Sectional and Ricci Curvature, Variations of Arc-Length and Diameter of Riemannian Manifold

§1. Sectional Curvature of Riemannian Manifold

We motivate the introduction of the notion of sectional curvature of a Riemannian manifold from two considerations. One is the generalization of the Gauss curvature of a surface in $\mathbb{R}^3$ as a function at a point to the case of a Riemannian manifold when a 2-dimensional linear subspace of the tangent space at a point is specified. The other is the role of the sectional curvature in the problem of the convergence of divergence of the family of geodesics emanating from a point.

For the first motivation, we recall that the Gauss curvature is defined for a surface in $\mathbb{R}^3$ or an abstract surface with the first fundamental form

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2.$$

In Riemann's generalization of the Gauss curvature to the Riemann curvature tensor, the Riemann curvature tensor is defined by

$$R(\xi,\eta) = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}$$

(where ξ,η are vector fields and $[\xi,\eta]=\xi\eta-\eta\xi$ and ∇_ξ (respectively ∇_η and $\nabla_{[\xi,\eta]}$) means covariant derivative with respect to ξ (respectively η and $[\xi,\eta]$) so that $R(\xi,\eta)$ maps a vector field ζ to the vector field $R(\xi,\eta)\zeta$. In local coordinates, if $\xi=(\xi^k)$, $\eta=(\eta^\ell)$, and $\zeta=(\zeta^i)$, then

$$\big(R(\xi,\eta)\zeta\big)^j = \sum_{i,j,k} R_i{}^j{}_{k\ell}\,\zeta^i\xi^k\eta^\ell.$$

When we specialize to the 2 -dimensional case of a surface in $\mathbb{R}^3$, for the local coordinates $(u,v)=(u^1,u^2)$ in the surface under consideration

$$-\left(\nabla_u\nabla_v\vec{r}_u - \nabla_v\nabla_u\vec{r}_u\right)\cdot\vec{r}_v = -\left\langle R\left(\frac{\partial}{\partial u^1},\frac{\partial}{\partial u^2}\right)\left(\frac{\partial}{\partial u^1}\right),\frac{\partial}{\partial u^2}\right\rangle = -R_{1212}.$$

Since

$$EG-F^2 = \|\vec{r}_u\|^2\|\vec{r}_v\|^2 - |\vec{r}_u\cdot\vec{r}_v|^2,$$

the Gauss curvature is equal to

$$-\frac{R_{1212}}{EG-F^2}=\frac{\langle R_{\vec{r}_v,\vec{r}_u}\vec{r}_u,\vec{r}_v\rangle}{\|\vec{r}_u\|^2\|\vec{r}_v\|^2-|\vec{r}_u\cdot\vec{r}_v|^2}.$$

This motivates the definition of the sectional section of a general Riemannian manifold M for the two directions ξ and η in the tangent space $T_{M,P}$ at a point P as

$$-\frac{\langle R(\xi,\eta)\xi,\eta\rangle}{\langle\xi,\xi\rangle\langle\eta,\eta\rangle-\langle\xi,\eta\rangle^2}.$$

This definition depends only on the 2-dimensional linear subspace V of the tangent space $T_{M,P}$ at the point P which is spanned by ξ and η and is independent of the choice of the generators ξ and η of the 2-dimensional linear subspace V. We leave this verification to a homework problem. For each element $v\in V$ of unit length let C_v be a geodesic of M emanating from P in the direction of v, the totality $\bigcup_{v\in v}C_v$ forms a local 2-dimensional submanifold Y of M at P, as a result of the implicit function theorem and the definition of a geodesic as a curve satisfying the condition of the vanishing of the covariant derivative of its unit tangent vector with respect to itself. The sectional curvature of M for the 2-dimensional linear subspace V is equal to the Gauss curvature of Y at P with its first fundamental form induced from the Riemannian metric of M. We also leave this verification to a homework problem.

