\, C^\sharp \sup_{W'_\lambda} \left(\frac{1}{(\pi r_1^2)^{n+1}} \left| \hat{\xi}_\lambda \right|^2 dV_{z^{(\lambda)},t} \right) \right).$$

Adding up, we get

$$\frac{1}{(\pi r_1^2)^{n+1}} \int_{W'_\lambda} \left(\log \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{m_1}\, \tilde{\sigma}_l \right|^2 \right) dV_{z^{(\lambda)},t}$$

$$\leqslant \frac{1}{(\pi r_1^2)^{n+1}} \int_{W'_\lambda} \left(\log \max_{1 \leqslant k \leqslant N_1} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda\, \tilde{s}_k^{(1)} \right|^2 \right) dV_{z^{(\lambda)},t}$$

$$+ (m_1 - 1) \log \left(\tilde{N}\, C^\natural\, C^\sharp \sup_{W'_\lambda} \left(\frac{1}{(\pi r_1^2)^{n+1}} \left| \hat{\xi}_\lambda \right|^2 dV_{z^{(\lambda)},t} \right) \right).$$

1.6.4 Use of sub-mean-value property of logarithm of absolute-value of holomorphic functions.

By the sub-mean-value property of plurisubharmonic functions

$$\sup_{W_\lambda} \log \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{m_1}\, \tilde{\sigma}_l \right|^2$$

$$\leqslant \frac{1}{\left(\pi\, (r_1 - r_0)^2 \right)^{n+1}} \int_{W_\lambda'} \left(\log \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{m_1}\, \tilde{\sigma}_l \right|^2 \right) dV_{z^{(\lambda)},t}.$$

Choose a positive number $C^\spadesuit$ such that $\frac{1}{m_0} \log C^\spadesuit$ is no less than

$$\left(\frac{r_1}{r_1 - r_0} \right)^{2(n+1)} \log \left(\tilde{N}\, C^\natural\, C^\sharp \sup_{W_\lambda'} \left(\frac{1}{(\pi r_1^2)^{n+1}} \left| \hat{\xi}_\lambda \right|^2 dV_{z^{(\lambda)},t} \right) \right)$$

for every $1 \leqslant \lambda \leqslant \Lambda$ and $1 \leqslant p \leqslant m_1 - 1$. Since $\tilde{N}$ is bounded independently of l, we can choose $C^\spadesuit$ to be independent of l. Let $\hat{C}$ be defined by

$$\log \hat{C} = \frac{1}{(\pi r_1)^{n+1}} \int_{W_\lambda'} \left(\log \max_{1 \leqslant k \leqslant N_1} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda\, \tilde{s}_k^{(1)} \right|^2 \right) dV_{z^{(\lambda)},t}.$$

Then

$$\sup_{1 \leqslant \lambda \leqslant \hat{\Lambda}} \sup_{W_\lambda} \left| \hat{\tau}_{\lambda,A}\, \hat{\xi}_\lambda^{l\, m_0}\, \tilde{\sigma}_l \right|^2 \leqslant \hat{C}\, \left(C^\spadesuit \right)^{l\, m_0 - 1}.$$

1.6.5 Final step of construction of limit metric.

For $1 \leqslant \lambda \leqslant \hat{\Lambda}$, let χ_λ be the function on W_λ which is the upper semi-continuous envelope of

$$\limsup_{l \to \infty} \log \left| \hat{\xi}_\lambda^{l\, m_0}\, \tilde{\sigma}_l \right|^{2/l}$$

which is $\leqslant m_0 \log C^\spadesuit$. From the definition of χ_λ and the fact that $\tilde{\sigma}_l$ is the extension of $\left(s^{(m_0)} \right)^l s_A$, we have

$$\sup_{X_0 \cap W_\lambda} \left(\left| \hat{\xi}_\lambda^{m_0}\, s^{(m_0)} \right|^2 e^{-\chi_\lambda} \right) \leqslant 1$$

for $1 \leqslant \lambda \leqslant \hat{\Lambda}$. Let $e^{-\chi} = |\hat{\xi}_\lambda|^2 e^{-\frac{\chi_\lambda}{m_0}}$ be the metric of K_X on $X \cap \{|t| < r\}$ so that the square of the pointwise norm of a local section σ of K_X on W_λ is $|\sigma \hat{\xi}_\lambda|^2 e^{-\chi_\lambda}$.

Let $\{\rho_\lambda\}_{1 \leqslant \lambda \leqslant \tilde{\Lambda}}$ be a partition of unity subordinate to the open cover $\{X_0 \cap W_\lambda\}_{1 \leqslant \lambda \leqslant \tilde{\Lambda}}$ of X_0. Since

$$\left| s^{(m_0)} \right|^2 e^{-(m_0-1)\chi} = \left| s^{(m_0)} \right|^2 \left| \tilde{\xi}_\lambda \right|^{2(m_0-1)} e^{-\left(\frac{m_0-1}{m_0} \right)\chi_\lambda}$$

$$= \left| s^{(m_0)}\, \tilde{\xi}_\lambda^{m_0} \right|^2 \left(\frac{e^{-\left(\frac{m_0-1}{m_0} \right)\chi_\lambda}}{|\tilde{\xi}_\lambda|^2} \right) \leqslant \frac{e^{-\left(\frac{m_0-1}{m_0} \right)\chi_\lambda}}{|\tilde{\xi}_\lambda|^2}$$

on $X_0 \cap W_\lambda$ for $1 \leqslant \lambda \leqslant \tilde{\Lambda}$, it follows that

$$\int_{X_0} \left| s^{(m_0)} \right|^2 e^{-(m_0-1)\chi} = \sum_{\lambda=1}^{\tilde{\Lambda}} \int_{X_0 \cap W_\lambda} \rho_\lambda \left| s^{(m_0)} \right|^2 e^{-(m_0-1)\chi}$$

$$\leqslant \sum_{\lambda=1}^{\tilde{\Lambda}} \int_{X_0 \cap W_\lambda} \rho_\lambda \left(\frac{e^{-\left(\frac{m_0-1}{m_0} \right)\chi_\lambda}}{|\tilde{\xi}_\lambda|^2} \right) < \infty.$$

Since the curvature current of the metric $e^{-\chi}$ of K_X on $X \cap \{|t| < r\}$ is nonnegative, by the extension theorem of Ohsawa-Takegoshi type (1.4.3), we can extend $s^{(m_0)}$ to an element of $\Gamma\left(X \cap \{|t| < r\}, m_0 K_X\right)$. Finally, by the coherence of the zeroth direct image of $\mathscr{O}_X(m_0 K_X)$ under the projection $\pi\colon X \to \Delta$ and by the Steinness of Δ, we can extend $s^{(m_0)}$ to an element of $\Gamma\left(X \cap \{|t| < r\}, m_0 K_X\right)$. This concludes the proof of the deformational invariance of the plurigenera for the general case of projective algebraic manifolds not necessarily of general type.

