Hyperbolicity in Complex Geometry

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Summary. A complex manifold is said to be hyperbolic if there exists no nonconstant holomorphic map from the affine complex line to it. We discuss the techniques and methods for the hyperbolicity problems for submanifolds and their complements in abelian varieties and the complex projective space. The discussion is focussed on Bloch's techniques for the abelian variety setting, the recent confirmation of the longstanding conjecture of the hyperbolicity of generic hypersurfaces of high degree in the complex projective space, and McQuillan's techniques for compact complex algebraic surfaces of general type.

0 Introduction

A complex manifold X is said to be hyperbolic if there exists no nonconstant holomorphic map $\mathbf{C} \to X$. The hyperbolicity problem in complex geometry studies the conditions for a given complex manifold X to be hyperbolic. Hyperbolicity problems have a long history and trace back to the small Picard theorem and the hyperbolicity of compact Riemann surfaces of genus ≥ 2 from Liouville's theorem and the uniformization theorem. The hyperbolicity of compact Riemann surfaces of genus ≥ 2 led to the study of compact complex manifolds X with positive canonical line bundle K_X. The small Picard theorem led to the study of the the complement of a hypersurface Y in a compact algebraic manifold X with conditions related to the positivity of $Y + K_X$. For a long time a great part of the research on the hyperbolicity problems has been focussed on the following two environments.

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(1) The setting of abelian varieties. The hyperbolicity problem concerns a subvariety in an abelian variety or concerns the complement of a hypersurface in an abelian variety.

(2) The setting of the complex projective space $\mathbf{P}_n$. The hyperbolicity problem concerns a generic hypersurface of sufficiently high degree or concerns the complement of a generic hypersurface of sufficiently high degree.

In this survey article we present the essential techniques from function theory developed over the years and especially recently for the investigation of hyperbolicity problems for these two settings. There are three main parts in this survey paper. The first part concerns the techniques of Bloch for the setting of abelian varieties. The second part concerns the most recent methods for the setting of the complex projective space which were developed from the techniques of Clemens-Ein-Voisin from algebraic geometry. The third part concerns the techniques of McQuillan for hyperbolicity problems for surfaces of general type.

The main emphasis of this article is to explain the techniques and methods and not to present a comprehensive list of results. The background and the history of the problems of hyperbolicity together with lists of references can be found in other survey articles such as [Si95][Si99].

Notations and terminology

The structure sheaf of a complex space Y is denoted by $\mathcal{O}_Y$. The bundle of holomorphic p-forms on a complex manifold Z is denoted by Ω^p_Z.

For a complex manifold X denote by $J_k(X)$ the space of all k-jets of X. At a point P_0 of X an element of $J_k(X)$ is the k-th jet of a holomorphic map $f : U \to X$ at 0 and is determined by the derivatives of f up to order k at 0 with respect to the coordinate of U, where U is an open neighborhood of 0 in $\mathbf{C}$ and $f(0) = P_0$. For nonnegative integers $\ell < k$ there is a natural projection $J_k(X) \to J_\ell(X)$ and $J_0(X) = X$.

A k-jet differential on X with local coordinates $z_1, \cdots, z_n$ is locally a polynomial in $d^\ell z_j$ $(1 \leq \ell \leq k,\ 1 \leq j \leq n)$ of homogeneous weight, when the weight of $d^\ell z_j$ is defined to be ℓ. For a complex hypersurface Y in X locally defined by a holomorphic function g, a k-jet differential on X with logarithmic poles along Y is locally a polynomial in $d^\ell z_j,\ d^\ell \log g$ $(1 \leq \ell \leq k,\ 1 \leq j \leq n)$.

For a holomorphic map $\varphi : \mathbf{C} \to X$ the k-th differential $d^k\varphi$ of φ is a map from $J^k(\mathbf{C})$ to $J_k(X)$. Most of the time, we just regard $d^k\varphi$ as a map from $\mathbf{C}$ to $J_k(X)$, which assigns to every point of $\mathbf{C}$ the k-jet of the curve in X defined by φ at that point. Another way to interpret it is that we map $\mathbf{C}$ to $J_k(\mathbf{C})$ by taking the k-jet of the curve in $\mathbf{C}$ defined by the identity map of $\mathbf{C}$ and take the composite of it with $d^k\varphi : J_k(\mathbf{C}) \to J_k(X)$ to get a map from $\mathbf{C}$ to $J_k(X)$. When a k-jet differential ω of weight m on X, as a function on $J_k(X)$, is pulled back by $d^k\varphi : J_k(\mathbf{C}) \to J_k(X)$, one gets a k-jet $\varphi^*\omega$ of weight m on $\mathbf{C}$ and is a polynomial P in $d^j\zeta$ $(1 \leq j \leq k)$ of weight m. When $d^k\varphi$ is regarded as a map from $\mathbf{C}$ to $J_k(X)$, the pullback $\varphi^*\omega$ is a function on $\mathbf{C}$ which is the coefficient of $(d\zeta)^m$ in the polynomial P. Most of the time we will use the latter interpretation. For statements involving the vanishing of $\varphi^*\omega$, its vanishing in the second interpretation for all choices of the coordinate ζ in $\mathbf{C}$ implies its vanishing in the first interpretation.

For a holomorphic map $\varphi : \mathbf{C} \to X$ and a real closed $(1,1)$-current ω on X, the Nevanlinna characteristic function is $T(r,\varphi,\omega) = \int_{\rho=0}^{r} \frac{d\rho}{\rho} \int_{|\zeta|<\rho} \varphi^*\omega$, where ζ is the coordinate of $\mathbf{C}$. When ω is positive definite and the context requires only the growth order of $T(r,\varphi,\omega)$ up to a positive constant factor, we drop ω from $T(r,\varphi,\omega)$ and simply use $T(r,\varphi)$. When Y is a complex hypersurface of X, the counting function is $N(r,\varphi,Y) = \int_{\rho=0}^{r} n(\rho,\varphi,Y) \frac{d\rho}{\rho}$, where $n(\rho,\varphi,Y)$ is the number of points ζ in $\{|\zeta|<\rho\} \cap \varphi^{-1}(Y)$ with multiplicities counted. The truncated counting function $N_k(r,\varphi,Y)$ means $\int_{\rho=0}^{r} n_k(\rho,\varphi,Y) \frac{d\rho}{\rho}$, where $n_k(\rho,\varphi,Y)$ is the number of points ζ in $\{|\zeta|<\rho\} \cap \varphi^{-1}(Y)$ counted under the rule that the multiplicity is replaced by k if it is greater than k. Let σ_Y be a function on X which is locally equal to a smooth positive function times the absolute value of a local holomorphic function defining Y. The proximity function is $m(r,\varphi,Y) = \oint_{|\zeta|=r} \log^+ \frac{1}{(\sigma_Y)\circ\varphi}$, where $\oint$ means the average over the circle $|\zeta| = r$. The growth behavior of the proximity function is determined up to a bounded additive term.

A meromorphic function F on $\mathbf{C}$ defines a holomorphic map $\varphi_F : \mathbf{C} \to \mathbf{P}_1$ and $T(r,F)$ means $T(r,\varphi_F,\omega_{\mathbf{P}_1})$, where $\omega_{\mathbf{P}_n}$ is the Fubini-Study form on $\mathbf{P}_n$. The proximity term $m(r,\varphi_F,\infty)$ up to a bounded additive term is equal to $\oint_{|\zeta|=r} \log^+|F|$, which we also denote by $m(r,F,\infty)$ or simply $m(r,F)$. Another description of $T(r,F)$ up to a bounded additive term is $N(r,F,\infty) + m(r,F,\infty)$. When $\varphi : \mathbf{C} \to \mathbf{P}_n$ is defined by entire holomorphic functions $F_0,\cdots,F_n$ on $\mathbf{C}$ without common zeros, $T\left(r,\frac{F_j}{F_0}\right) \leq T(r,\varphi,\omega_{\mathbf{P}_n}) + O(1) \leq \sum_{j=1}^{n} T\left(r,\frac{F_j}{F_0}\right) + O(1)$.

For a holomorphic map $\varphi : \mathbf{C} \to X$ and an ample divisor Y whose cohomology class is represented by a smooth closed positive $(1,1)$-form ω, the First Main Theorem of Nevanlinna states that $T(r,\varphi,\omega) = m(r,\varphi,Y) + N(r,\varphi,Y) + O(1)$. The defect $\delta(\varphi,Y)$ is the limit inferior of $\frac{m(r,\varphi,Y)}{T(r,\varphi,\omega)}$ as $r \to \infty$. The defect $\delta(\varphi,Y)$ can only take value in the closed interval $[0,1]$. When the defect $\delta(\varphi,Y)$ is < 1, the image of the holomorphic map φ must intersect Y. What is usually referred to as a Second Main Theorem is an estimate of the error when the limit is taken in the definition of the defect, such as $m(r,\varphi,Y) \leq \delta(\varphi,Y)\, T(r,\varphi,\omega) + E(r)$ for r outside a set with finite measure with respect to $\frac{dr}{r}$, where $E(r)$ is an error term, for example, of the order $o(T(r,\varphi,\omega))$ in a weaker version and of the order $O(\log T(r,\varphi,\omega) + \log r)$. An estimate of the defect or a Second Main Theorem is a more quantitative description of the hyperbolicity of $X - Y$.

For a meromorphic function F on $\mathbf{C}$, Nevanlinna's logarithmic derivative lemma states that $m\left(r,(\log F)'\right) \leq O(\log T(r,F) + \log r)$ for r outside a set with finite measure with respect to $\frac{dr}{r}$, where $(\log F)'$ is the derivative of $\log F$ with respect to the coordinate of $\mathbf{C}$. When we invoke the logarithmic lemma, the condition that r is outside a set with finite measure with respect to $\frac{dr}{r}$ is understood and will not be explicitly mentioned.

The symbol A is reserved to denote an abelian variety of complex dimension n. An algebraic subvariety of $J_k(A) = A \times \mathbf{C}^{nk}$ means a subvariety which is algebraic along the factor $\mathbf{C}^{nk}$. In other words, it is defined by local functions on $J_k(A) = A \times \mathbf{C}^{nk}$ which are holomorphic in the local coordinates of A and are polynomials in the nk global coordinates of $\mathbf{C}^{nk}$. Algebraic Zariski closures in $J_k(A)$ are defined using such algebraic subvarieties.

The position-forgetting map $\pi_k : J_k(A) \to \mathbf{C}^{nk}$ denotes the projection $J_k(A) = A \times \mathbf{C}^{nk} \to \mathbf{C}^{nk}$ onto the second factor. For $\ell > k$ denote by $p_{k,\ell} : \mathbf{C}^{n\ell} \to \mathbf{C}^{nk}$ the natural projection so that $\pi_k = p_{k,\ell} \circ \pi_\ell$. For an ample divisor D of A we denote by θ_D be the theta function on $\mathbf{C}^n$ whose divisor is D. For a holomorphic map φ from $\mathbf{C}$ to A, let $W_{k,\varphi}$ be the algebraic Zariski closure of $\pi_k\left(\mathrm{Im}\left(d^k\varphi\right)\right)$ in $\mathbf{C}^{nk}$. Clearly, $p_{k,\ell}(W_{\ell,\varphi}) = W_{k,\varphi}$ for $\ell > k$. When Z_k is a proper subvariety of $W_{k,\varphi}$, the inverse image $p_{k,\ell}^{-1}(Z_k)$ is a proper subvariety of $W_{\ell,\varphi}$ for $\ell > k$.