The second motivation is the problem of the convergence or divergence of geodesics emanating from one point in the Riemannian manifold M. Let us first look at a family of geodesics given by $C_\lambda(s)$ mapping $s\in(a,b)$ to M parametrized by λ in an open interval Λ in $\mathbb{R}$, where s is the arc-length of the geodesic $C_\lambda(s)$. For fixed λ let

$$T=\frac{d}{ds}C_\lambda(s)$$

be the unit tangent vector of C_λ so that $\nabla_T T=0$ on C_λ. Let V be the tangent vector of M defined by $\frac{\partial}{\partial\lambda}C_\lambda(s)$. From the commutativity

$$\frac{\partial}{\partial s}\frac{\partial}{\partial\lambda}=\frac{\partial}{\partial s}\frac{\partial}{\partial\lambda}$$

of partial differentiation with respect to local coordinate functions we have $[T,V]=0$. The torsion-free property of the Levi-Civita connection of the

Riemannian metric of M implies that the vanishing the torsion tensor $\nabla_T V - \nabla_V T - [T, V] = 0$, which means that $\nabla_T V = \nabla_V T$. In other for the curvature tensor to occur, we have to apply covariant differentiation twice and use the commutator of the double covariant differentiation. So we consider

$$\begin{aligned}
&\nabla_T \nabla_T V = \nabla_T \nabla_V T \quad \text{from } \nabla_T V = \nabla_V T \\
&= \nabla_T \nabla_V T - \nabla_T \nabla_V T \quad \text{from } \nabla_T T = 0 \\
&\qquad \text{(due to } \; T \text{ being unit tangent vector of geodesic)} \\
&= R(T, V)T.
\end{aligned}$$

In words, the vector field V along a geodesic is the infinitesimal displacement of the geodesic in a family of geodesics. This introduction of the vector field V leads to the following definition of a Jacobi field along a geodesic.

Definition (Jacobi Field). Suppose T is the unit tangent vector of a geodesic C in a Riemannian manifold M. A vector field V given at every point of C is called a Jacobi field along the geodesic C is the following Jacobi equation is satisfied.

$$\nabla_T \nabla_T V = R(T, V)T,$$

where R is the curvature tensor of M.

Since the Jacobi equation is a second-order ordinary differentiation with the arc-length of the geodesic as the single variable, the existence and uniqueness of a local Jacobi field along a geodesic for a prescribed initial data of the position V and the derivative $\nabla_T V$ at one point are guaranteed. The vector fields $T(s)$ and $sT(s)$ are two special examples of Jacobi fields along the geodesic with unit tangent vector $T(s)$ and arc-length s, with the initial condition

$$\begin{cases} T(s)\big|_{s=0} = T(0), \\ \nabla_T T(s)\big|_{s=0} = 0 \end{cases}$$

for the Jacobi vector field $T(s)$ at $s = 0$ and the initial condition

$$\begin{cases} (sT(s))\big|_{s=0} = 0, \\ \nabla_T (sT(s))\big|_{s=0} = T(0) \end{cases}$$

for the Jacobi vector field $sT(s)$ at $s = 0$.

A Jacobi field V along a geodesic with unit tangent vector T is called normal if V is perpendicular to T at every point of the geodesic. Any Jacobian vector field V can be decomposed as the sum of the (tangential) Jacobi vector field $\alpha T(s) + \beta s T(s)$ (for some $\alpha, \beta \in \mathbb{R}$) and a normal Jacobi field.

To study the convergence or divergence of geodesics emanating from one point, the family of geodesics emanating from one point P gives rise to Jacobi vector field V along any one geodesic C in the family with V vanishing at that point. The question of convergence or divergence is a study of the rate of change of $\|V\|$ as one moves along C. To study the rate of change of $\|V\|$, we first normalize V to assume without loss of generality that $\nabla_T V = \frac{d}{ds} V$ is a unit vector and we seek to expand $\|V\|^2 = \langle V, V\rangle$ as a power series in s with $s(P)$ assumed to be 0. Using $V(0) = 0$, we apply $\nabla_T = \frac{d}{ds}$ a number of times to $\|V\|^2 = \langle V, V\rangle$ to get