1.7   Maximally regular metric for the canonical line bundle

As we have seen in the above proof, the main idea of the proof of the deformational invariant of the plurigenera is to produce a metric $e^{-\chi}$ for K_X with χ plurisubharmonic so that a given element $s^{(m_0)} \in \Gamma\left(X_0, m_0 K_X\right)$ satisfies

$$\int_{X_0} \left|s^{(m_0)}\right|^2 e^{-(m_0-1)\chi} < \infty. \tag{1.7.0.1}$$

The metric $e^{-\chi}$ is constructed as

$$\frac{1}{\sum_j \left|\tau_j\right|^2},$$

where each τ_j is a multi-valued holomorphic section of K_X (in the sense that $(\tau_j)^q$ for some appropriate positive integer q is a global holomorphic section of $q K_X$ over X). The metric $e^{-\chi}$ is singular at points of common zeroes of τ_j. In order for $(1.7.0.1)$ to be satisfied, it is natural to choose τ_j so that $\sum_j \left|\tau_j\right|^2$ is as large as possible, or equivalently, the metric $e^{-\chi}$ is as regular as possible.

A maximally regular metric for the canonical line bundle K_{X_0} of X_0 is given by Φ^{-1} with

$$\Phi = \sum_{m=1}^{\infty} \varepsilon_m \left(\sum_{j=1}^{q_m} \left|\sigma_j^{(m)}\right|^2\right)^{1/m},$$

where

$$\sigma_1^{(m)}, \cdots, \sigma_{q_m}^{(m)} \in \Gamma\left(X_0, m K_{X_0}\right)$$

form a basis over $\mathbb{C}$ and ε_m is a sequence of positive numbers which decrease to 0 fast enough to make the above defining series for Φ converge. With the maximally regular metric Φ^{-1} for the canonical line bundle K_{X_0} of X_0 the condition $(1.7.0.1)$ is automatically satisfied. The metric Φ^{-1} is only for K_{X_0} and not for K_X. With the introduction of a holomorphic line bundle A on X which is sufficiently positive for the global generation of multiplier ideal sheaves (1.4.2), the "twin tower argument" above could give us

$$\hat{s}_1^{(m)}, \cdots, \hat{s}_{\hat{q}_m}^{(m)} \in \Gamma\left(X, m K_X + A\right)$$

whose restrictions to X_0 would give a basis of $\Gamma\left(X_0, m K_{X_0} + A\right)$ over $\mathbb{C}$. We could form the metric $\left(\hat{\Phi}_m\right)^{-1}$ for $mK + A$ on X with

$$\hat{\Phi}_m = \sum_{j=1}^{\hat{q}_m} \left|\hat{s}_j^{(m)}\right|^2.$$

The condition

$$\int_{X_0} \left| s^{(m_0)} s_A \right|^2 \left(\hat{\Phi}_{m_0-1} \right)^{-1} h_A < \infty$$

is clearly satisfied.

A natural way to get rid of the undesirable summand A is to use the limit of

$$\left(\hat{\Phi}_{(m_0-1)l} \right)^{-1/l}$$

as a metric of $(m_0 - 1) K_X$ when some sequence l of positive integers goes to infinity. The main obstacle is the convergence of the limit. The extension of orthonormal basis from the initial fibers to the whole family in general would not preserve the orthonormality property. The use of sub-mean-value property of the logarithm of the absolute value of holomorphic functions to get uniform bounds would involve shrinking the domain, which might eventually disappear completely in an infinite inductive process. The known techniques of estimates could not yield the needed convergence.

A special method was introduced in ?[Siu22]? to solve this problem in the special case when manifolds X_t in the family $\pi \colon X \to \Delta$ are of general type. For the special case of general type we can write $a K_X = A + D$ for some positive integer a and some effective divisor D of X not containing any fiber X_t (after replacing Δ by a smaller disk if necessary). Let s_D be the canonical section of the divisor D. We can use

$$\left(\hat{\Phi}_{m_0 l} \left| s_D \right|^2 \right)^{-\frac{m_0-1}{m_0 l + a}}$$

as the metric for $(m_0 - 1) K_X$ for l sufficiently large without passing to the limit with $l \to \infty$. When l is sufficiently large, the condition $(1.7.0.1)$ follows from using Hölder's inequality to separate the factor $\left(\left| s_D \right|^2 \right)^{-\frac{m_0-1}{m_0 l + a}}$ into an integral of a power of it. The use of this particular step circumvents the obstacle of proving the convergence of the limit.

Finally a solution of the convergence problem came in ref. [32] when it was realized that in the construction of the metric for $(m_0 - 1) K_X$ the priority should be given to the convergence condition instead of to the finiteness condition $(1.7.0.1)$. Instead of using the most natural maximally regular metric for the canonical line bundle, in ref. [32] the most singular metric for the canonical line bundle is chosen which still fulfils the condition $(1.7.0.1)$.

Though surprisingly it turns out that the most singular metric should be used for the deformational invariance of the plurigenera, the maximally regular metric of the canonical line bundle will play an important role in the problem of the finite generation of the canonical ring which we will discuss below.