Denote by $\lceil u \rceil$ the smallest integer no less than u and denote by $\lfloor u \rfloor$ the greatest integer not exceeding u. Denote by $\mathbf{N}$ the set of all nonnegative integers.

1 The Setting of Abelian Varieties

For hyperbolicity problems in the setting of abelian varieties there are three important techniques from function theory, all introduced by Bloch [B26].

(1) *The use and construction of holomorphic jet differentials.* Bloch's construction uses the map from the jet-space induced by the position-forgetting map from the jet-space of an abelian variety and by using the field extension of meromorphic functions from a holomorphic map between compact algebraic manifolds with generically finite fibers.

(2) *The Schwarz lemma on the vanishing of the pullback, to* $\mathbf{C}$*, by a holomorphic map, of a holomorphic jet differential on a compact algebraic manifold which vanishes on some ample divisor.* Bloch's proof uses the logarithmic derivative lemma and its idea agrees that of nowaday's best prevailing proof.

(3) *Differential equations constructed from position-forgetting maps of different order.* These differential equations link hyperbolicity to natural conjectured geometric properties. In the case treated in Bloch's paper the link is the translational invariance of the Zariski closure of an entire holomorphic curve in an abelian variety, which is nowadays referred to as Bloch's theorem.

The language and the style of Bloch's paper are very different from those of papers published in today's mathematical journals. The description of arguments in his papers is extremely concise and is more conversational in style.

He considered only the case of surfaces, but his techniques are easily adaptable to general dimensions. Here I will first present Bloch's three techniques formulated in modern mathematical terms. Then I will explain where the three techniques are located in Bloch's paper and how Bloch originally presented them.

1.1 First Technique of Bloch

Lemma 1 (Construction of holomorphic jet differentials by generically finite maps). *Let $\mathcal{X}$ be an algebraic subvariety of the space $J_k(A)$ of k-jets of an n-dimensional abelian variety A such that $\sigma_k : \mathcal{X} \to \mathbf{C}^{nk}$ induced by the position-forgetting map $\pi_k : J_k(A) \to \mathbf{C}^{nk}$ is generically finite onto its image. Let $\tau : \mathcal{X} \to A$ be induced by the natural projection $J_k(A) \to A$. Let D be an ample divisor of A. Then there exists a polynomial P of $d^\ell z_j$ $(1 \leq \ell \leq k, 1 \leq j \leq n)$ with constant coefficients which vanishes on $\tau^{-1}(D)$ but does not vanish identically on $\mathcal{X}$. In particular, P defines a holomorphic jet differential on A whose pullback to $\mathcal{X}$ vanishes on the divisor $\tau^{-1}(D)$ of $\mathcal{X}$.*

Proof. Take a meromorphic function F on A whose pole divisor is a positive multiple of D. By assumption on τ and σ_k, $F \circ \tau$ belongs to a finite extension of the field of all rational functions of $\mathbf{C}^{nk}$. There exist polynomials P_j $(0 \leq j \leq p)$ with constant coefficients in the variables $d^\ell z_\nu$ $(1 \leq \ell \leq k, 1 \leq \nu \leq n)$ such that $\sum_{j=0}^{p} (\sigma_k^* P_j)(\tau^* F)^j = 0$ on $\mathcal{X}$ and $\sigma_k^* P_p$ is not identically zero on $\mathcal{X}$. Then P_p must vanish on $\tau^{-1}(D)$ and the holomorphic jet differential P_p on $\mathcal{X}$ must vanish on $\tau^{-1}(D)$. We need only set $P = P_p$.

1.2 Second Technique of Bloch

Lemma 2 (Schwarz Lemma on vanishing of pullback of holomorphic jet differentials vanishing on ample divisors to entire holomorphic curves). *Let X be a compact complex algebraic manifold and ω be a holomorphic jet differential on X vanishing on an ample divisor D of X. Let $\varphi : \mathbf{C} \to X$ be a holomorphic map. Then the pullback $\varphi^*\omega$ of ω by φ is identically zero on $\mathbf{C}$.*

Proof. Assume $\varphi^*\omega$ not identically zero on $\mathbf{C}$. We embed X into some $\mathbf{P}_N$ so that the hyperplane section line bundle of $\mathbf{P}_N$ pulls back to a positive multiple ℓ of D. By applying the defect relation of Ahlfors-Cartan [Ah41][Ca33] to $\mathbf{C} \to \mathbf{P}_N$ induced by φ, for any $0 < \varepsilon < 1$ we can find a hyperplane in $\mathbf{P}_N$ whose intersection E with X satisfies $N(r, \varphi, E) \geq (1 - \varepsilon) T(r, \varphi, E)$. Let s_D (respectively s_E) be the holomorphic section associated to D (respectively E) so that the divisor of s_D (respectively s_E) is D (respectively E). By replacing ω by $s_E (s_D)^{-\ell} \omega$, we can assume that $\ell = 1$ and $D = E$.

Let k and m be respectively the order and weight of ω. Using global meromorphic functions as local coordinates on X, we can find global meromorphic functions $F_{\ell,j}$ on X such that $|\omega| \leq C \sum_{1 \leq \ell \leq q, 1 \leq j \leq k} \left| d^k \log F_{\ell,j} \right|^{\frac{m}{k}}$ as functions on $J_k(X)$ for some $C > 0$. By the logarithmic derivative lemma, $T(r, \varphi^* \omega) = o(T(r, \varphi))$ when $\varphi^* \omega$ is considered as a function on $\mathbf{C}$. On the other hand, the vanishing of ω on D implies that $T(r, \varphi^* \omega)$ dominates the counting function $N(r, \varphi, D)$, which yields the contraction that $(1 - \varepsilon) T(r, \varphi^* \omega) = o(T(r, \varphi))$.

This proof by logarithmic derivative lemma is taken from [SY97] and is already implicit in a preprint of Min Ru and Pit-Mann Wong [RW95]. The Schwarz lemma and its proof can straightforwardly be adapted to the case of jet differentials with logarithmic poles along divisors.

Remark 1. The intuitive reason for the Schwarz lemma is that $\mathbf{C}$ does not admit a metric with curvature bounded from above by a negative number, even when certain degeneracies of the metric are allowed. The statement holds also for jet metrics. A jet metric assigns a value to a k-jet instead of to a tangent vector as is the case with a usual metric. One can use a holomorphic jet differential ω which vanishes on an ample divisor D to define a jet metric on $\mathbf{C}$ with curvature bounded from above by a negative number to give a contradiction if the pullback of the jet differential is not identically zero. The idea is to use the pullback of $\frac{|\omega|^2}{|s_D|^2}$, where $|s_D|$ is the pointwise norm of the canonical section s_D of the line bundle L associated to D computed with respect to a positively curved metric of L, and $|\omega|^2$ is just the usual absolute-value square without the use of any metric. A rigorous proof by curvature arguments can be found in [SY97].

Remark 2. The proof of Lemma 2 actually gives the following stronger result. Let X be a compact complex algebraic manifold and Y be a complex hypersurface in X (or the empty set). Let $\varphi : \mathbf{C} \to X - Y$ be a holomorphic map and V be the algebraic Zariski closure of the image of $d^k \varphi$ in $J_k(X)$. Let ω be a locally defined holomorphic k-jet differential on X with logarithmic poles along Y which vanishes on an ample divisor D of X and which is globally defined as a function on V. Then the pullback $\varphi^* \omega$ of ω by φ is identically zero on $\mathbf{C}$.

1.3 Third Technique of Bloch

Bloch's system of differential equations constructed from position-forgetting maps of different order. Let $\varphi : \mathbf{C} \to A$ be a holomorphic map from $\mathbf{C}$ to an n-dimensional abelian variety A whose universal cover $\mathbf{C}^n$ has coordinates $z_1, \cdots, z_n$. Let $\mathcal{X}_k$ be the algebraic closure of the image of $d^k \varphi$ in $J_k(A)$. Let $\Phi_k : \mathcal{X}_k \to \mathbf{C}^{kn}$ be the map induced by the position-forgetting map $\pi_k : J_k(A) = A \times \mathbf{C}^{kn} \to \mathbf{C}^{kn}$. Bloch introduced a system of differential equations from the properties of the kernel and image of $d\Phi_k$ and their relations to Φ_{k+1}. For the result (known now as Bloch's theorem) treated in Bloch's paper the specific property involving the kernel and image of $d\Phi_k$ is the finiteness of the generic fiber of Φ_k (for $k = 2$ in Bloch's case of a surface in a 3-dimensional abelian variety).

Lemma 3. (Translational invariance from differential equations). *Let $X = \mathcal{X}_0$ which is the algebraic Zariski closure of the image of φ in A. Assume that X is of complex dimension $m < n$. If the fibers of $\Phi_m : \mathcal{X}_m \to \mathbf{C}^{mn}$ are not generically finite, then there exists a nontrivial constant tangent vector v of A such that the translation operator $T_v : A \to A$ defined by v maps X to X.*

Proof. We assume that $\varphi(0)$ is a regular point of X where $z_1, \cdots, z_m$ are local coordinates of X. Write $dz_\alpha = \sum_{\nu=1}^m \omega_{\alpha\nu} dz_\nu$ on X for some meromorphic functions $\omega_{\alpha\nu}$ on X which are holomorphic near $\varphi(0)$ for $m + 1 \leq \alpha \leq n$.

Assume the fibers of $\Phi_m : \mathcal{X}_m \to \mathbf{C}^{mn}$ not generically finite. Since $\mathcal{X}_m \subset J_m(X)$, for $\zeta_0 \in \mathbf{C}$ close to 0 some nonzero tangent vector T to $J_m(X)$ at $Q := \varphi(\zeta_0)$ is mapped to zero by $J_m(A) \to \mathbf{C}^{mn}$. We can represent T by $\left(\left(\partial_\zeta^j \partial_t (z_\alpha \circ \psi)\right)(\zeta, 0)\right)_{0 \leq j \leq m, 1 \leq \alpha \leq n}$ for some holomorphic map $\psi(\zeta, t)$ from a neighborhood of $(\zeta_0, 0)$ in $\mathbf{C}^2$ to X with $\psi(\zeta, 0) = \varphi(\zeta)$. From the vanishing of $\left(\partial_\zeta^j \partial_t (z_\alpha \circ \psi)\right)(\zeta_0, 0)$ $(1 \leq j \leq m,\ 1 \leq \alpha \leq n)$ and $dz_\alpha = \sum_{\nu=1}^m \omega_{\alpha\nu} dz_\nu$ $(m + 1 \leq \alpha \leq n)$ it follows from the chain rule that $\sum_{\nu=1}^m (\partial_\zeta^j (\omega_{\alpha\nu} \circ \varphi)) \partial_t (z_\nu \circ \psi) = 0$ at $(\zeta_0, 0)$ for $1 \leq j \leq m$ and $1 \leq \alpha \leq n$. Since the m complex numbers $(\partial_t (z_\nu \circ \psi))(\zeta_0, 0)$ $(1 \leq \nu \leq m)$ are not all zero due to the nontriviality of T, it follows that the rank of the $((n - m)m) \times m$ matrix $M := ((\partial_\zeta^j (\omega_{\alpha\nu} \circ \varphi))(\zeta_0))$ is less than m when $1 \leq \nu \leq m$ is regarded as the row index and the double index (j, α) with $1 \leq j \leq m$ and $m + 1 \leq \alpha \leq n$ is regarded as the column index.