$$
\begin{aligned}
&\langle V, V\rangle|_{s=0} = 0, \\
&\frac{d}{ds}\langle V, V\rangle\bigg|_{s=0} = 2\left\langle V, \frac{dV}{ds}\right\rangle\bigg|_{s=0}, \\
&\frac{d^2}{ds^2}\langle V, V\rangle\bigg|_{s=0} = 2\left\langle \frac{dV}{ds}, \frac{dV}{ds}\right\rangle\bigg|_{s=0} + 2\left\langle \frac{d^2V}{ds^2}, V\right\rangle\bigg|_{s=0} = 2\left\langle \frac{dV}{ds}, \frac{dV}{ds}\right\rangle\bigg|_{s=0} = 2, \\
&\frac{d^3}{ds^3}\langle V, V\rangle\bigg|_{s=0} = 6\left\langle \frac{d^2V}{ds^2}, \frac{dV}{ds}\right\rangle\bigg|_{s=0} + 2\left\langle \frac{d^3V}{ds^3}, V\right\rangle\bigg|_{s=0} = 0 \\
&\qquad\qquad\left(\text{due to } \frac{d^2V}{ds^2}\bigg|_{s=0} = R(T,V)T|_{s=0} = 0\right), \\
&\frac{d^4}{ds^4}\langle V, V\rangle\bigg|_{s=0} = 8\left\langle \frac{d^3V}{ds^3}, \frac{dV}{ds}\right\rangle\bigg|_{s=0} + 6\left\langle \frac{d^2V}{ds^2}, \frac{d^2V}{ds^2}\right\rangle\bigg|_{s=0} + 2\left\langle \frac{d^4V}{ds^4}, V\right\rangle\bigg|_{s=0} \\
&\qquad\qquad = 8\left\langle R\left(T, \frac{dV}{ds}\right)T, V\right\rangle\bigg|_{s=0},
\end{aligned}
$$

because

$$
\begin{aligned}
\frac{d^3V}{ds^3}\bigg|_{s=0} &= \nabla_T\left(R(T,V)T\right)|_{s=0} \\
&= \left((\nabla_T R)(T,V)T\right)|_{s=0} + R\left(T, \frac{dV}{ds}\right)T\bigg|_{s=0} \\
&= R\left(T, \frac{dV}{ds}\right)T\bigg|_{s=0}.
\end{aligned}
$$

Thus, we obtain the following Taylor series expansion of $\|V(s)\|^2$ in the variable s at $s=0$

$$\|V(s)\|^2 = s^2 - \frac{1}{3}\left\langle R\left(\frac{dV}{ds}, T\right)T, \frac{dV}{ds}\right\rangle\Big|_{s=0} s^4 + \mathrm{O}(s^5)$$

to arrive at the conclusion that the geodesics converge or divergent according as the sectional curvature

$$\frac{\left\langle R\left(\frac{dV}{ds}, T\right)T, \frac{dV}{ds}\right\rangle}{\|T\|^2\left\|\frac{dV}{ds}\right\|^2 - \left\langle T, \frac{dV}{ds}\right\rangle^2}$$

of M for the 2-dimensional linear subspace

$$\mathbb{R}T + \mathbb{R}\frac{dV}{ds}$$

at P is positive or negative.

§2. Ricci Curvature of Riemannian Manifold

The Ricci curvature Ric of an n-dimensional Riemannian manifold M at a point P of M is a quadratic form on the tangent space $T_{M,P}$ of M at P such that its value $\mathrm{Ric}(\xi, \xi)$ at $\xi \in T_{M,P}$ of unit length is $n-1$ times the average of the sectional curvature of M at $\mathbb{R}\xi + \mathbb{R}\eta$ over all $\eta \in T_{M,P}$ which are perpendicular to ξ. In terms of local coordinates $x^1, \cdots, x^n$ of M, the Ricci curvature is given by

$$\sum_{j,k=1}^n R_{jk}\, dx^j dx^k$$

with

$$R_{jk} = \sum_{i,\ell=1}^n g^{i\ell} R_{ijk\ell},$$

where

$$(R(\sigma, \tau)\zeta)^j = \sum_{i,k,\ell=1}^n R_i{}^j{}_{k\ell}\, \sigma^k \tau^\ell \zeta^i$$

is the Riemannian curvature tensor and $g^{i\ell}$ is the inverse of the Riemannian metric tensor $g_{i\ell}$ and

$$R_{ijk\ell} = \sum_{q=1}^n g_{jq} R_i{}^q{}_{k\ell}.$$

Here we introduce the Ricci curvature for two reasons. One is its relation to the diameter of a complete (in particular compact) Riemannian manifold in the form of the theorem of Bonnet-Myers when the Ricci curvature is positive. The other is its relation to the vanishing of the first Betti number in the form of the theorem of Bochner.