1.8 Finite generation of canonical ring and Skoda's estimates for ideal generation

We now turn to the problem of the finite generation of the canonical ring. A very powerful tool for this problem is the following estimate of Skoda for ideal generation[34].

1.8.1 **Theorem (Skoda).** Let Ω be a pseudoconvex domain spread over $\mathbb{C}^n$ and ψ be

a plurisubharmonic function on Ω. Let $g_1, \cdots, g_p$ be holomorphic functions on Ω. Let $\alpha > 1$ and $q = \min(n, p-1)$. Then for every holomorphic function f on Ω such that

$$\int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda < \infty,$$

there exist holomorphic functions $h_1, \cdots, h_p$ on Ω such that

$$f = \sum_{j=1}^p g_j h_j$$

and

$$\int_\Omega |h|^2 |g|^{-2\alpha q} e^{-\psi} d\lambda \leqslant \frac{\alpha}{\alpha - 1} \int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda,$$

where

$$|g| = \left(\sum_{j=1}^p |g_j|^2 \right)^{1/2}, \quad |h| = \left(\sum_{j=1}^p |h_j|^2 \right)^{1/2},$$

and $d\lambda$ is the Euclidean volume element of $\mathbb{C}^n$.

1.8.2   Skoda's estimate applied to sections of line bundles.   By using the procedure which reduces the situation of sections of line bundles over compact algebraic manifolds to functions on Stein domains over $\mathbb{C}^n$ with L^2 estimates, we can translate Skoda's estimate into the following algebraic geometric formulation.

1.8.3   Theorem.   Let X be a compact complex algebraic manifold of dimension n, L a holomorphic line bundle over X, and E a holomorphic line bundle on X with metric $e^{-\varphi}$ (φ possibly assuming $-\infty$ values) such that ψ is plurisubharmonic. Let $k \geqslant 1$ be an integer, $G_1, \cdots, G_p \in \Gamma(X, L)$, and $|G|^2 = \sum_{j=1}^p |G_j|^2$. Let $\mathcal{I} = \mathcal{I}_{(n+k+1) \log |G|^2 + \varphi}$ and $\mathcal{J} = \mathcal{J}_{(n+k) \log |G|^2 + \varphi}$. Then

$$\Gamma \left(X, \mathcal{I} \otimes \left((n+k+1)L + E + K_X \right) \right)$$
$$= \sum_{j=1}^p G_j \Gamma \left(X, \mathcal{J} \otimes \left((n+k)L + E + K_X \right) \right).$$

Proof.   Clearly the right-hand side is contained in the left-hand side of the equality in the conclusion. To prove the opposite direction, we take $F \in \Gamma(X, \mathcal{I} \otimes (n+k+1)L + E + K_X)$.

Let S be a non identically zero meromorphic section of E on X. We take a branched cover map $\pi\colon X \to \mathbf{P}_n$. Let Z_0 be a hypersurface in $\mathbf{P}_n$ which contains the infinity hyperplane of $\mathbf{P}_n$ and the branching locus of π in $\mathbf{P}_n$ such that $Z := \pi^{-1}(Z_0)$ contains the divisor of G_1 and the zero-set of S and the pole-set of S. Let $\Omega = X - Z$.

Let

$$g_j = G_j / G_1 \quad (1 \leqslant j \leqslant p)$$

and define f by

$$\frac{F}{G_1^{n+k+1} S} = f dz_1 \wedge \cdots \wedge dz_n,$$

where $z_1, \cdots, z_n$ are the affine coordinates of $\mathbb{C}^n$. Use $\alpha = \frac{n+k}{n}$. Let $\psi = \varphi - \log |S|^2$. It follows from $F \in \mathcal{I}_{(n+k+1)\log |G|^2 + \varphi}$ that

$$\int_X \frac{|F|^2}{|G|^{2(n+k+1)}} e^{-\varphi} < \infty,$$

which implies that

$$\int_\Omega \frac{|f|^2}{|g|^{2(n+k+1)}} e^{-\psi} = \int_\Omega \frac{\left| \frac{F}{G_1^{n+k+1}S} \right|^2}{\left| \frac{G}{G_1} \right|^{2(n+k+1)}} e^{-\psi} = \int_\Omega \frac{|F|^2}{|G|^{2(n+k+1)}} e^{-\varphi} < \infty,$$

where $|g|^2 = \sum_{j=1}^p |g_j|^2$. By Theorem (1.8.1) with $q = n$ (which we assume by adding some

$$F_{p+1} \equiv \cdots \equiv F_{n+1} \equiv 0$$

if $p < n + 1$) so that

$$2\alpha q + 2 = 2 \cdot \frac{n+k}{n} \cdot n + 2 = 2(n + k + 1).$$

Thus there exist holomorphic functions $h_1, \cdots, h_p$ on Ω such that $f = \sum_{j=1}^p g_j h_j$ and

$$\sum_{j=1}^p \int_\Omega \frac{|h_j|^2}{|g|^{2(n+k)}} e^{-\psi} < \infty.$$

Define

$$H_j = G_1^{n+k} h_j S dz_1 \wedge \cdots \wedge dz_n.$$

Then

$$\int_\Omega \frac{|H_j|}{|G|^{2(n+k)}} e^{-\varphi} = \int_\Omega \frac{|h_j|}{|g|^{2(n+k)}} e^{-\psi} < \infty$$

so that H_j can be extended to an element of $\Gamma\left(X, \mathcal{J} \otimes ((n+k)L + E + K_X)\right)$. Q.E.D.

As an illustration of how Skoda's estimate could be used to yield readily results on finite generation, we give the following statement in the case of globally free line bundles.

1.8.4   Theorem (generation for the ring of sections of multiples of free bundle).   Let F be a holomorphic line bundle over a compact projective algebraic manifold X of complex dimension n. Let $a > 1$ and $b \geqslant 0$ be integers such that aF and $bF - K_X$ are globally free over X. Then the ring $\oplus_{m=0}^\infty \Gamma(X, mF)$ is generated by $\oplus_{m=0}^{(n+2)a+b-1} \Gamma(X, mF)$.