We now apply the usual Wronskian type argument simultaneously to the $(n-m)m$ sets of m functions which are the column vectors of M and conclude that there exist constants c_ν $(1 \leq \nu \leq m)$ not all zero, independent of ζ, such that $\sum_{\nu=1}^m c_\nu (\partial_\zeta^j (\omega_{\alpha\nu} \circ \varphi))(\zeta) \equiv 0$ for $1 \leq j \leq m$ and $m + 1 \leq \alpha \leq n$ and for all ζ close to 0. Hence the 1-parameter group action on A defined by the vector field $\sum_{\mu=1}^n c_\mu \frac{\partial}{\partial z_\mu}$ maps X to itself.

Let B be the set of all elements a of A such that X is invariant under translation of A by elements of a. By induction on the dimension of A and replacing A by A/B and replacing $\varphi : \mathbf{C} \to A$ by the composite of φ with the quotient map $A \to A/B$, we get the following theorem of Bloch.

Theorem 1. [B26] *Let A be an abelian variety and $\varphi : \mathbf{C} \to A$ be a holomorphic map. Let X be the Zariski closure of the image of φ. Then X is the translate of an abelian subvariety of A.*

The techniques for Bloch's theorem of translational invariance can easily be adapted to give the following generalization for the space of jets of entire holomorphic curves.

Theorem 2. *Let A an abelian variety and $\varphi : \mathbf{C} \to A$ be a holomorphic map whose image is Zariski dense in A. Let k be a positive integer and $\mathrm{Var}_{\varphi,k}$ be the algebraic Zariski closure of the k-th order jet of φ in the space $J_k(A)$ of k-jets of A. Then there is a finite unramified covering $\pi : \tilde{A} \to A$ of A and a decomposition $\tilde{A} = B_1 \times B_2 \times \cdots \times B_s$ of $\tilde{A}$ as a product of positive-dimensional abelian varieties such that the lift $\pi^*\mathrm{Var}_{\varphi,k}$ is invariant under translation by elements in B_1, and, in general, $\rho_i(\pi^*\mathrm{Var}_{\varphi,k})$ is invariant under translation by elements in B_{j+1}, where $\rho_j : J_k\left(\tilde{A}\right) \to J_k\left(\tilde{A}/(B_1 \times \cdots \times B_j)\right) = J_k(B_{j+1} \times \cdots \times B_s)$ is the natural projection map ($1 \leq j < s$).*

This theorem is in the "Correction" of [SY96a] and is a modification of Theorem 2.2 of [SY96a]. The original Theorem 2.2 of [SY96a] is stronger than what was demonstrated in its proof, as pointed out to us by Paul Vojta in an e-mail message on July 13, 2000. A trivial counter-example to the original Theorem 2.2 of [SY96a] is that A is the product of two copies of any elliptic curve and φ is induced by the map $\zeta \to (z_1, z_2) = \left(\zeta, \zeta^2\right)$ from $\mathbf{C}$ to the universal cover $\mathbf{C}^2$ of A.

Remark 3. Bloch's theorem (Theorem 1) gives the hyperbolicity of any complex submanifold Y in an abelian variety which does not contain any translate of an abelian subvariety. The problem concerning the Schottky radius for the hyperbolicity of Y is still open. The Schottky radius is the largest r such that there exists a holomorphic map from the 1-disk of radius r to Y when the differential of the map is normalized at the origin. The problem is to express or estimate the Schottky radius effectively in terms of numbers computable from the geometry and topology of Y.

1.4 Bloch's Original Presentation of the Three Techniques

To see how the three techniques are originally presented in Bloch's paper, first let us look at the setting of Bloch's paper, which is described on line 15 of page 36, "Soit une surface algébrique S d'irrégularité supérieure à 2, que nous supposons placée dans l'espace à trois dimensions, où elle a pour équation $f(x, y, z) = 0$. Les coordonnées d'un point P de la surface sont supposées fonctions méromorphes, dans tout le plan complexe, d'une variable t. Soient U, V, W trois integrales simples de premiere espèce linéairement indépendantes de la surface."

So the abelian variety A in Bloch's setting has complex dimension 3 and U, V, W are the coordinates of the universal cover $\mathbf{C}^3$ of A. His algebraic surface S is inside A and at the same time inside $\mathbf{P}_3$ defined by $f(x, y, z) = 0$ with inhomogeneous coordinates x, y, z of $\mathbf{P}_3$.

Bloch used the notation $\mathrm{gm}\,(r,f)$ to denote Nevanlinna's characteristic function $T\,(r,f)$ for a meromorphic function f on $\mathbf{C}$. When P is a point on and He considered a holomorphic map φ from $\mathbf{C}$ to a complex manifold X of complex dimension n and considered a point P on X where some global meromorphic functions $g_1,\cdots,g_n$ on X are used to define local coordinates of X at P. He used $\widehat{gm}\,(r,P)$ to denote the maximum of $gm\,(r;g_j\circ\varphi)$ for $1\leq j\leq n$. So far as the growth order up to sandwiching by two constant multiplies is concerned, $\widehat{gm}\,(r,P)$ is independent of the choice of $g_1,\cdots,g_n$. The use of P in the symbol $\widehat{gm}\,(r,P)$ simply emphasizes the fact that P is a variable point on a manifold X which is a target of a holomorphic map $\varphi:\mathbf{C}\to X$.

Bloch's Construction of Holomorphic Jet Differentials

On page 37, lines 10–11 from the bottom of [B26], he introduced holomorphic jet differentials by forming algebraic functions of dU,dV,dW,d^2U,d^2V,d^2W. His statement that "on obtiendra pour x et y des fonctions algébriques de dU,dV,dW,d^2U,d^2V,d^2W" means that the meromorphic functions x and y on X satisfies some polynomial equation $\sum_{j=0}^k P_j\xi^k=0$, where $P_\ell=P_\ell\left(dU,dV,dW,d^2U,d^2V,d^2W\right)$ is a polynomial of dU,dV,dW,d^2U,d^2V,d^2W. The equation actually is meant only to hold on $\mathcal{X}_2$, as one can see from "Si, donc, cette résolution est possible, comme cela aura lieu *au géneral*, on sera conduit à une impossibilité en supposant x, y, z fonctions méromorphes de t" on line 10 from the bottom of page 37 of [B26].

Bloch's Proof of Schwarz Lemma

From the preceding equation $\sum_{j=0}^k P_j\xi^k=0$ Bloch argued to get the contradiction that $\widehat{gm}\,(r,P)$ is dominated by $\log\widehat{gm}\,(r,P)$. He did not explicitly mention that the holomorphic jet differential P_0 vanishes on the ample divisor defined by the zero-set of x. Without much details he simply referred to Lemme III of [B26], which is an argument corresponding to the logarithmic derivative lemma. Since in the proof of Lemme III on lines 9–13 on page 33 of [B26], when he compared the characteristic functions of e^U and $\varPhi$, he used the comparison of the counting functions of the zero-sets of e^U-j and $\varPhi$ for some judiciously chosen point j, the natural way to fill in the details which he left out is that he would use the same method of comparison of the counting functions of the zero-sets. In other words, he would use the fact that the holomorphic jet differential P_0 vanishes on the zero-divisor of x. By this comparison he would get the domination of $T\,(r,x\circ\varphi)$ by a positive constant times $T\,(r,\varphi^*P_0)$. Thus by the use of the logarithmic derivative lemma $T\,(r,\varphi^*P_0)$ is dominated by a positive constant times $\widehat{gm}\,(r,P)$.

For this step the proof of the Schwarz lemma (Lemma 2) applies the logarithmic derivative lemma to $\frac{dx}{x},\frac{dy}{y},\frac{d^2x}{x},\frac{d^2y}{y}$ and compare $|P_0|$ to the absolute value of a polynomial of $\frac{dx}{x},\frac{dy}{y},\frac{d^2x}{x},\frac{d^2y}{y}$. Bloch did not apply the logarithmic derivative lemma the same way, but used some essentially equivalent argument. He applied the differentiation of Poisson's formula to $dU = \frac{d(e^U)}{e^U}$ and $\frac{d^2(e^U)}{e^U}$ and similar expressions for V, W. Since the function e^U is not a meromorphic function on the abelian variety A, he had to use the argument that the maximum of $T\left(r, e^U \circ \varphi\right)$, $T\left(r, e^V \circ \varphi\right)$, $T\left(r, e^W \circ \varphi\right)$ dominates a positive constant times $T(r, g \circ \varphi)$ for any meromorphic function g on the 3-dimensional abelian variety A. For the case of an elliptic curve C whose universal cover has coordinate U, he argued that there is a covering map $\pi : \mathbf{C} - 0 \to C$ with e^U as the coordinate of $\mathbf{C} - 0$. On lines 9 to 13 on page 33, by comparing the counting function for the zero-set of $e^U - j$ for a judiciously chosen point j of $\mathbf{C} - 0$ and the counting function of the zero-set of $\Phi - \pi(j)$ on C, he concluded that $T\left(r, e^U \circ \psi\right)$ dominates a positive constant times $T(r, \Phi \circ \psi)$ for any holomorphic map $\psi : \mathbf{C} \to \mathbf{C}$. He then adapted the argument for the elliptic curve C to the 3-dimensional abelian variety A.

Bloch's Differential Equations

In Bloch's paper the differential equations are given as follows. Consider the position-forgetting map defined by dU, dV, dW, d^2U, d^2V, d^2W. Bloch's paper (in equation (6) on page 38 of [B26]) states that if the position-forgetting map from the algebraic Zariski closure $\mathcal{X}_2$ in $J^2(A)$ to $\mathbf{C}^6$ is not generically of finite fiber, then for any local coordinates ξ, η of S one has the vanishing on $\mathcal{X}_2$ of the Jacobian determinant

$$\frac{\partial\left(d^2U, d^2V, d^2W, d^3U, d^3V, d^3W\right)}{\partial\left(d\xi, d\eta, d^2\xi, d^2\eta, d^3\xi, d^3\eta\right)} \tag{1}$$

when $d^2U, d^2V, d^2W, d^3U, d^3V, d^3W$ are regarded only as functions of $d\xi$, $d\eta$, $d^2\xi$, $d^2\eta$, $d^3\xi$, $d^3\eta$ with ξ, η regarded as constants for the purpose of forming the Jacobian determinant. In [B26] $\xi = x$ and $\eta = y$ are used. Since the vanishing of (1) on $\mathcal{X}_2$ is independent of the choice of local coordinates ξ, η, the notations ξ and η are used here just to avoid confusion in notations later. In the notation of Bloch's third technique in (1.3) with $m = 2$ and $n = 3$, this corresponds to the vanishing of the determinant of the 2×2 matrix $\left((\partial_\zeta^j(\omega_{\alpha\nu} \circ f))(\zeta_0)\right)_{j=1,2;\, \nu=1,2}$ with $\alpha = 3$. When we choose $U = z_1, V = z_2, W = z_3$ and the local coordinates $\xi = z_1, \eta = z_2$, the 2×2 matrix becomes

$$\begin{pmatrix} dW_\xi & dW_\eta \\ d^2W_\xi & d^2W_\eta \end{pmatrix} \tag{2}$$

pulled back to $\mathcal{X}_2$. Straightforward computation shows that on $\mathcal{X}_2$ the vanishing of (1) agrees with the vanishing of (2). Bloch then used the usual Wronskian argument to show that there are constants a, b, c such that $adU + bdV + cdW$ on S, which the equation on line 16 on page 39 of [B26]. The vanishing of (1) on $\mathcal{X}_2$ is the system of differential equations from the link between $d\Phi_k$ and Φ_{k+1}.