§3. Second Variation of Arc-Length with Fixed End-Points and Index Forms

This is to prepare for the proof of the theorem of Bonnet-Myers which we will discuss next. We start out with a geodesic $\gamma : [a, b] \to M$ in a Riemannian manifold M and we perturb it with two parameters $v, w \in (-\varepsilon, \varepsilon)$ (with $\varepsilon > 0$) and label the perturbed curve $\gamma_{v,w}$ (which in general is not a geodesic) so that our initial geodesic γ is $\gamma_{0,0}$. We parametrize the initial geodesic $\gamma = \gamma_{0,0}$ by its arc-length and use it also to parametrize the perturbed curve $\gamma_{v,w}$. Note that in general s is no longer the arc-length of $\gamma_{v,w}$ with $(v, w) \neq (0, 0)$. Let T by the image of $\frac{\partial}{\partial s}$ under $(s, v, w) \to \gamma_{v,w}(s) \in M$. Since $\gamma = \gamma_{0,0}$ is a geodesic, we have $\|T\| \equiv 1$ and $\nabla_T T \equiv 0$ for $(v, w) = (0, 0)$. Let V (respectively W) be the $\frac{\partial}{\partial v}$ (respectively $\frac{\partial}{\partial w}$) under $(s, v, w) \to \gamma_{v,w}(s) \in M$, which means that V (respectively W) is the vector field representing the infinitesimal perturbation by v (respectively w). The arc-length $L(v, w)$ of $\gamma_{v,w}$ is given by

$$L(v, w) = \int_{s=a}^{b} \|T\| ds.$$

We now compute the second variation

$$\frac{\partial^2}{\partial v \partial w} L(v, w)$$

of the arc-length $L(v,w)$ and evaluate the result at $(v,w)=(0,0)$ as follows.

$$\begin{aligned}\frac{\partial}{\partial v}L(v,w) &= \frac{\partial}{\partial v}\int_{s=a}^{b}\sqrt{\|T\|^2}ds \\ &= \int_{s=a}^{b} V\left(\sqrt{\|T\|^2}\right)ds \\ &= \int_{s=a}^{b}\frac{1}{2\sqrt{\|T\|^2}}V\left(\|T\|^2\right)ds \\ &= \int_{s=a}^{b}\frac{1}{2\sqrt{\|T\|^2}}\,2\left\langle\nabla_V T, T\right\rangle ds \\ &= \int_{s=a}^{b}\frac{\left\langle\nabla_T V, T\right\rangle}{\|T\|}ds,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product with respect to the Riemannian metric of M and the identity $\nabla_T V=\nabla_V T$ comes from the torsion-free property of the Levi-Civita connection of M and from $[T,V]=0$ as the image of $\left[\frac{\partial}{\partial s},\frac{\partial}{\partial v}\right]=0$.

$$\begin{aligned}&\frac{\partial^2}{\partial v\partial w}L(v,w) \\ &= \frac{\partial}{\partial w}\int_{s=a}^{b}\frac{\left\langle\nabla_T V, T\right\rangle}{\|T\|}ds \\ &= \int_{s=a}^{b}\left(\frac{\left\langle\nabla_W\nabla_T V, T\right\rangle+\left\langle\nabla_T V,\nabla_W T\right\rangle}{\|T\|}-\frac{\left\langle\nabla_T V, T\right\rangle\left\langle\nabla_W T, T\right\rangle}{\|T\|^3}\right)ds \\ &= \int_{s=a}^{b}\left(\frac{\left\langle R(W,T)V, T\right\rangle+\left\langle\nabla_T\nabla_W V, T\right\rangle+\left\langle\nabla_T V,\nabla_W T\right\rangle}{\|T\|}-\frac{\left\langle\nabla_T V, T\right\rangle\left\langle\nabla_W T, T\right\rangle}{\|T\|^3}\right)ds.\end{aligned}$$