Proof.   For $0 \leqslant l < a$, let $E_l = (b + l)F - K_X$ and $L = aF$. Let $G_1, \cdots, G_p$ be a basis of $\Gamma(X, L) = \Gamma(X, aF)$. Let $H_1, \cdots, H_q$ be a basis of $\Gamma(X, bF - K_X)$. We give E_l the metric

$$\left(\sum_{j=1}^p |G_j|^{2l/a} \right)^{-1} \left(\sum_{j=1}^q |H_j|^2 \right)^{-1}.$$

Since both $\mathcal{I}$ and $\mathcal{J}$ (from Theorem (1.8.3) when E is set to be E_l) are unit ideal sheaves due to the global freeness of aF and $bF - K_X$, it follows from Theorem (1.8.3) that

$$\Gamma(X, (n+k+1)L + E_l + K_X) = \sum_{j=1}^p G_j \Gamma(X, (n+k)L + E + K_X)$$

for $k \geqslant 1$ and $0 \leqslant l < a$, which means that

$$\Gamma\left(X, ((n+k+1)a + l + b)F\right) = \sum_{j=1}^{p} G_j \Gamma\left(X, ((n+k)a + l + b)F\right)$$

for $k \geqslant 1$ and $0 \leqslant l < a$. Thus $\oplus_{m=0}^{(n+2)a+b-1} \Gamma(X, mF)$ generates the ring $\oplus_{m=0}^{\infty} \Gamma(X, mF)$.   Q.E.D.

1.9   Stable vanishing order

The use of Skoda's estimates in the simple case of finite generation for the ring of sections of multiples of free line bundles suggests that the finite generation of the canonical ring for the case of finite type depends on properties of the stable vanishing order which is defined as follows.

Let

$$s_1^{(m)}, \cdots, s_{q_m}^{(m)} \in \Gamma\left(X, mK_X\right)$$

be a basis over $\mathbb{C}$. Let

$$\Phi = \sum_{m=1}^{\infty} \varepsilon_m \left(\sum_{j=1}^{q_m} \left| s_j^{(m)} \right|^2 \right)^{1/m},$$

where ε_m is a sequence of positive numbers decreasing fast enough to guarantee convergence of the series in the definition of Φ.

We would like to remark that $1/\Phi$ is the metric for K_X which was introduced earlier in our discussion of the deformational invariance of plurigenera.

One main difficulty of applying Skoda's estimates to prove the conjecture on the finite generation of the canonical ring is the common zero-set of Φ and the vanishing orders of Φ at its points. A more precise definition of the vanishing order of Φ is the Lelong number of the closed positive $(1, 1)$-current

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \Phi$$

(see ref. [35] for the definition and properties of Lelong numbers). We call the vanishing orders of Φ the *stable vanishing orders.* When the canonical ring is finitely generated, it is clear that all vanishing orders of Φ are rational. As an intermediate step of the proof of the finite generation of the canonical ring, one has the following conjecture on the rationality of the stable vanishing orders.

1.9.1   Conjecture on rationality of stable vanishing orders.   If X is of general type, then all the vanishing orders of Φ are rational.

1.9.2   Approach to rationality by degenerate complex Monge-Ampère equation.   One approach to the conjecture on the rationality of stable vanishing orders is by degenerate complex Monge-Ampère equations and indicial equations. Let Ω be the open subset of X consisting of all points where Φ is positive. The degenerate complex Monge-Ampère equation is

$$(\sqrt{-1}\,\partial\overline{\partial} \log\,\omega)^n = \omega \ \text{ on } \Omega \qquad\qquad (1.9.2.1)$$

with the boundary condition that the quotient ω/Φ is bounded from above and below by positive constants on $\Omega \cap U$ for some neighborhood U of $\partial\Omega$, where the unknown ω is a positive smooth (n, n)-form on Ω so that $\sqrt{-1}\,\partial\bar{\partial}\log\omega$ is strictly positive on Ω.

The motivation of considering the degenerate complex Monge-Ampère equation comes from the following two considerations.

(i) According to an observation of Demailly, another way of getting a metric of K_X comparable to Φ^{-1} is to consider the maximum φ among all metrics $e^{-\varphi}$ of K_X with φ locally plurisubharmonic so that $e^{-\varphi}$ is bounded by 1 at one point with respect to a fixed chosen local trivialization of K_X. The reason is that one can use L^2 estimates of $\bar{\partial}$ to get global holomorphic sections of a line bundle L with semipositive curvature current after twisting L by a sufficiently positive line bundle A independent of L. Conversely, global sections can be used to define a metric with semipositive curvature current.

(ii) By the work of Bedford and Taylor on the Dirichlet problem for complex Monge-Ampère equations[36], a solution of the complex Monge-Ampère equation can be obtained by the Perron method of maximization subject to certain normalization.

We would like to remark that our degenerate complex Monge-Ampère equation (1.9.2.1) is very different from the kind of degenerate complex Monge-Ampère equations considered by Yau in the last part of his paper[37].

1.9.3   Indicial equations of regular singular ordinary differential equations

In order to use the degenerate complex Monge-Ampère equation to conclude the rationality of the stable vanishing orders, one compares the situation to the method of using undetermined coefficients to get vanishing orders of solutions of regular singular ordinary differential equation.

In the case of the regular singular ordinary differential equation

$$x^2 y'' + x a(x) y' + b(x) y = 0 \qquad (1.9.3.1)$$

of a single real dependent variable y and a single real independent variable x, for a solution of the form

$$y = x^r \sum_{j=0}^{\infty} c_j x^j$$

with undetermined coefficients c_j to satisfy (1.9.3.1) the exponent r, known as the *index*, has to satisfy the *indicial equation*

$$r(r-1) + r a(0) + b(0) = 0. \qquad (1.9.3.2)$$

The indicial equation is *quadratic* because in general there are *two* solutions of the second-order ordinary equation.

1.9.4   Analog of indicial equation for degenerate complex Monge-Ampère equation

For our *degenerate* complex Monge-Ampère equation (1.9.2.1), the analog of the indicial equation (1.9.3.2) is a system of equations whose unknowns are the stable vanishing

orders.