Remark 4. In Bloch's third technique the order of the jet space used is the same as the dimension of X. The zero-dimensionality of the generic fiber of $\Phi_k : \mathcal{X}_k \to \mathbf{C}^{kn}$ corresponds to the following statement used in diophantine approximation (Lemma 5.1 of [Voj96]).

Suppose A is an abelian variety and X is a subvariety of A which is not invariant under the translation of any nonzero element of A. Then for any $m > \dim X$ the map $X^{\times m} \to A^{\times \left(\frac{m(m-1)}{2}\right)}$ defined by $(x_j)_{1 \leq j \leq m} \mapsto (x_j - x_k)_{1 \leq j < k \leq m}$ is generically finite onto its image.

The jet space of order $m-1$ in Bloch's function theory case is replaced by a product of m copies of X in Vojta's case of diophantine approximation. The use of differences in diophantine approximation corresponds to the use of differentials in function theory. McQuillan [McQ96] used this correspondence to give a completely new proof of Bloch's theorem. Vojta's article [Voj99] gives a comprehensive survey on the relation between Nevanlinna theory and diophantine approximation.

1.5 Complement Case in the Setting of Abelian Varieties

Though hyperbolicity for the complement of a hypersurface in an abelian variety was not considered in Bloch's paper [B26], Bloch's techniques are also applicable to the complement case. We treat here the Second Main Theorem with truncated multiplicity which is the quantitative version and is the strongest known result for this setting. The result is in the Addendum of [SY97]. The system of differential equations from Bloch's third technique (1.3) is a bit unwieldy. In the paper proper of [SY97] we attempted to skirt their use by using the semi-continuity of cohomology groups in deformations. Noguchi, Winkelmann, and Yamanoi [NWY00] drew our attention to a difficulty in that part of [SY97]. The Addendum of [SY97] restores the use of Bloch's third technique (1.3) of a system of differential equations.

Theorem 3. *Let A be an abelian variety of complex dimension n and D be an ample divisor of A. Inductively let $k_0 = 0$ and $k_1 = 1$ and $k_{\ell+1} = k_\ell + 3^{n-\ell-1}\left(4(k_\ell+1)\right)^\ell D^n$ for $1 \leq \ell < n$. If $\varphi : \mathbf{C} \to A$ is a holomorphic map whose image is not contained in any translate of D, then $m(r,\varphi,D) + (N(r,\varphi,D) - N_{k_n}(r,\varphi,D)) = O(\log T(r,\varphi,D) + \log r)$ for r outside some set whose measure with respect to $\frac{dr}{r}$ is finite. In particular, the defect $\delta(\varphi,D)$ is zero and the complex manifold $A - D$ is hyperbolic.*

Proof. We observe that it suffices to show that

$$\pi_{k_n}(J_{k_n}(D)) \text{ does not contain } W_{k_n,\varphi}. \tag{3}$$

The reason is as follows. By (3) there exists a polynomial P in the variables $d^\mu z_\nu$ $(1 \leq \mu \leq k_n, 1 \leq \nu \leq n)$ with coefficients in $\mathbf{C}$ such that P is not identically zero on $W_{k_n,\varphi}$ and $\pi_{k_n}^* P$ is identically zero on $J_{k_n}(D)$. Let ∇ be the covariant differential operator for D defined from a metric of D whose curvature form has constant coefficients. We can write P as $\sum_{\mu=0}^{k_n} \tilde{\rho}_\mu \left(\nabla^\mu \theta_D\right)$ so that $\tilde{\rho}_\mu$ is a polynomial in $d^\mu z_\nu$ $(1 \leq \mu \leq k_n, 1 \leq \nu \leq n)$ whose coefficients are bounded smooth functions on $\mathbf{C}^n$. Let $\tilde{\varphi} : \mathbf{C} \to \mathbf{C}^n$ be the lifting of φ. By the logarithmic derivative lemma, both $m\left(r, \tilde{\varphi}^* P, \infty\right)$ and $m\left(r, \tilde{\varphi}^*\left(\frac{P}{\theta_D}\right), \infty\right)$ are of order $O\left(\log T\left(r, \varphi, D\right) + \log r\right)$. From the expression $P = \sum_{\mu=0}^{k_n} \tilde{\rho}_\mu \left(\nabla^\mu \theta_D\right)$ it follows that $N\left(r, \tilde{\varphi}^*\left(\frac{P}{\theta_D}\right), \infty\right)$ is dominated by $N_{k_n}\left(r, \varphi, D\right)$. Our conclusion then follows from the domination of $m\left(r, \varphi, D\right) + N\left(r, \varphi, D\right) = T\left(r, \tilde{\varphi}^*\left(\frac{1}{\theta_D}\right)\right) + O(1)$ by $T\left(r, \tilde{\varphi}^*\left(\frac{P}{\theta_D}\right)\right) + T\left(r, \tilde{\varphi}^*\left(\frac{1}{P}\right)\right) + O(1)$.

To prove (3) we define the following meromorphic connection for D. Choose $\tau_j \in \Gamma(A, jD)$ for $j = 3, 4$ with the property that for $1 \leq k \leq k_n$ there exists a proper subvariety Z_k of $W_{k,\varphi}$ such that for $\gamma_k \in W_{k,\varphi} - Z_k$ the set $\{\tau_3 \tau_4 = 0\} \cap J_k(D) \cap \pi_k^{-1}(\gamma_k)$ is a nowhere dense subvariety of $J_k(D) \cap \pi_k^{-1}(\gamma_k)$. Define $\mathcal{D}^k \theta_D = (\tau_4)^{k+1} d^k \left(\frac{\theta_D \tau_3}{\tau_4}\right)$ for $1 \leq k \leq k_n$.

Let $0 \leq \ell_0 \leq n$ be the largest integer such that the following $(A)_\ell$ holds with $\ell = \ell_0$.

$(A)_\ell$. There exists a proper subvariety E_ℓ of $W_{k_\ell,\varphi}$ such that the complex dimension of the common zero-set of $\left\langle \mathcal{D}^j \theta_D, \gamma \right\rangle$ $(0 \leq j \leq k_\ell)$ in $A - \{\tau_3 \tau_4 = 0\}$ is no more than $n - 1 - \ell$ for $\gamma \in W_{k_\ell,\varphi} - E_\ell$.

The definition of ℓ_0 implies that $(B)_\ell$ stated below holds with $\ell = \ell_0 + 1$ for the following reason. For a local complex subspace Y defined by holomorphic functions g_j $(0 \leq j \leq q - 1)$ and for a holomorphic germ g_q, if both the dimension and the multiplicity of the subspace defined by g_j $(0 \leq j \leq q)$ are the same of those of Y at a generic point Q of the highest-dimensional branches of Y, then g belongs to the ideal generated by $g_1, \cdots, g_q$ at Q.

We apply this to $g_j = \left\langle \mathcal{D}^j \theta_D, \gamma \right\rangle$ $(0 \leq j \leq q)$ for a generic $\gamma \in W_{q,\varphi}$ and note that $\left\langle \mathcal{D}^j \theta_D, \gamma \right\rangle$ depends only on $p_{j,q}\gamma$. We bound the multiplicity of Y by using $n - \ell_0 - 1$ generic global sections of the very ample line bundle $3D$ over A to get a zero-dimensional dimensional intersection with Y. We end up with $3^{n-\ell_0-1}\left(4\left(k_{\ell_0}+1\right)^{\ell_0}\right) D^n$ for the multiplicity. The factor $4\left(k_{\ell_0}+1\right)$ occurs, because we use ℓ_0 elements of $\Gamma\left(A, \left(4\left(k_{\ell_0}+1\right)\right) D\right)$ to locally define Y which are generic $\mathbf{C}$-linear combinations of $\sigma_j g_j$ $(0 \leq j \leq k_{\ell_0})$ for generic $\sigma_0 \in \Gamma\left(A, 4\left(k_{\ell_0}+1\right)\right)$ and $\sigma_j \in \Gamma\left(A, 4\left(k_{\ell_0}-j\right)\right)$ $(1 \leq j < k_{\ell_0})$ and $\sigma_{k_{\ell_0}} \equiv 1$. The sections σ_j are introduced so that all $\sigma_j g_j$ are sections of the same line bundle to enable us to take their $\mathbf{C}$-linear combinations.

$(B)_\ell$. There exist $k_{\ell-1} < q \leq k_\ell$, a subvariety V_ℓ of $J_q(D) \cap \pi_q^{-1}(W_{q,\varphi})$ containing $J_q(D) \cap \{\tau_3 \tau_4 = 0\}$, and a proper subvariety F_ℓ of $W_{q,\varphi}$, such that

(i) $(J_q(D) - V_\ell) \cap \pi_q^{-1}(\gamma_q)$ is a nonempty $(n - \ell)$-dimensional subvariety of $J_q(A) - V_\ell$ for every $\gamma_q \in W_{q,\varphi} - F_\ell$, and

(ii) on $J_q(A) - \pi_q^{-1}(F_\ell) - V_\ell$ the function $\mathcal{D}^q\theta_D$ locally belongs to the ideal sheaf which is locally generated by the functions $\mathcal{D}^j\theta_D$ $(0 \leq j \leq q-1)$ and by the pullbacks to $J_q(A)$ of the local holomorphic functions on $\mathbf{C}^{nq}$ vanishing on $W_{q,\varphi}$.

There exists some $\zeta_0 \in \mathbf{C}$ such that $\pi_q((d^q\varphi)(\zeta_0)) \in W_{q,\varphi} - F_{\ell_0+1}$. By $(B)_{\ell_0+1}$(i) there exists some point $Q_0 \in D$ such that when we define $g(\zeta) = (Q_0 - \varphi(\zeta_0)) + \varphi(\zeta) \in A$ for $\zeta \in \mathbf{C}$, we have $(d^q g)(\zeta_0) \in J_q(D) - V_{\ell_0+1} - \pi_q^{-1}(F_{\ell_0+1})$. Since $\pi_q((d^q g)(\zeta)) = \pi_q((d^q\varphi)(\zeta)) \in W_{q,\varphi}$ for $\zeta \in \mathbf{C}$, from $(B)_{\ell_0+1}$(ii) it follows that $\theta_D \circ g$ satisfies a differential equation $\left(\frac{d^q}{d\zeta^q}\right)(\theta_D \circ g)(\zeta) = \sum_{j=0}^{q-1} h_j(\zeta)\left(\frac{d^j}{d\zeta^j}\right)(\theta_D \circ g)(\zeta)$ on some open neighborhood U of ζ_0 in $\mathbf{C}$ for some holomorphic functions $h_0(\zeta), \cdots, h_{q-1}(\zeta)$ on U. By the uniqueness part of the fundamental theorem of ordinary differential equations, $(\theta_D \circ g)(\zeta)$ is identically zero for $\zeta \in \mathbf{C}$. Hence the image of φ is contained in the translate of D by $Q_0 - \varphi(\zeta_0)$. This contradicts the assumption of the image of φ not contained in any translate of D and finishes the verification of (3).

Remark 5. The same arguments work for semi-abelian varieties with straightforward modifications.

2 Setting of the Complex Projective Space

What has been guiding the investigations for this setting are the following two conjectures.

Conjecture 1. A generic hypersurface of degree at least $2n-1$ in $\mathbf{P}_n$ is hyperbolic for $n \geq 3$.

Conjecture 2. The complement in $\mathbf{P}_n$ of a generic hypersurface of degree at least $2n+1$ is hyperbolic.