Specializing to $(v,w)=(0,0)$, we get $\|T\|=1$ and $\nabla_T T=0$ and

$$\begin{aligned}&\frac{\partial^2}{\partial v\partial w}L(v,w)\Big|_{(v,w)=(0,0)} \\ &= \int_{s=a}^{b}\left(\left\langle\nabla_T V,\nabla_T W\right\rangle-\left\langle R(W,T)T,V\right\rangle+T\left\langle\nabla_W V,T\right\rangle-T\left\langle V,T\right\rangle T\left\langle W,T\right\rangle\right)ds \\ &= \left\langle\nabla_W V,T\right\rangle\Big|_a^b+\int_{s=a}^{b}\left(\left\langle\nabla_T V,\nabla_T W\right\rangle-\left\langle R(W,T)T,V\right\rangle-T\left\langle V,T\right\rangle T\left\langle W,T\right\rangle\right)ds.\end{aligned}$$

If V and W both vanish at the end-points $s = a$ and $s = b$, then

$$
\begin{aligned}
&\frac{\partial^2}{\partial v \partial w} L(v,w)\Big|_{(v,w)=(0,0)} \\
&= \int_{s=a}^{b} \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle - T\langle V, T\rangle\, T\langle W, T\rangle \right) ds.
\end{aligned}
$$

If in addition both V, W are Jacobi fields along $\gamma = \gamma_{0,0}$, then both $T\langle V, T\rangle$ and $T\langle W, T\rangle$ are constant and after integrating out

$$
T\langle V, T\rangle\, T\langle W, T\rangle = T\left(\langle V, T\rangle\, T\langle W, T\rangle\right),
$$

we obtain

$$
\frac{\partial^2}{\partial v \partial w} L(v,w)\Big|_{(v,w)=(0,0)} = \int_{s=a}^{b} \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle \right) ds.
$$

We define the *index form* $I(V, W)$ by

$$
I(V,W) = \int_{s=a}^{b} \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle \right) ds
$$

for Jacobi fields V and W along a geodesic parametrized by its arc-length with unit tangent vector T.

§4. Theorem of Bonnet-Myers

The Bonnet-Myers theorem connects a Riemannian manifold's curvature to its diameter. It states that

if (M, g) is a complete Riemannian manifold of dimension n with Ricci curvature as a quadratic form satisfying $\mathrm{Ric} \geq (n-1)K > 0$ for some constant K, then the diameter of M is at most $\frac{\pi}{\sqrt{K}}$.

In other words, if the Ricci curvature is bounded below by a positive constant, then the completer manifold must be compact and have a finite diameter. Its fundamental group must be finite from consideration of its universal covering manifold. Here "complete" means complete as a metric space with its metric defined by the minimum of the total length of broken geodesics. The theorem can be stated in terms of the mean curvature vector field H as well: if $|H| \geq c > 0$ for some constant c, then the diameter of M is at most $\frac{n-1}{c}$.

Proof. Assume that the diameter of M is $> \frac{n-1}{\sqrt{K}}$. There are two points P, Q in M whose distance is $\ell > \frac{n-1}{\sqrt{K}}$. We are going to derive a contradiction. Since M is complete, any two points of M can be joined by a geodesic (according to the argument given in the Appendix I at the end of this set of lecture notes). In particular, there exists a geodesic $\gamma : [0, \ell] \to M$ parametrized by its arc-length, with $\gamma(0) = P$ and $\gamma(\ell) = Q$. Let $\{E_i\}_{i=1}^n$ be an orthonormal basis of parallel vector fields along γ such that E_n is the unit tangent vector T of γ. In other words, $\{E_i\}_{i=1}^n$ is a Darboux frame-field. Let $W_i(t) = \sin\left(\frac{\pi t}{\pi}\right) E_i(t)$ for $1 \leq i \leq n$. The purpose of introducing the factor $\sin\left(\frac{\pi t}{\pi}\right)$ in the definition of W_i is to make sure that it vanishes at both ends $t = 0$ and $t = \ell$ of the geodesic γ. We compute the index $I(W_i, W_i)$ for $1 \leq i \leq n-1$ to get

$$\begin{aligned} I(W_i, W_i) &= -\int_{t=0}^{\ell} \left\langle W_i, \nabla_T^2 W_i, R(W_i, T)T \right\rangle dt \\ &= \int_{t=0}^{\ell} \sin\left(\frac{\pi t}{\pi}\right)^2 \left(\frac{\pi^2}{\ell^2} - \langle R(E_i, T)T, E_i\rangle\right) dt. \end{aligned}$$