Because the solution of the degenerate complex Monge-Ampère equation is expected to be unique, instead of the quadratic indicial equation (1.9.3.2) from the ordinary differential equation, the system of equations for the stable vanishing orders is expected to be linear with rational coefficients, independent enough to give the rationality of the stable vanishing orders.

1.9.5 Finite generation of canonical ring with assumption of rationality of stable vanishing orders

Suppose we are able to show that all stable vanishing orders are rational. How far away are we from concluding the finite generation of the canonical ring? To answer this question, we introduce first the notion of Lelong sets.

For a positive number c, the Lelong set $E_c(\Phi)$ of Φ (or more precisely, $\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\Phi$) means the set of all points x of X such that the Lelong number of the closed positive $(1,1)$-current $\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\Phi$ is $\geqslant c$ at the point x. Note that $E_c(\Phi)$ is always a complex-analytic subvariety[35].

An irreducible Lelong set E of Φ means a branch of $E_c(\Phi)$ for some $c>0$. The generic Lelong number on a Lelong set E is the Lelong number at a generic point of E (which is independent of the choice of the generic point of E). The following statement holds.

1.9.6 Statement.   If the number of all irreducible Lelong sets of Φ is finite and if their generic Lelong numbers are all rational, then the canonical ring is finitely generated.

So after the verification of the rationality of all stable vanishing orders, what remains is the problem of handling an infinite number of irreducible Lelong sets. For that one uses the fact an additional copy of the canonical line bundle is added to play the role of the volume form when one considers vanishing theorems and L^2 estimates to solve the $\bar\partial$ equation for global holomorphic sections. This additional copy of the canonical line bundle has to be used to enable one to ignore the effect of sufficiently small Lelong numbers. Note that the "twin tower argument" for the deformational invariance of the plurigenera also depends crucially on an additional copy of the canonical line bundle to go down and up the two towers.

2 Application of algebraic geometry to analysis

We now consider the other direction which is the application of algebraic geometry to subelliptic estimates.

2.1 Regularity, subelliptic estimates, and Kohn's multiplier ideals

Let us start out with the setting of the problem of subelliptic estimates.

2.1.1 Setting of regularity of the $\bar\partial$-problem.   Let Ω be a bounded domain in $\mathbb{C}^n$ with

C^∞ boundary $\partial\Omega$ which is defined by a single C^∞ real-valued function r on some open neighborhood of the topological closure $\bar\Omega$ of Ω in $\mathbb{C}^n$ so that $\Omega = \{r < 0\}$ and $\partial\Omega = \{r = 0\}$ and the differential dr of r is nowhere zero on $\partial\Omega$. We assume that $\partial\Omega$ is weakly pseudoconvex in the sense that the Hermitian form $\sqrt{-1}\partial\bar\partial r$ on $T_{\partial\Omega}^{1,0} = T_{\mathbb{C}^n}^{1,0} \cap (T_{\partial\Omega} \otimes \mathbb{C})$ is semipositive, where $T_{\partial\Omega}$ is the real tangent space of the manifold $\partial\Omega$ and $T_{\mathbb{C}^n}^{1,0}$ is the $\mathbb{C}$-vector space of all tangent vectors of type $(1,0)$ of $\mathbb{C}^n$.

2.1.2 Global regularity problem for (0,1)-forms. The problem of global regularity of $\bar\partial$ for $(0,1)$-forms asks whether, given a C^∞ $\bar\partial$-closed $(0,1)$-form f on Ω which is L^2 with respect to the Euclidean metric of $\mathbb{C}^n$, the solution u of the equation $\bar\partial u = f$ on Ω which is orthogonal to all L^2 holomorphic functions on Ω with respect to the Euclidean metric of $\mathbb{C}^n$ is C^∞ up to the boundary of Ω. The discussion here applies, with appropriate adaptation, also to $(0,q)$-form for general $1 \leqslant q \leqslant n-1$. For notational simplicity we will confine ourselves here only to the case of $(0,1)$-forms.

2.1.3 Subelliptic estimates. The global regularity problem is a consequence of the subelliptic estimate which is defined as follows. The *subelliptic estimate*, with subellipticity order $\varepsilon > 0$, holds at a boundary point P of Ω if, for some open neighborhood U_P of P in $\mathbb{C}^n$ and some positive number C,

$$\||\varphi\||_\varepsilon^2 \leqslant C \left(\|\bar\partial\varphi\|^2 + \|\bar\partial^*\varphi\|^2 + \|\varphi\|^2 \right)$$

for every test $(0,1)$-form φ supported in $U_P \cap \bar\Omega$ which is in the domain of $\bar\partial$ and in the domain of $\bar\partial^*$, where $\|\cdot\|$ is the usual L^2 norm on Ω and $\||\varphi\||_\varepsilon^2$ is the Sobolev L^2 norm on Ω for derivatives up to order ε in the direction tangential to $\partial\Omega$ (i.e. directions annihilated by dr).

2.1.4 Kohn's multipliers and their ideals. To quantitatively measure the failure of subelliptic estimates Kohn introduced multipliers[3]. For a point P in the boundary of Ω, a *Kohn multiplier* is a C^∞ function germ F on $\mathbb{C}^n$ at P such that, for some open neighborhood U_P of P in $\mathbb{C}^n$ and some positive numbers C and ε (which may depend on F),

$$\||F\varphi\||_\varepsilon^2 \leqslant C \left(\|\bar\partial\varphi\|^2 + \|\bar\partial^*\varphi\|^2 + \|\varphi\|^2 \right)$$

for every test $(0,1)$-form φ supported in $U_P \cap \bar\Omega$ which is in the domain of $\bar\partial$ and in the domain of $\bar\partial^*$.

The set of all Kohn multipliers forms an ideal known as the *Kohn multiplier ideal* which we denote by I_P. By definition it is clear that the subelliptic estimate holds if and only if I_P is the unit ideal (which means that the function identically 1 belongs to the ideal), in which case the order ε of subellipticity can be chosen to be the positive number ε for the identically 1 function.