Remark 6. The number $2n+1$ is used in Conjectures 1 and 2, because when one replaces the generic hypersurface in Conjecture 2 by a union of hyperplanes in general position, Green [Gr75] proved that the image of a holomorphic map from $\mathbf{C}$ to $\mathbf{P}_n$ minus $n+k$ hyperplanes in general position is contained in a linear subspace of dimension no more than the integral part $\left\lfloor \frac{n}{k} \right\rfloor$ of $\frac{n}{k}$, where the bound $\left\lfloor \frac{n}{k} \right\rfloor$ is sharp.

Some partial results are known. The set of hyperbolic hypersurfaces in $\mathbf{P}_n$ is open in the Hausdorff topology on the moduli space of hypersurfaces [Za89]. The complement in $\mathbf{P}_2$ of a generic curve of sufficiently high degree is hyperbolic [SY96]. A generic surface of degree at least 36 in $\mathbf{P}_3$ is hyperbolic [McQ99]. A generic surface of degree at least 21 in $\mathbf{P}_3$ and the complement of a generic curve of degree at least 21 are both hyperbolic [DE00]. For the surface case the new important techniques come from [McQ98]. We will discuss some of those techniques in Section 3. Improvements in the degree bound in [DE00] are mainly due to the use of a more restricted class of jet differentials called Semple jet differentials so that one needs fewer independent holomorphic jet differentials vanishing on ample divisors in this class in order to use the Schwarz lemma to force the image of every holomorphic map to lie in a proper subvariety. The Semple jet differentials are those whose pullbacks to any local complex curve depend only on the first differential of the local coordinate of the curve and not on any of its higher order differentials. Examples of Semple jet differentials are the determinants of the form $\left(d^j f_\ell\right)_{1\leq j,\ell\leq k}$, where $f_1, \cdots, f_k$ are functions.

The most recent result for Conjecture 1 for general dimension is the following. We will discuss the main ideas in its proof. The method can also be adapted for Conjecture 2, but we will not discuss its adaptation here.

Theorem 4. [Si02] *There exists a positive integer δ_n such that a generic hypersurface in $\mathbf{P}_n$ of degree $\geq \delta_n$ is hyperbolic.*

The method of proof is motivated by the work of Clemens-Ein-Voisin [C86][E88][Voi96] and uses the following ingredients: (1) the global generation of the bundle of meromorphic vector fields on the vertical jet space of the universal family of hypersurfaces with low pole order in the fiber direction;; (2) the method of counting monomials, motivated by the theorem of Riemann-Roch and the lower bound of the negativity of jet-differential bundles of the projective space, to produce holomorphic families of vertical jet differentials on fibers vanishing on ample divisors; (3) the action of low pole-order vector fields on the vertical jet space of the universal family of of hypersurfaces to produce independent vertical jet differentials on generic hypersurfaces.

The method is generalizable to complete intersections, but is unable to handle the conjecture that entire holomorphic curves in a compact algebraic manifold of general type must be contained in a fixed proper subvariety.

By Schwarz lemma (1.2), the hyperbolicity problem is reduced to the following two steps: (1) the construction of holomorphic jet differentials vanishing on some ample divisor; (2) to make sure that there are enough independent such jet differentials.

Besides Bloch's method (1.1) for the abelian setting, some available methods of construction of jet differentials are as follows: (1) the explicit construction similar to $\frac{P(x,y)dx}{R_y(x,y)} = -\frac{P(x,y)dy}{R_x(x,y)}$ for holomorphic 1-forms on a regular plane curve $R(x,y)=0$ with $\deg P(x,y) \leq \deg R(x,y)-3$; (2) the use of the theorem of Riemann-Roch (sometimes coupled with the stability properties of tangent bundles for more information on Chern classes). The last method is applicable only to complex surfaces because of the inability to handle the terms of the higher cohomology groups in the general dimension case.

One new technique introduced in the proof of Theorem 4 is to use a method motivated by the lower bound of the negativity of the jet differential bundle on $\mathbf{P}_n$ to take care of the terms of the higher cohomology groups in the application of the theorem of Riemann-Roch to hypersurfaces in $\mathbf{P}_n$.

Since our method is motivated by the result of Clemens-Ein-Vosin [C86] [E88] [Voi96], we first discuss their method.

2.1 Clemens's Result and Its Techniques (with Generations by Ein and Voisin)

Let $S = \mathbf{P}_N$ be the moduli space of all hypersurfaces in $\mathbf{P}_n$ of degree δ, where $N = \binom{\delta+n}{n} - 1$. Let $\mathcal{X} \to S$ be the universal family, defined by $f = \sum_{\nu \in \mathbf{N}^{n+1}, |\nu|=\delta} \alpha_\nu z^\nu$, which is nonsingular, of bidegree $(\delta, 1)$ in $\mathbf{P}_n \times \mathbf{P}_N$. Here for a multi-index $\nu = (\nu_0, \cdots, \nu_n)$ we denote $\nu_0 + \cdots + \nu_n$ by $|\nu|$.

Theorem 5 (Clemens [C86]). *If X is a generic hypersurface of degree $\delta \geq 2n-1$ in $\mathbf{P}_n$, then there does not exist any ratoinal curve (respectively elliptic curve) in X if $\delta \geq 2n-1$ (respectively $\delta \geq 2n$).*

The key step is the following lemma.

Lemma 4. *The $(1,0)$-twisted tangent bundle of $\mathcal{X}$ is globally generated.*

Proof. For $0 \leq p \leq n$ let e_p denote the $(n+1)$-tuple whose only nonzero entry is the integer 1 in the p-th position. Let L be any homogeneous polynomial L of degree 1 in the set of variables $\{\alpha_\nu\}_{|\nu|=\delta}$. If $0 \leq p \neq q \leq n$ and $\nu, \mu \in \mathbf{N}^{n+1}$ with $\nu + e_p = \mu + e_q$, then the $(1,0)$-twisted vector field $L\left(z_q\left(\frac{\partial}{\partial \alpha_\nu}\right) - z_p\left(\frac{\partial}{\partial \alpha_\mu}\right)\right)$ on $\mathbf{P}_n \times \mathbf{P}_N$ is tangential to $\mathcal{X}$. For any given $a_{j,k}$ $(0 \leq j,k \leq n)$, the vector field $\sum_{j,k=0}^n a_{j,k} z_j \frac{\partial}{\partial z_k} + \sum_{|\nu|=\delta} \beta_\nu \frac{\partial}{\partial \alpha_\nu}$ is tangential to $\mathcal{X}$ when $\beta_\nu = -\sum_{0 \leq j,k \leq n, j \neq k} \alpha_{\Phi_{j,k}(\nu)} a_{j,k}(\nu_k + 1) - \sum_{j=0}^n \alpha_{\nu_0,\nu_1,\cdots,\nu_n} a_{j,j} \nu_j$, where $\Phi_{j,k} : \mathbf{N}^{n+1} \to \mathbf{N}^{n+1}$ is defined by $\left(\Phi_{j,k}(\nu)\right)_\ell = \nu_\ell$ for $\ell \neq j,k$ with $\left(\Phi_{j,k}(\nu)\right)_j = \nu_j - 1$ and $\left(\Phi_{j,k}(\nu)\right)_k = \nu_k + 1$. Thus the $(1,0)$-twisted tangent bundle of $\mathcal{X}$ is globally generated.

Proof (of Theorem 5). Suppose there exists a curve C of genus 0 or 1 in a generic X. From such curves C we define a subvariety $\mathcal{C}$ of $\mathcal{X}$ so that for a generic X the restriction to X of the normal bundle $N_{\mathcal{X},\mathcal{C}}$ (of $\mathcal{C}$ in $\mathcal{X}$) equals the normal bundle $N_{X,C}$ (of C in X). By Lemma 4, the $(1,0)$-twisted normal bundle $N_{\mathcal{X},\mathcal{C}}$ is globally generated. Thus $N_{C,X} \otimes \mathcal{O}_{\mathbf{P}_n}(1)$ is globally generated. The adjunction formula implies that the genus of C is no less than $\delta - 2n + 1$.

2.2 Spaces of Vertical Jets and Vector Fields on Their Universal Space

The global generation of the $(1,0)$-twisted tangent bundle of $\mathcal{X}$ in Lemma 4 implies the global generation of all q-jets in the $(q+1,q)$-twisted tangent bundle of $\mathcal{X}$. If there exists a holomorphic jet differential ω on the part of $\mathcal{X}$ over an affine open subset of S which vanishes to sufficiently high order on some ample divisors of the fibers, then the derivatives of ω with respect to low pole-order vector fields on $\mathcal{X}$ would give enough independent holomorphic jet differentials on generic fibers vanishing on ample divisor. However, there is one difficulty with this approach to proving the hyperbolicity of generic fibers.

The difficulty is that one can find holomorphic jet differentials on each fiber of $\mathcal{X}$ but not on the total space $\mathcal{X}$ itself, because a holomorphic jet differential on $\mathcal{X}$ is different from a holomorphic family of fiber-wise jet differentials. The latter can be obtained by sufficient twisting in the moduli space, but there is no way to construct the former. To illustrate the difficulty, consider the situation of a family of plane curves $R(x,y,\alpha) = 0$. On each curve with parameter α, one can get the holomorphic 1-form $\frac{dx}{R_x(x,y,\alpha)} = -\frac{dy}{R_x(x,y,\alpha)}$ from $dR = 0$ when α is regarded as a constant. However, when α is a variable, $d\alpha$ gives us trouble in $dR = R_x dx + R_y dy + R_\alpha d\alpha$ and we cannot use this method to get a holomorphic 1-form on the total space of plane curves even when we confine α to the open unit disk in $\mathbf{C}$.

To overcome this difficulty one introduces the space $J_{n-1}^{\text{vert}}(\mathcal{X})$ of vertical $(n-1)$-jets which is defined by $f = df = \cdots = d^{n-1}f = 0$ in $J_{n-1}(\mathcal{X})$ with the coefficients α_ν of f regarded as constants when forming $d^j f$ $(1 \leq j \leq n-1)$.

The proof of the following proposition is along the lines of the proof of Lemma 4, but the bookkeeping of indices and the algorithms used are very involved. The two constants c_n, c'_n in the proposition can be explicitly written.

Proposition 1 (Existence of low pole-order vector fields). *There exist $c_n, c'_n \in \mathbf{N}$ such that the (c_n, c'_n)-twisted tangent bundle of the projectivization of $J_{n-1}^{\text{vert}}(\mathcal{X})$ is globally generated.*

To avoid considering singularities of weighted projective spaces, instead of using the projectivization of $J_{n-1}^{\text{vert}}(\mathcal{X})$, for the proof of Proposition 1 and its application we actually use functions whose restrictions to each fiber are polynomials, of homogeneous weight, in the differentials of the homogeneous coordinates of $\mathbf{P}_n$.

2.3 Construction of Jet Differentials on Hypersurfaces

The motivation for the new method of construction of holomorphic jet differentials on hypersurfaces of high degree is from the theorem of Riemann-Roch and the lower bound of the negativity of jet bundle of hypersurface X. Since the twisted cohomology groups of $\mathbf{P}_n$ can be computed from counting the number of certain monomials, in our construction we use the method of directly counting certain monomials instead of going through the theorem of Riemann-Roch. One key step is the following lemma on the nonvanishing of the restrictions, to a hypersurface of high degree, of jet differentials given by polynomials of low weight in the differentials of the inhomogeneous coordinates of $\mathbf{P}_n$. By using the high degree of the hypersurface and the vanishing of certain sheaf cohomology of the complex projective space minus a certain linear subspace, one proves the lemma by showing that the identical vanishing of the restriction of the jet differential would imply that the low-weight polynomial can be expressed in an impossible way in terms of the differentials of the high-degree polynomial f.