Since

$$\sum_{i=1}^{n-1} I(W_i, W_i) = \int_{t=0}^{\ell} \sin\left(\frac{\pi t}{\pi}\right)^2 \left((n-1)\frac{\pi^2}{\ell^2} - \mathrm{Ric}(T, T)\right) dt$$

is negative due to $\ell > \frac{\pi}{\sqrt{K}}$ and $\mathrm{Ric}(T, T) \geq (n-1)K$, it follows that $I(W_i, W_i)$ is negative for some $1 \leq i \leq n-1$, which means that the second variation of arc-length in the direction of W_i is negative. When the geodesic γ is deformed in the direction W_i, a nearly deformation still joining P to Q has length $< \ell$, contradicting the fact that the distance between P and Q is ℓ. Q.E.D.

§5. Formula of Bochner

The relation of the Ricci curvature to the vanishing of the first Betti number is one of the results obtained by a technique introduced by Bochner in the form of the following formula

$$(3.1) \qquad (\Delta\omega, \omega)_M = \|\nabla\omega\|_M^2 + (\mathrm{Ric}\,\omega,\, \omega)_M$$

for any smooth 1-form ω on a compact Riemannian manifold M, where Δ is the Laplacian operator of M and Ric is the Ricci curvature operator and

$(\cdot, \cdot)_M$ is the global inner product with respect to the Riemannian metric tensor of M. We leave the derivation of (3.1) to a homework problem. As a consequence of (3.1), the first Betti number of a compact Riemannian manifold is zero if its Ricci curvature is strictly positive as a quadratic form at every point. Here, the statement is needed that the deRham cohomology group defined by d-closed p-forms modulo d-exact p-forms is isomorphic to the space of all d-harmonic p-forms (which are the d-closed and d^*-closed p-forms). This statement is proved in the posted lecture notes on "Hodge Decomposition." For the result of Bochner used here only the case of $p = 1$ is needed. For general p, the identity corresponding to (3.1) is given in Appendix II at the end of this set of lecture notes.

APPENDIX I: Distance Between Two Points in Complete Riemannian Manifold Realized by Geodesic

Let M be a complete Riemannian manifold and $P_0 \in M$ and let $R > 0$. Let Ω be the open ball of radius R in M centered at P_0 (which means the set of all points of M of distance $< R$ from P) and $\tilde{\Omega}$ be the open ball of radius $3R$ in M centered at P_0. Assume that $0 < \delta < \frac{R}{2}$ satisfies the condition that for every point P of $\tilde{\Omega}$ the exponential map $\exp_P$ from $T_{M,P}$ to M is diffeomorphic from the ball of radius δ centered at P in $T_{M,P}$ (when $T_{M,P}$ as the finite-dimensional Hilbert space $\mathbb{R}^n$ is given the Riemannian metric of M at P) onto the ball of radius δ centered at P in M.

Let P and Q be two distinct points in Ω. Let μ be the infimum of the lengths of all C^∞ curves in M joining P and Q. Let γ_ν ($\nu \in \mathbb{N}$) be a sequence of C^∞ curves in M joining P to Q such that the limit of the length $L(\gamma_\nu)$ of γ_ν is μ as $\nu \to \infty$. Let $\ell_\nu = L(\gamma_\nu)$. Assume without loss of generality that $\ell_\nu < 2\mu$ for $\nu \in \mathbb{N}$ and that γ_ν is a C^∞ map from $[0, \ell_\nu]$ to M parametrized by its arc-length. Let N be the integral part of $\frac{2\mu}{\delta}$. Choose

$$0 = t_{\nu,0} < t_{\nu,1} < \cdots < t_{\nu,j} < t_{\nu,j+1} < \cdots < t_{\nu,N} = \ell_\nu$$

such that $t_{\nu,j+1} - t_{\nu,j} < \delta$ for $0 \leq j \leq N-1$. Replace the image of $[t_{\nu,j}, t_{\nu,j+1}]$ under γ_ν by the geodesic between $\gamma_\nu(t_{\nu,j})$ and $\gamma_\nu(t_{\nu,j+1})$ to form the piecewise smooth curve $\tilde{\gamma}_\nu$.