Kohn[3] introduced the following procedures of generating multipliers.

(i) $r \in I_P$.

(ii) The coefficient of $\partial r \wedge \bar\partial r \wedge \left(\partial\bar\partial r\right)^{n-1}$ belongs to I_P.

(iii) I_P equals its $\mathbb{R}$-radical $\sqrt[\mathbb{R}]{I_P}$ in the sense that if $g \in I_P$ and $|f|^m \leqslant |g|$ for some $m \geqslant 1$, then $f \in I_P$.

(iv) If $f_1, \cdots, f_{n-1}$ belong to I_P and $1 \leqslant j \leqslant n-1$, then the coefficients of

$$\partial f_1 \wedge \cdots \wedge \partial f_j \wedge \partial r \wedge \bar{\partial} r \wedge \left(\partial \bar{\partial} r\right)^{n-1-j}$$

belong to I_P.

2.2   Finite type and subellipticity

The problem is how to use geometric properties of the boundary of Ω to conclude that the above procedures of Kohn generate the unit ideal.

2.2.1   Finite type.

One natural geometric property is the property of finite type which is defined as follows[38,39]. The *type* m at a point P of the boundary of Ω is the supremum of the normalized touching order

$$\frac{\mathrm{ord}_0\left(r \circ \varphi\right)}{\mathrm{ord}_0 \varphi},$$

to $\partial\Omega$, of all local holomorphic curves $\varphi \colon \Delta \to \mathbb{C}^n$ with $\varphi(0) = P$, where Δ is the open unit 1-disk and ord_0 is the vanishing order at the origin 0. A point P of the boundary of Ω is said to be of *finite type* if the type m at P is finite. The whole boundary $\partial\Omega$ is of finite type if the supremum over the types of all points in the boundary of Ω is finite.

2.2.2   Results of Kohn and Catlin.

Kohn's original goal of developing his theory of multiplier ideals to show that his procedures of generating multipliers will yield the unit ideal if $\partial\Omega$ is of finite type.

For the case where r is real-analytic, Kohn showed[3], by using a result of Diederich-Fornaess[40], that if his multiplier ideal is not the unit ideal, it would lead to a contradiction that $\partial\Omega$ contains some local holomorphic curves. His method of argument by contradiction does not yield an effective ε as an explicit function of the finite type m and the dimension n of Ω.

Later Catlin[41−43], using another approach of multitypes and approximate boundary systems, showed that subelliptic estimates hold for smooth bounded weakly pseudoconvex domains of finite type.

The problem remains whether Kohn's procedures are enough, or some other procedures need to be found, to effectively generate the unit ideal in the case of finite type.

Catlin's use of multitype suggests the possibility of additional procedures different from Kohn's which is geared to the approach of multitype and which may be related to the "Weierstrass systems" for ideals in the sense of Hironaka[44,45] or Grauert[46,47].

2.2.3   Multiplier modules.

The listing of Kohn's procedures of generating multipliers can be better organized with the introduction of *multiplier modules*. For a point P in the boundary of Ω, a *multiplier-form* is a C^∞ germ θ of $(1,0)$-form on $\mathbb{C}^n$ at P such that, for some open neighborhood U_P of P in $\mathbb{C}^n$ and some positive numbers C and ε (which may

depend on θ),

$$\||\theta \cdot \varphi\||_\varepsilon^2 \leqslant C \left(\left\|\bar{\partial}\varphi\right\|^2 + \left\|\bar{\partial}^*\varphi\right\|^2 + \|\varphi\|^2 \right)$$

for every test $(0, 1)$-form φ supported in $U_P \cap \bar{\Omega}$ which is in the domain of $\bar{\partial}$ and in the domain of $\bar{\partial}^*$, where $\theta \cdot \varphi$ is the inner product between the $(1, 0)$-form θ and the $(0, 1)$-form φ with respect to the Euclidean metric of $\mathbb{C}^n$. The *multiplier module*, denoted by A_P, is the set of all multiplier-forms at P. We break up Kohn's list of procedures into the following three groups.

(A) Initial membership.

(i) $r \in I_P$.

(ii) $\partial\bar{\partial}_j r$ belongs to A_P for every $1 \leqslant j \leqslant n - 1$ if $\partial r = \partial z_n$ at P for some local coordinate system $(z_1, \cdots, z_n)$, where ∂_j means $\frac{\partial}{\partial z_j}$.

(B) Generation of new members.

(i) If $f \in I_P$, then $\partial f \in A_P$.

(ii) If $\theta_1, \cdots, \theta_{n-1} \in A_P$, then the coefficient of

$$\theta_1 \wedge \cdots \wedge \theta_{n-1} \wedge \partial r$$

is in I_P.

(C) Real radical property.

If $g \in I_P$ and $|f|^m \leqslant |g|$, then $f \in I_P$.

2.3   Algebraic formulation for special domains

A *special domain* Ω in $\mathbb{C}^{n+1}$ (with coordinates $z_1, \cdots, z_n, w$) means a bounded domain defined by

$$\operatorname{Re} w + \sum_{j=1}^N |h_j(z_1, \cdots, z_n)|^2 < 0,$$

where $h_j(z_1, \cdots, z_n)$ is a holomorphic function defined on some open neighborhood of the closure $\bar{\Omega}$ which depends only on the first n variables $z = (z_1, \cdots, z_n)$.

2.3.1   Finite type condition of special domain.   The condition of finite type at a boundary point P of Ω can be formulated in terms of the ideal generated by $h_1, \cdots, h_N$. To describe the formulation, we can assume without loss of generality that P is the origin of $\mathbb{C}^{n+1}$. Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the ring of all holomorphic function germs of $\mathbb{C}^n$ at the origin of $\mathbb{C}^n$ and $\mathfrak{m}_{\mathbb{C}^n,0}$ be the maximum ideal of $\mathcal{O}_{\mathbb{C}^n,0}$. Finite type at the origin means that

$$(\mathfrak{m}_{\mathbb{C}^n,0})^p \subset \sum_{j=1}^N \mathcal{O}_{\mathbb{C}^n,0} h_j$$

for some positive integer p. The number p is related to the type m in the following way.