Lemma 5 (Nonvanishing of restrictions of low-weight jet differentials). *Let $1 \leq k \leq n-1$ and let f be a polynomial of degree δ in the inhomogeneous coordinates $x_1, \cdots, x_n$ of $\mathbf{P}_n$ so that the zero-set of f defines a complex manifold X in $\mathbf{P}_n$. Let Q be a non identically zero polynomial in the variables $d^j x_\ell$ $(0 \leq j \leq k, 1 \leq \ell \leq n)$. Assume that Q is of degree m_0 in $x_1, \cdots, x_n$ and is of homogeneous weight m in the variables $d^j x_\ell$ $(1 \leq j \leq k, 1 \leq \ell \leq n)$ when the weight of $d^j x_\ell$ is assigned to be j. If $m_0 + 2m < \delta$, then Q is not identically zero on the space of k-jets of X.*

Lemma 5 enables us to use the method of counting certain monomials to construct holomorphic jet differentials on hypersurfaces of sufficiently high degree which vanish on an ample divisor. The following proposition gives the precise statement on the existence of such jet differentials.

Proposition 2 (Existence of holomorphic jet differentials). *Let X be a nonsingular hypersurface of degree δ in $\mathbf{P}_n$ defined by a polynomial $f(x_1, \cdots, x_n)$ of degree δ in the affine coordinates $x_1, \cdots, x_n$ of $\mathbf{P}_n$. Suppose ϵ, ϵ', θ_0, θ, and θ' are numbers in the open interval $(0,1)$ such that $n\theta_0 + \theta \geq n + \epsilon$ and $\theta' < 1 - \epsilon'$. Then there exists an explicit positive number $A = A(n, \epsilon, \epsilon')$ depending only on n, ϵ, and ϵ' such that for $\delta \geq A$ and any nonsingular hypersurface X in $\mathbf{P}_n$ of degree δ there exists a non identically zero $\mathcal{O}_{\mathbf{P}_n}(-q)$-valued holomorphic $(n-1)$-jet differential ω on X of total weight m with $q \geq \delta^{\theta'}$ and $m \leq \delta^\theta$. Here, with respect to a local holomorphic coordinate system $w_1, \cdots, w_{n-1}$ of X, the weight of ω is in the variables $d^j w_\ell$ $(1 \leq j \leq n-1,\ \ 1 \leq \ell \leq n-1)$ with the weight j assigned to $d^j w_\ell$. Moreover, for any affine coordinates $x_1, \cdots, x_n$ of $\mathbf{P}_n$, when $f_{x_1} = 1$ defines in a nonsingular hypersurface in X, the $(n-1)$-jet differential ω can be chosen to be of the form $\frac{Q}{f_{x_1} - 1}$, where $q = \left\lfloor \delta^{\theta'} \right\rfloor$ and Q is a polynomial in $d^j x_1, \cdots, d^j x_n$ $(0 \leq j \leq n-1)$ which is of degree $m_0 = \left\lceil \delta^{\theta_0} \right\rceil$ in $x_1, \cdots, x_n$ and is of homogeneous weight $m = \left\lceil \delta^\theta \right\rceil$ in $d^j x_1, \cdots, d^j x_n$ $(1 \leq j \leq n-1)$ when the weight of $d^j x_\ell$ is assigned to be j.*

The proof of Proposition 2 consists of counting the number of coefficients of Q and counting the number of equations needed for the jet differential on X defined by Q to vanish on an ample divisor in X of high degree which is defined by the vanishing of a polynomial $g =$ in $\mathbf{P}_n$. For the counting of equations, we use $f = df = \cdots = d^{n-1} f = 0$ (with the coefficients α_ν treated as constants) to eliminate one coordinate and its differentials. One has to use the nonvanishing result from Lemma 5.

2.4 Argument of Genericity and Final Step of Proof of Hyperbolicity

Before we do the last step of the proof of hyperbolicity which is done by applying low-pole order vector fields to produce enough independent holomorphic jet differentials, we first explain where the genericity of the hypersurface comes from in our argument. A part of the genericity comes from the following well-known statement.

Proposition 3. *Let $\tilde{\pi} : \mathcal{Y} \to S$ be a flat holomorphic family of compact complex spaces and $\mathcal{L}$ be a holomorphic vector bundle over $\mathcal{Y}$. Then there exists a proper subvariety Z of S such that for $s \in S - Z$ the restriction map $\Gamma(U_s, \mathcal{L}) \to \Gamma\left(\tilde{\pi}^{-1}(s), \mathcal{L}|_{\tilde{\pi}^{-1}(s)}\right)$ is surjective for some open neighborhood U_s of s in S.*

For our application $S = \mathbf{P}_N$ is the parameter space of hypersurfaces of degree δ in $\mathbf{P}_n$ and the fiber $Y_s = \tilde{\pi}^{-1}(s)$ of the flat holomorphic family $\pi : \mathcal{Y} \to S$ of compact complex spaces is the weighted projective space defined from the space $J_{n-1}(X_s)$ of $(n-1)$-jets of the hypersurface X_s with moduli s, where the weight is chosen to correspond to the weight of $d^j x_\ell$ being j. The restriction to Y_s of the holomorphic vector bundle $\mathcal{L}$ corresponds to the bundle of $(n-1)$-jet differentials of weight m twisted by $\mathcal{O}_{\mathbf{P}_n}(-q)$.

When $s_0 \in S - Z$ the hypersurface satisfies the following weaker form of hyperbolicity (also known as the algebraic degeneracy of entire holomorphic curves): there exists a proper subvariety X'_{s_0} in X_{s_0} such that the image of any nonconstant holomorphic map $\varphi : \mathbf{C} \to X_{s_0}$ must be contained in X'_{s_0}. The reason is as follows.

On a generic hypersurface X_{s_0} the jet differential $\frac{Q}{g}$ can be extended holomorphically to ω on $\bigcup_{s \in U_{s_0}} X_s$ with values in $\mathcal{O}_{\mathbf{P}_n}(-q)$. Consider ω as a function on $J_{n-1}(X_s)$. The vanishing order of this function ω as a function of the variables $d^j x_\ell$ $(1 \leq \ell \leq n-1, \;\; 1 \leq j \leq n-1)$ is no more than the weight m of ω at a generic point of X_{s_0} outside the zero-section of $J_{n-1}(X_{s_0})$. We use meromorphic vector fields $v_1, \cdots, v_p$ on $J^{\text{vert}}_{n-1}(\mathcal{X})$ of low pole order in the fiber direction from Proposition 2. We now differentiate this function ω no more than m times by such meromorphic vector fields $v_1, \cdots, v_p$ on $J^{\text{vert}}_{n-1}(\mathcal{X})$ to form $v_1 \cdots v_p \, \omega$. The restriction of $v_1 \cdots v_p \, \omega$ to X_{s_0} defines a holomorphic jet differential on X_{s_0} which vanishes on an ample divisor, because the vanishing order of ω in the fiber direction is far greater than the pole order of v_j in the fiber direction. The collection of holomorphic jet differentials on X_{s_0} defined by $v_1 \cdots v_p \, \omega$ would have no common zeros on $J_{n-1}(X_{s_0})$ over a generic point of X_{s_0} outside the zero-section of $J_{n-1}(X_{s_0})$.

For the proof of hyperbolicity in its original sense, we need another part of the definition of genericity which concerns the maximum possible touching order of jet differentials of low degree and weight. More precisely, there exists some proper subvariety Z' of S with the following property.

Lemma 6 (Generic low vanishing order of constructed jet differentials). *Suppose $s \in S - Z'$ and Q is a polynomial in $d^j x_\ell$ ($1 \leq \ell \leq n$, $0 \leq j \leq n-1$) of degree no more than m_0 in $x_1, \cdots, x_n$ and of homogeneous weight no more than m in $d^j x_\ell$ ($1 \leq \ell \leq n$, $1 \leq j \leq n-1$). Then the vanishing order of the pullback of Q to X_s is no more than $a_{\delta,m_0,m}$ when $\delta \geq \delta_{n,m_0,m}$, where the numbers $a_{\delta,m_0,m}$ and $\delta_{n,m_0,m}$ can be explicitly written and the product of $a_{\delta,m_0,m}$ and the maximum fiber pole order of the meromorphic vector fields from Proposition 1 is far less than the minimum fiber vanishing order of the holomorphic vector field ω from Proposition 2 and from extension by Proposition 3.*

Once we have this additional part of genericity, the collection of holomorphic jet differentials on X_{s_0} defined by $v_1 \cdots v_p \, \omega$ would have no common zeros on $J_{n-1}(X_{s_0})$ at any point of X_{s_0} outside the zero-section of $J_{n-1}(X_{s_0})$. Let us know look at the proof of Lemma 6. Low vanishing order is a Zariski open condition. It suffices to prove the existence of one point $s_0 \in S - Z'$. The idea is to embed $\mathbf{P}_n$ generically into $\mathbf{P}_{\tilde{n}}$ by a high degree map of degree d and then pull back the holomorphic jet differential on a hypersurface $\tilde{X}$ in $\mathbf{P}_{\tilde{n}}$ to the intersection X of $\tilde{X}$ with $\mathbf{P}_n$. This method enables us to control the vanishing order of the coefficients at the expense of increasing the order of the jet differential. One drawback is that the degree of the hypersurface X must now contain the factor d. To overcome this drawback, we use a generic embedding of $\mathbf{P}_n$ into $\mathbf{P}_{2n+1} \times \mathbf{P}_{2n+1}$ by a map of appropriate bidegree (d_1, d_2) and construct jet differentials on a hypersurface $\tilde{X}$ in $\mathbf{P}_{2n+1} \times \mathbf{P}_{2n+1}$ and pull them back to the intersection X of $\tilde{X}$ with $\mathbf{P}_n$. This concludes the discussion of the ideas of the proof of Theorem 4.

3 McQuillan's Techniques for Hyperbolicity of Surfaces

McQuillan introduced a number of completely new techniques to treat the hyperbolicity problems for compact complex algebraic surfaces of general type [McQ98]. His techniques were motivated by concepts from diophantine approximation. Here we highlight some of his techniques, recast them in more analytical formulations, and sketch a purely function-theoretical approach which uses the familiar tools of Nevanlinna theory.

3.1 Three Techniques of McQuillan

We select three techniques from [McQ98]. The last two are relevant to our sketch of a purely function-theoretical approach which uses the familiar tools of Nevanlinna theory. The first one is an interpretation of Nevanlinna theory in terms of cohomology and intersection theory which may point the way to other possibilities of Nevanlinna theory in other types of intersection theory.