Construct a subsequence ν_k by requiring that for each fixed $0 \leq j \leq N-1$ the sequence $\gamma_{\nu_k}(t_{\nu_k,j})$, as well as the sequences $\gamma'_{\nu_k}(t_{\nu_k,j} + 0)$ and $\gamma'_{\nu_k}(t_{\nu_k,j+1} - 0)$, converges as $k \to \infty$. Then as $k \to \infty$ the piecewise smooth curve $\tilde{\gamma}_\nu : [0, \ell_\nu] \to M$ converges to a smooth geodesic joining P to Q. The small details omitted here are left to a homework assignment

Remark. The technique, given in Problem 1, of constructing a geodesic joining two points is motivated by Perron's method of using the Poisson integral to replace a function v inside a circle K by a harmonic function $\mathfrak{M}_K v$ inside the circle K with the same boundary value as v on K. See p.43, lines 10-11, of

Oskar Perron, Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$. *Math. Zeitschr* **18** (1923), 42 – 54.

APPENDIX II: Bochner's Formula for Laplacian of p-Forms on Riemannian Manifold

Let M be a Riemannian manifold of dimension n with Riemannian metric g_{ij}. Let φ be a p-form on M with components in terms of local coordinates $x_1, \cdots, x_n$ of M given by

$$\varphi = \frac{1}{p!} \sum_{j_1, \cdots, j_p} \varphi_{j_1 \cdots j_p} dx_{j_1} \wedge \cdots \wedge dx_{j_p}.$$

We have

$$\begin{aligned} d\varphi &= \frac{1}{p!} \sum_{j_0, j_1, \cdots, j_p} (\partial_{j_0} \varphi_{j_1 \cdots j_p}) dx_{j_0} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_p} \\ &= \frac{1}{(p+1)!} \sum_{j_0, j_1, \cdots, j_p} \sum_{\nu=0}^{p} (-1)^\nu (\partial_{j_\nu} \varphi_{j_0, \cdots, j_{\nu-1}, j_{\nu+1}, \cdots, j_p}) dx_{j_0} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_p}, \end{aligned}$$

where the last equality represents skew-symmetrization in the indices $j_0, \cdots, j_p$ which involves only the transposition of j_0, j_ν for $0 \leq \nu \leq p$ and the factor $\frac{1}{p+1}$, because $\partial_{j_0} \varphi_{j_1 \cdots j_p}$ is already skew-symmetric in $j_1, \cdots, j_p$. It means that the components of $d\varphi$ are given by

$$(d\varphi)_{j_0, \cdots, j_p} = \sum_{\nu=0}^{p} (-1)^\nu \partial_{j_\nu} \varphi_{j_0, \cdots, j_{\nu-1}, j_{\nu+1}, \cdots, j_p}.$$

The computation of

$$(d^* \varphi)_{j_1 \cdots j_{p-1}} = -\sum_{s,t} g^{st} \nabla_s \varphi_{t\, j_1 \cdots j_{p-1}}$$

is straightforward without the additional step of skew-symmetrization. Putting the two computations together, we have the following formula for the application of the Laplacian $\Delta = dd^* + d^*d$ to the p-form φ.

$$\begin{aligned} (dd^* + d^*d)\varphi)_{j_1, \cdots, j_p} &= -\sum_{s,t} g^{st} \nabla_s \nabla_t \varphi_{j_1, \cdots, j_p} + \sum_{i,j,j_1, \cdots, j_p} R_{ij} \varphi^{i j_2 \cdots j_p} \varphi^{j}{}_{j_2 \cdots j_p} \\ &\quad + \frac{p-1}{2} \sum_{i,j,k,\ell, j_1, \cdots, j_p} R_{ijk\ell} \varphi^{i j j_3 \cdots j_p} \alpha^{k\ell}{}_{j_3 \cdots j_p}, \end{aligned}$$

where the raising of indices for φ is done with the use of the inverse matrix of the matrix $(g_{ij})_{1\leq i,j\leq n}$ of the Riemannian metric of the manifold. The curvature terms enter into the expansion of $(dd^* + d^*d)\varphi$, because of the differentiation of $\nabla\varphi$ in $d^*\varphi$ and the skew-symmetrization when d is applied to $d^*\varphi$.