The inequality

$$|z|^q \leqslant C \sum_{j=1}^{N} |h_j(z)|$$

holds for some positive constant C on some open neighborhood of the origin in $\mathbb{C}^n$ when $q \leqslant m \leqslant 2q$. By Skoda's theorem (1.8.1), p can be chosen so that $p \leqslant q \leqslant (n+2)p$.

2.3.2   Inductively defined ideals and functions.   We introduce the following inductively defined ideals and positive-valued functions.

For $\nu \in \mathbb{N}$ we inductively define ideals J_ν and $\tilde{J}_\nu$ of $\mathcal{O}_{\mathbb{C}^n,0}$ and positive-valued functions γ_ν on J_ν and positive-valued functions $\tilde{\gamma}_\nu$ on $\tilde{J}_\nu$ as follows.

The ideal J_1 is generated by all elements f such that f is defined by

$$dg_1 \wedge \cdots \wedge dg_n = f\, dz_1 \wedge \cdots \wedge dz_n,$$

where each of $g_1, \cdots, g_n$ is a $\mathbb{C}$-linear combination of $h_1, \cdots, h_N$.

The value $\gamma_1(f)$ is equal always to $1/8$ for $f \in J_1$. The ideal $\tilde{J}_1$ is the radical of J_1. For $f \in \tilde{J}_1$ the value $\tilde{\gamma}_1(f)$ of the function $\tilde{\gamma}_1$ at f is equal to $\frac{1}{8m}$, where m is the smallest positive integer m such that $f^m \in J_1$.

For $\nu \in \mathbb{N}$ and $\nu \geqslant 2$ the ideal J_ν is generated by all elements f such that f is either an element of $\tilde{J}_{\nu-1}$ or an element defined by an expression of the form

$$dg_1 \wedge \cdots \wedge dg_n = f\, dz_1 \wedge \cdots \wedge dz_n, \qquad (2.3.2.1)$$

where $0 \leqslant k \leqslant n$ and each of $g_1, \cdots, g_k$ is a $\mathbb{C}$-linear combination of $h_1, \cdots, h_N$ and each of $g_{k+1}, \cdots, g_n$ is an element of $\tilde{J}_{\nu-1}$.

For $f \in J_\nu$ we assign the largest possible positive number $a(f)$ which can be obtained in one of the following ways. If f is in $\tilde{J}_{\nu-1}$, the number $a(f)$ is the same as $\tilde{\gamma}_{\nu-1}(f)$. If f is given by (2.3.2.1),

(i) the number $a(f)$ is the minimum of

$$\frac{1}{8}, \frac{1}{2}\tilde{\gamma}_{\nu-1}(g_{k+1}), \cdots, \frac{1}{2}\tilde{\gamma}_{\nu-1}(g_n)$$

when $1 \leqslant k < n$; and

(ii) the number $a(f)$ is the minimum of

$$\frac{1}{2}\tilde{\gamma}_{\nu-1}(g_1), \cdots, \frac{1}{2}\tilde{\gamma}_{\nu-1}(g_n)$$

when $k = 0$; and

(iii) the number $a(f)$ is equal to $1/8$ when $k = n$.

For $f \in \tilde{J}_\nu$ the value $\tilde{\gamma}_\nu(f)$ of the function $\tilde{\gamma}_\nu$ at f is equal to $\frac{1}{m\gamma_\nu(f^m)}$, where m is the smallest positive integer m such that $f^m \in J_\nu$.

The following statement is the algebraic formulation of using Kohn's procedures to effectively generate the unit ideal in the case of special domains of finite type.

2.3.3 Statement. There exists a positive number ε which depends only on n and p by an explicit expression such that for some $\nu \in \mathbb{N}$ there exists some $f \in \tilde{J}_\nu$ with f nonzero at the origin and $\tilde{\gamma}_\nu(f) \geqslant \varepsilon$.

From the point of view of local complex-analytic geometry, the above statement (2.3.3) can be regarded as generalizing in a very elaborate manner, to the case of ideals of holomorphic functions of n complex variables, the trivial statement that the vanishing order of the differential of a holomorphic function germ of a single complex variable at a point is equal to its vanishing order minus 1. For this very elaborate generalization, the process of taking differential is replaced by the process of taking the Jacobian determinant of n holomorphic function germs.

2.4 Interpretation as Frobenius theorem over Artinian subschemes

The problem of using Kohn's procedures to effectively generate the unit ideal in the case of general type can be interpreted as a Frobenius theorem over Artinian subschemes.

The usual Frobenius theorem for $\mathbb{R}^m$ is the following. Let U be an open subset of $\mathbb{R}^m$. Consider a smooth distribution $x \mapsto V_x \subset T_{\mathbb{R}^m} = \mathbb{R}^m$ of k-dimensional subspaces of $T_{\mathbb{R}^m}$.

The Frobenius theorem can be formulation in terms of Lie brackets or in terms of differential forms. The Lie bracket formulation states that the distribution V_x is integrable (i.e. each V_x is the tangent space of a submanifold of dimension k in an $(m-k)$-parameter family of such k-folds) if and only if V_x is closed under taking the Lie brackets of any two of its elements (i.e. the Lie bracket $[V_x, V_x]$ is contained in V_x for all $x \in U$).

The differential form formulation states that the distribution V_x is integrable if and only if $d\omega_j = \sum_{l=1}^{m-k} \omega_l \wedge \eta_l$ for some 1-forms $\eta_1, \cdots, \eta_{m-k}$, where $\omega_1, \cdots, \omega_{m-k}$ are smooth 1-forms defining V_x (i.e. V_x is the intersection of the kernels of $\omega_1, \cdots, \omega_{m-k}$).

By an Artinian subscheme we mean an unreduced subspace supported at a single point. In other words, it is a multiple point. For example, the ringed space $(0, \mathcal{O}_{\mathbb{C}^n} / \mathcal{I})$, with

$$\left(\mathfrak{m}_{\mathbb{C}^n, 0} \right)^N \subset \mathcal{I}$$

for some integer $N \geqslant 1$, is an Artinian subscheme.