Currents Associated to an Entire Holomorphic Curve

Let X be a compact complex manifold of complex dimension n and $\varphi : \mathbf{C} \to X$ be a holomorphic map. Let H be an ample line bundle over X with positive smooth curvature form Θ_H and R be a sequence of positive real numbers with ∞ as limit. One defines the positive $(n-1, n-1)$-current $[\varphi]$ on X as follows. For a smooth $(1,1)$-form ω on X, the value $[\varphi](\omega)$ at ω is defined as the limit of $\frac{1}{T(r,\varphi,\Theta_H)} \int_{\rho=0}^{r} \frac{d\rho}{\rho} \int_{|\zeta|<\rho} \varphi^* \omega$ as $r \in R$ goes to ∞. McQuillan [McQ98] verified that for a suitable choice of the coordinate ζ of $\mathbf{C}$ and of the sequence R the positive $(n-1, n-1)$-current $[\varphi]$ is closed on X. When Y is a complex hypersurface in X, since the closed positive $(1,1)$-current $[Y]$ defined by (integration over the regular part of) Y is not smooth, it is not meaningful to evaluate the closed positive $(n-1, n-1)$-current $[\varphi]$ at $[Y]$. However, one can still define the value of $[\varphi]$ at $[Y]$ as the limit of $\frac{1}{T(r,\varphi,\Theta_H)} \int_{\rho=0}^{r} n(r, \varphi, Y) \frac{d\rho}{\rho}$ as $r \in R$ goes to ∞. In general this value may be different from the cup product of the cohomology class of $[\varphi]$ and the cohomology class of $[Y]$. As a matter of fact, when the cohomology class $[Y]$ agrees with the first Chern class of some ample line bundle E, the two values agree if and only if the defect $\delta(\varphi, Y)$ for φ and Y is 0. In [McQ98] this technique is on pages 127–132.

Remark 7. This technique gives an enlightening interpretation of Nevanlinna theory in terms of the discrepancy between the intersection number of the current of an entire holomorphic curve and a divisor and the cup product of the two cohomology classes defined by both. It leads naturally to possibilities of analogs of Nevanlinna theory for other products of cohomology classes and the corresponding intersection theory, for example, the quantum cohomology theory (see *e.g.,* [Va92][MS94]). We take three subvarieties Y_1, Y_2, Y_3 in a compact complex algebraic manifold X. In quantum cohomology one counts the number of holomorphic maps f from $\mathbf{P}_1$ with three marked points P_1, P_2, P_3 to X so that $P_j \in Y_j$ for $1 \leq j \leq 3$ and f satisfies certain topological conditions. We replace one of the subvarieties by the disk of radius r in an entire holomorphic curve and pass to limit as $r \to \infty$ after normalization to get a number. The analog of Nevanlinna theory is to study how this number varies when the other two subvarieties vary in their respective cohomology classes.

From Parabolic Leaf in Foliation to Defect for Reduced Points

We now consider the case of a compact complex algebraic surface X. Suppose $\mathcal{F}$ is an algebraic foliation in X with possible singularities. Assume that the singularity set Z of $\mathcal{F}$ at worst consist of reduced points. It means that $\mathcal{F}$ is locally defined by $f(x,y)dx + g(x,y)dy = 0$ for local coordinates x, y so that the common zero-set of the two holomorphic functions $f(x,y)$ and $g(x,y)$ consist of points with reduced structure (*i.e.,* every stalk of the ideal sheaf generated by $f(x,y)$ and $g(x,y)$ is either the unit ideal or the maximum ideal). Associated to $\mathcal{F}$ we have an sequence $0 \to N \to \Omega^1_X \to L \to 0$, where N and

L are holomorphic line bundles over X, so that the sequence is exact outside Z.

We give X some Kähler metric so that it induces a Hermitian metric along the fibers of the line bundle N on $X - Z$. Let Θ_N be the curvature of N. Now assume that the holomorphic map φ is along the leaves of the algebraic foliation $\mathcal{F}$ in the sense that the local holomorphic 1-form $f(x,y)dx+g(x,y)dy$ defining $\mathcal{F}$ is pulled back to 0 by φ. Since the transition functions for N on $X - Z$ can be chosen to be locally independent of the coordinate along the leaves of $\mathcal{F}$, one can choose a closed $(1,1)$-form $\tilde{\Theta}_N$ on $X-Z$ representing the Chern class of N on $X-Z$ so that its pullback to the leaves of $\mathcal{F}$ vanishes (*i.e.,* its pullback by φ vanishes). In particular, $T\left(r, \varphi, \tilde{\Theta}_N\right) = 0$. If Z is empty, then the difference of the two smooth $(1,1)$-forms Θ_N and $\tilde{\Theta}_N$ is exact on X and, by a lemma of Kodaira, is equal to the $\partial\bar{\partial}$ of some smooth function on X. So under the assumption of Z being empty, we have $T\left(r, \varphi, \Theta_N\right) = O(1)$. When Z is a nonempty set of reduced points, the same argument done carefully with estimates near the points of Z gives us $T\left(r, \varphi, \Theta_N\right) = m\left(r, \varphi, Z\right) + O(1)$, where $m\left(r, \varphi, Z\right)$ is the proximity function to Z defined as $\oint_{|\zeta|=r} \log^+ \frac{1}{(\mathrm{dist}_Z)\circ\varphi}$ with dist_Z denoting the distance function to Z. In [McQ98] this technique is in the section on residue heights on pages 144–146.

Defect Relation for Reduced Points

In [McQ98] McQuillan introduced his “refined tautological inequality” to give the defect relation for reduced points on page 167. We are going to give the roughly equivalent traditional function-theoretical formulation for the defect relation for points with condition on the cotangent bundle. This is in contrast to the defect relation for hypersurfaces with condition on the canonical line bundle. The idea of a condition on the cotangent bundle for defect for points is compatible with the intuitive geometric expectation in Nevanlinna theory.

Theorem 6 (DEFECT RELATION FOR POINTS). *Let H be an ample line bundle over a compact complex manifold Y of complex dimension n. Let Z be a finite subset of Y. Let $\varphi : \mathbf{C} \to Y$ be a holomorphic map. Let α be a positive rational number and ℓ be a positive integer such that $\alpha\ell$ is an integer. Let $\sigma \in \Gamma\left(Y, \mathrm{Sym}^{\ell}\left(\Omega^1_Y\right) \otimes \left(\alpha\ell H\right)\right)$ such that $\varphi^*\sigma$ is not identically zero on $\mathbf{C}$. Let W be the zero divisor of σ in the projectivization $\mathbf{P}\left(T_Y\right)$ of the tangent bundle T_Y of Y when σ is naturally regarded as the section of a line bundle over $\mathbf{P}\left(T_Y\right)$. Then $\frac{1}{\ell} N\left(r, d\varphi, W\right) + \oint_{|\zeta|=r} \log^+ \left|\frac{1}{\mathrm{dist}_Z(\varphi(\zeta))}\right| \leq \alpha\, T(r,\varphi,H) + O\left(\log T(r,\varphi,H) + \log r\right)$ for r outside a set with finite measure with respect to $\frac{dr}{r}$, where dist_Z is the distance function to Z with respect to any smooth metric of Y. The conclusion still holds when τ is replaced by a global holomorphic section of $\mathrm{Sym}^{\ell}\left(\Omega^1_Y\right) \otimes \left(\alpha\ell H\right)$ over only the algebraic Zariski closure of the image of $d\varphi$.*

For the proof of Theorem 6 we need the following simple lemma for the pullback of holomorphic forms by the blow-up map for a point.

Lemma 7. *Let G be an open neighborhood of 0 in $\mathbf{C}^n$ and $\pi : \tilde{G} \to G$ be the blow-up map for blowing up the point 0. Let $D = \pi^{-1}(0)$. Then $\pi^*\left(\mathcal{O}_G\left(\Omega^1_G\right)\right) \subset \mathcal{I}_D\left(\Omega^1_{\tilde{G}}(\log D)\right)$, where $\mathcal{I}_D$ is the ideal sheaf for the divisor D and $\Omega^1_{\tilde{G}}(\log D)$ is the bundle of 1-forms on $\tilde{G}$ with logarithmic poles along D.*

Proof. The blowup $\tilde{G}$ is the complex submanifold of $G \times \mathbf{P}_{n-1}$ defined by $w_j z_k = w_k z_j$ for $1 \leq j \neq k \leq n$, where $(z_1, \cdots, z_n)$ is the coordinate of $\mathbf{C}^n$ and $[w_1, \cdots, w_n]$ is the homogeneous coordinate of $\mathbf{P}_{n-1}$. Let U_k be the affine subset of $\mathbf{P}_{n-1}$ defined by $w_k \neq 0$ $(1 \leq k \leq n)$. Let $\widetilde{U_k}$ be $\tilde{G} \cap (G \times U_k)$. On $\widetilde{U_k}$ we choose the coordinate $z_k, \frac{w_j}{w_k}$ $(1 \leq j \leq n,\, j \neq k)$ so that we have $dz_j = z_k d\left(\frac{w_j}{w_k}\right) + \left(\frac{w_j}{w_k}\right) dz_k$ on $\widetilde{U_k}$ for $1 \leq j \leq n$ and $j \neq k$. The subvariety $D \cap \widetilde{U_k}$ of $\widetilde{U_k}$ is defined by the single equation $z_k = 0$. Since for $1 \leq j \neq k \leq n$ on $\widetilde{U_k}$ we can write $dz_k = z_k\left(\frac{dz_k}{z_k}\right)$ and $dz_j = z_k\left(d\left(\frac{w_j}{w_k}\right) + \left(\frac{w_j}{w_k}\right)\left(\frac{dz_k}{z_k}\right)\right)$ for $j \neq k$, it follows that the restriction of $\pi^*(dz_\ell)$ to $\widetilde{U_k}$ as a differential with log-pole along D has coefficients vanishing on D for $1 \leq \ell \leq n$.

Proof (of Theorem 6). Let $\pi : \tilde{Y} \to Y$ be the blow-up of Z and let $E = \pi^{-1}(Z)$. Let $\tilde{\varphi} : \mathbf{C} \to \tilde{Y}$ be the lifting of φ and let $\tau = \pi^*\sigma$. By Lemma 7, τ is a holomorphic section of $\mathrm{Sym}^\ell\left(\Omega^1_{\tilde{X}}(\log E)\right) \otimes \pi^*(\alpha\ell H)$ over $\tilde{Y}$ and vanishes to order at least ℓ on E. Let $e^{-\chi}$ be a smooth metric for H.

We denote also by E the line bundle on $\tilde{Y}$ associated to the divisor E. Let s_E be the canonical section of E and $e^{-\kappa}$ be a smooth Hermitian metric along the fibers of E. Let $\tilde{\tau} = \frac{\tau}{s_E^\ell}$. Then $\tilde{\tau}$ is a holomorphic section of $\mathrm{Sym}^\ell\left(\Omega^1_{\tilde{X}}(\log E)\right) \otimes \pi^*(\alpha\ell H) \otimes (-\ell E)$ over $\tilde{Y}$.

By applying the logarithmic derivative lemma and using global meromorphic functions as local coordinates, we know that $\oint_{|\zeta|=r} \log^+ \varphi^*\left(|\tau|^2 e^{-\alpha\ell\chi}\right)$ and $\oint_{|\zeta|=r} \log^+ \tilde{\varphi}^*\left(|\tilde{\tau}|^2 e^{\ell\kappa - \alpha\ell\chi}\right)$ are both of the order $O(\log T(r,\varphi) + \log r)$. Using $\log x = \log^+ x - \log^+ \frac{1}{x}$ for any $x > 0$ and the identity $-\log \Phi(0) + \oint_{|\zeta|=r} \log \Phi = \int_{\rho=0}^r \frac{d\rho}{\rho} \int_{|\zeta|<\rho} \frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log \Phi$, we obtain

$$
\begin{aligned}
2\ell\, m(r, \tilde{\varphi}, E) &= \oint_{|\zeta|=r} \log^+ \frac{1}{\tilde{\varphi}^*\left(\left|s_E^\ell\right|^2 e^{-\ell\kappa}\right)} + O(1) \\
\leq \oint_{|\zeta|=r} \log^+ \frac{1}{\varphi^*\left(|\tau|^2 e^{-\alpha\ell\chi}\right)} &+ \oint_{|\zeta|=r} \log^+ \tilde{\varphi}^*\left(|\tilde{\tau}|^2 e^{\ell\kappa - \alpha\ell\chi}\right) + O(1) \\
&= -\oint_{|\zeta|=r} \log \varphi^*\left(|\tau|^2 e^{-\alpha\ell\chi}\right) + \oint_{|\zeta|=r} \log^+ \varphi^*\left(|\tau|^2 e^{-\alpha\ell\chi}\right)
\end{aligned}
$$

$$+\oint_{|\zeta|=r}\log^{+}\tilde{\varphi}^{*}\left(|\tilde{\tau}|^{2}e^{\ell\kappa-\alpha\ell\chi}\right)+O(1)$$
$$=-2\,N\left(r,\operatorname{div}\left(\varphi^{*}\tau\right)\right)+2\alpha\ell\,T\left(r,\varphi,H\right)+O\left(\log T\left(r,\varphi\right)+\log r\right)$$
$$=-2\,N\left(r,d\varphi,W\right)+2\alpha\ell\,T\left(r,\varphi,H\right)+O\left(\log T\left(r,\varphi\right)+\log r\right),$$

where $N\left(r,\operatorname{div}\left(\varphi^{*}\tau\right)\right)$ means the counting function $\int_{\rho=0}^{r}n\left(\rho,\operatorname{div}\left(\varphi^{*}\tau\right)\right)\frac{d\rho}{\rho}$ of the divisor $\operatorname{div}\left(\varphi^{*}\tau\right)$ of $\varphi^{*}\tau$, with $n\left(\rho,\operatorname{div}\left(\varphi^{*}\tau\right)\right)$ denoting the number of points of the divisor of $\varphi^{*}\tau$ in $|\zeta|<\rho$ with multiplicities counted.

Remark 8. We now reconcile our Theorem 6 with McQuillan's refined tautological inequality in Theorem 3.3.2 *bis* on page 167 of [McQ98] in the following way. On $\mathbf{P}\left(T_{Y}\right)$ global sections of $\ell\,H_{\mathbf{P}(T_{Y})}\otimes\left(p\,H\right)$ over $\mathbf{P}\left(T_{Y}\right)$ are elements of $\Gamma\left(Y,\operatorname{Sym}^{\ell}\left(\Omega_{Y}^{1}\right)\otimes\left(p\,H\right)\right)$. Here $H_{\mathbf{P}(T_{Y})}$ is the line bundle over the projectivization $\mathbf{P}\left(T_{Y}\right)$ of the tangent bundle T_{Y} of Y so that the restriction of $H_{\mathbf{P}(T_{Y})}$ to each fiber of $\mathbf{P}\left(T_{Y}\right)$ over a point of Y is the hyperplane section line bundle of that fiber.

When we choose α sufficiently large, the line bundle $H_{\mathbf{P}(T_{Y})}\otimes\left(\alpha\,H\right)$ is ample over $\mathbf{P}\left(T_{Y}\right)$. When we choose ℓ sufficiently large, the line bundle $\ell\,H_{\mathbf{P}(T_{Y})}\otimes\left(\ell\,\alpha\,H\right)$ is very ample over $\mathbf{P}\left(T_{Y}\right)$. Then there exists $\sigma\in\Gamma\left(Y,\operatorname{Sym}^{\ell}\left(\Omega_{Y}^{1}\right)\otimes\left(\alpha\ell\,H\right)\right)$ not identically zero so that the defect for the divisor W of σ on $\mathbf{P}\left(T_{Y}\right)$ and for the map $d\varphi:\mathbf{C}\to\mathbf{P}\left(T_{Y}\right)$ is zero. From Theorem 6 and $T\left(r,d\varphi,H_{\mathbf{P}(T_{Y})}\right)=\frac{1}{\ell}N\left(r,d\varphi,W\right)-\alpha\,T(r,\varphi,H)$ it follows that $T\left(r,d\varphi,H_{\mathbf{P}(T_{Y})}\right)+m\left(r,\varphi,Z\right)\leq O\left(\log T(r,\varphi,H)+\log r\right)$. This is the tautological inequality in Theorem 3.3.2 *bis* on page 167 of [McQ98].

3.2 Nonexistence of Zariski Dense Parabolic Leaf in Surface of General Type

We consider the hyperbolicity problem for a compact complex algebraic surface X of general type (for example, for a surface X of sufficiently high degree in $\mathbf{P}_3$) for which the technique of Bogomolov ([Bo79], page 523 of [DE00]) produces an algebraic multi-foliation $\mathcal{F}$ so that every entire holomorphic curve in X must be along the leaves of $\mathcal{F}$. Here an algebraic multi-foliation means that it is locally defined by some local holomorphic 1-jet differentials. This reduces the problem of showing the nonexistence of Zariski dense entire holomorphic curves in X to the nonexistence of Zariski dense parabolic leaves in $\mathcal{F}$.

We now use the last two of the three techniques of McQuillan listed above to sketch a purely function-theoretic proof of the crucial step of the nonexistence of Zariski dense parabolic leaves in an algebraic multi-foliation in a compact complex algebraic surface of general type. To make the sketch easier, we consider the case of a surface X of sufficiently high degree in $\mathbf{P}_3$. Assume that there is such a leaf given by a holomorphic map $\varphi:\mathbf{C}\to X$ and we are going to derive a contradiction.

The technique of Miyaoka's almost ampleness theorem ([Mi82], [LY90], Th.0.2.2 on page 143 of [McQ98], Th.2.3 on page 524 of [DE00]) gives the bigness of the line bundle B of holomorphic 1-jets along an algebraic multi-foliation of a surface with $c_1^2 > 2c_2$. If we add some twisting to B and replace B by $B + \gamma K_X$ for some positive rational number γ, then for $(1+\beta)\, c_1^2 > c_2$ with $0 \leq \beta < 1$ Miyaoka's technique gives the bigness of $B + \gamma K_X$ when $\gamma^2 + 3\gamma - \frac{1}{2} > \beta$, which is satisfied when $\gamma > \frac{-3+\sqrt{11+\beta}}{2}$, and, in particular, when $\gamma \geq \frac{1}{6}$ when β is sufficiently small.

To make the presentation a little simpler, we assume at this point that the algebraic multi-foliation $\mathcal{F}$ is actually an algebraic foliation whose singular set Z consists only of reduced points. The general case can be handled by taking a branched cover of X and using monoidal transformations. By Theorem 6, the proximity function $m\,(r, \varphi, Z)$ for a set Z of reduced points in X is dominated by $\gamma T(r, \varphi, K_X) + O\,(\log T(r, \varphi) + \log r)$.

The Kähler metric of X induces also Hermitian metrics on the line bundle L on $X - Z$ and on the canonical line bundle K_X of X. Let Θ_{K_X} and Θ_L denote the curvature of K_X and L respectively. For any tangent vector ξ of type $(1, 0)$ in $X - Z$, we have $\Theta_{K_X}\left(\xi, \bar{\xi}\right) = \Theta_N\left(\xi, \bar{\xi}\right) + \Theta_L\left(\xi, \bar{\xi}\right)$, because the contributions, from the second fundamental form for a subbundle and the quotient bundle, to Θ_N and Θ_L are equal and of opposite sign when both are computed from the curvature of the cotangent bundle Ω_X^1 of X. Thus $T\,(r, \varphi, \Theta_{K_X}) = T\,(r, \varphi, \Theta_N) + T\,(r, \varphi, \Theta_L)$ for any holomorphic map $\varphi : \mathbf{C} \to X$ when we confine our consideration only to points of $X - Z$. Since the singularity set Z of $\mathcal{F}$ consists only of reduced points, the difference between $T\,(r, \varphi, \Theta_N)$ and $T\,(r, \varphi, N)$ and also the difference between $T\,(r, \varphi, \Theta_L)$ and $T\,(r, \varphi, L)$ are each no more than $m\,(r, \varphi, Z)$. We conclude that $T\,(r, \varphi, \Theta_L)$ dominates $(1 - 3\gamma - \varepsilon)\, T(r, \varphi, K_X)$ for any $\varepsilon > 0$.

To derive a contradiction, for some sufficiently large integer ℓ with $\ell\gamma$ also an integer, we take a non identically zero global section τ of $\mathrm{Sym}^\ell\left(\Omega_X^1\right) \otimes (\gamma\ell\Theta_{K_X})$ over the algebraic Zariski closure of the image of $d\varphi$ (which is the algebraic foliation $\mathcal{F}$). Let σ be the global section of $\ell L + \gamma\ell K_X$ over X which is induced by τ. We assume that ℓ is sufficiently large so that $\ell\gamma K_X$ is very ample. Let θ be a general element of $\Gamma\,(X, \ell\gamma K_X)$ with divisor D. Let $e^{-\chi}$ be a smooth metric for $\ell\gamma K_X$.

By using the logarithmic derivative lemma and using global meromorphic functions as local coordinates, we obtain $\oint_{|\zeta|=r} \log^+ \varphi^*\left(|\tau|^2 e^{-\chi}\right) = O\,(\log T(r, \varphi) + \log r)$. Let D be the zero-divisor of θ. Then $\oint_{|\zeta|=r} \log^+ \varphi^*\left(\left|\frac{\tau}{\theta}\right|^2\right)$ does not exceed

$$\begin{aligned}\oint_{|\zeta|=r} \log^+ \varphi^*\left(|\tau|^2 e^{-\chi}\right) + \oint_{|\zeta|=r} \log^+ \varphi^*\left(\frac{1}{|\theta|^2 e^{-\chi}}\right) + O(1) \\ = T\,(r, \varphi, \gamma\ell K_X) - N\,(r, \varphi, D) + O\,(\log T(r, \varphi) + \log r)\,.\end{aligned}$$

We compute $T\,(r, \varphi, \ell L + \gamma\ell K_X)$ by using the counting function and the proximity function of the zero-set of σ. We compute the Nevanlinna characteristic function $T\left(r, \varphi^*\left(\frac{\tau}{\theta}\right)\right)$ of the meromorphic function $\varphi^*\left(\frac{\tau}{\theta}\right)$ on $\mathbf{C}$ by using the counting function and the proximity function of its zero-set. By using Brody's result [Br78] to reparametrize the entire holomorphic curve $\varphi : \mathbf{C} \to X$, we can assume that the pointwise norm of $d\varphi$ is uniformly bounded on $\mathbf{C}$. We conclude from the above computations of $T(r, \varphi, \ell L + \gamma \ell K_X)$ and $T\left(r, \varphi^* \left(\frac{\tau}{\theta}\right)\right)$ that $T(r, \varphi, \ell L + \gamma \ell K_X) - N(r, \varphi, D) \leq T\left(r, \varphi^* \left(\frac{\tau}{\theta}\right)\right)$. Hence $T(r, \varphi, \ell L) \leq T\left(r, \varphi^* \left(\frac{\tau}{\theta}\right)\right)$ does not exceed

$$N\left(r, \varphi^* \left(\frac{\tau}{\theta}\right), \infty\right) + T(r, \varphi, \gamma \ell K_X) - N(r, \varphi, D) + O(\log T(r, \varphi) + \log r) \leq T(r, \varphi, \gamma \ell K_X) + O(\log T(r, \varphi) + \log r),$$

contradicting the domination of $(1 - 3\gamma - \varepsilon) T(r, \varphi, K_X)$ by $T(r, \varphi, \ell L)$ for any $\varepsilon > 0$.

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