Now instead of considering integrability of the distribution V_x over the open subset U of $\mathbb{R}^m$ we will consider integrability over some multiple point.

For our setting of bounded weakly pseudoconvex domain Ω with smooth boundary $M = \partial\Omega$, the distribution we consider is the space of all real tangent vectors of M which are the real parts of elements of $T_M^{(1,0)}$, where $T_M^{(1,0)}$ is the space of all complex-valued tangent vectors of M of type $(1, 0)$. The Lie bracket formulation of the integrability of this distribution over an open subset M' of M is the same as M' being Levi-flat.

Integrability of this distribution over an Artinian subscheme supported at a point P

of M means the existence of a local holomorphic curve touching M at P to an order corresponding to the Artinian subscheme.

The differential form formulation of the usual Frobenius theorem now corresponds to the generation of new multipliers by wedge product and exterior differentiation in Kohn's procedures.

2.5   Sums of squares and Kohn's counter-example

In the case of complex dimension two one way in which Kohn obtained subelliptic estimates for weakly pseudoconvex domains of finite type[48] is to use Hörmander's subelliptic estimates for sums of squares of real-valued vector fields whose iterated Lie brackets span the entire tangent space[49]. The operator used is $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. When one decouples the operator from the domain and asks for subellipticity from iterated Lie bracket conditions, one has to consider the case of complex-valued vector fields in the setting of sums of squares. Kohn recently produced the following counter-example[50]. Let M be the boundary of the domain $\operatorname{Re} z_2 + |z_1|^2 < 0$ in $\mathbb{C}^2$ (with coordinates z_1, z_2), which is biholomorphic to the complex 2-ball, so that M is given by $\operatorname{Re} z_2 = -|z_1|^2$. Let $x = \operatorname{Re} z_1$, $y = \operatorname{Im} z_1$, $z = x + \sqrt{-1}y$, and $t = \operatorname{Im} z_2$. Let

$$L = \frac{\partial}{\partial z_1} - 2\overline{z_1}\frac{\partial}{\partial z_2} = \frac{\partial}{\partial z} + \sqrt{-1}\bar{z}\frac{\partial}{\partial t},$$

$$\bar{L} = \frac{\partial}{\partial \overline{z_1}} - 2z_1\frac{\partial}{\partial z_2} = \frac{\partial}{\partial \bar{z}} - \sqrt{-1}z\frac{\partial}{\partial t},$$

and

$$X_{1k} = \overline{z_1}^k L, \quad X_2 = \bar{L}, \quad E_k = X_{1k}^* X_{1k} + X_2^* X_2.$$

The commutators of X_{1k}, X_2 of order $\leqslant k+1$ span the complexified tangent space of M at 0. Kohn proved that for $k > 0$ subellipticity does not hold for E_k, yet hypoellipticity holds for E_k with a loss of k derivatives in the supremum norm and a loss of $k-1$ derivatives in the Sobolev norm for $k > 1$.

2.5.1   Explanation of failure of subellipticity.

The reason for the failure of subellipticity can be illustrated by the following domain version corresponding to the boundary version in Kohn's counter-example. Consider the ball Ω defined by $|z_1 - 1|^2 + |z_2|^2 - 1 < 0$ in $\mathbb{C}^2$ and the vector field

$$L = \overline{z_2}\frac{\partial}{\partial z_1} + (1 - \overline{z_1})\frac{\partial}{\partial z_2}$$

which spans the space $T_{\partial\Omega}^{1,0}$ of complex-valued tangent vectors of $\partial\Omega$. Let

$$X_1 = \overline{z_1}L, \quad X_2 = \bar{L}.$$

Then the commutators of X_1, X_2 of order $\leqslant 2$ span the complexified tangent space of $\partial\Omega$ at 0. Let $E = X_1^* X_1 + X_2^* X_2$. Unlike $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ which corresponds to the situation with X_1 replaced by L, the operator E does not have subellipticity.

Take any $\varepsilon > 0$. To see the failure of subellipticity for E with subellipticity order $\varepsilon > 0$, take $p > 2$ and $C > 0$ so that, if g is a function on Ω with L^2 norm $\leqslant 1$ and if its

Sobolev L^2 norm on Ω for derivative up to order ε is also $\leqslant 1$, then the L^p norm of g on Ω is $\leqslant C$. For $0 < \eta < 1$ and $0 < \alpha < 1$, take a branch of $\varphi_\eta = 1/(z_1 + \eta)^\alpha$ on Ω. The L^2 norm of φ_η on Ω is bounded uniformly in $0 < \eta < 1$. Choose α so close to 1 that the L^p norm of φ_η on Ω is not bounded uniformly in $0 < \eta < 1$.

Because of the factor $\overline{z_1}$ in X_1, the L^2 norm of $X_1\varphi_\eta$ on Ω is bounded uniformly in $0 < \eta < 1$. Since $X_2\varphi_\eta$ is identically zero, if the subelliptic estimate holds for E with subellipticity order $\varepsilon > 0$, the Sobolev norm of φ_η on Ω for derivative up to ε would be bounded uniformly in $0 < \eta < 1$ and, as a consequence, the L^p norm of φ_η on Ω would be bounded uniformly for $0 < \eta < 1$, which contradicts the choice of α.

2.5.2 **Sums of squares of matrix-valued vector fields.** The subellipticity of $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ for $(0,1)$-forms holds on weakly pseudoconvex domains of finite type. The operator $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ for $(0,1)$-forms can be written as a sum of squares of matrix-valued vector fields. On the other hand, even for the strongly pseudoconvex case of the complex 2-ball, when the operator is decoupled from the domain and different from $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, Kohn's counter-example shows that, unlike the case of *real-valued* vector fields, in general subellipticity fails for a sum of squares of *complex-valued* vector fields with the iterated Lie bracket condition. An important natural problem is to understand what additional conditions are involved in the case of $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ for $(0,1)$-forms on weakly pseudoconvex domains of finite type which would give subellipticity for sums of squares of matrix-valued vector fields in general.

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