Lie Derivative

§1. Integral Function of Smooth Vector Field

Let $\vec{v} = (v^1, \cdots, v^k)$ be a C^k vector field (where $k \in [0, \infty]$ is chosen so that the regularity conditions required by the arguments presented here are satisfied) on an open neighborhood U of 0 in $\mathbb{R}^n$. To avoid the introduction of more notations, we agree to shrink U if necessary without changing the notation U. There exists some $\eta > 0$ such that for $a = (a^1, \cdots, a^n)$ with $|a| < \eta$ the ordinary differential equation

$$\frac{dx^j}{dt} = v^j(x^1(t), \cdots, x^j(t))$$

with initial condition $x^j(0) = a^j$ (for $1 \leq j \leq n$) can be solved for $0 \leq t \leq \eta$ by applying the fixed point theorem for contraction map in Banach space to its associated integral equation.

§2. Composite Rule for Integral Function of Smooth Vector Field

Denote the solution by

$$\varphi(t, a) = (\varphi^1(t, a), \cdots, \varphi^n(t, a)).$$

Take $0 < s < \eta$. Consider the composite map

$$\psi(t) := \varphi(t, \varphi(s, a))$$

We claim that

$$\psi(t) = \varphi(t + s, a),$$

which we verify by applying the uniqueness of the solution of the ordinary differential equation to $t \mapsto \psi(t)$. Let $b = \varphi(s, a)$. First, we have the ordinary differential equation

$$\frac{d\psi(t)}{dt} = \frac{d\varphi(t, b))}{dt} = \vec{v}(\varphi(t, b))$$

from the ordinary equation satisfied by $t \mapsto \varphi(t, b)$. Second, we have the initial condition

$$\psi(t)\Big|_{t=0} = \varphi(t, b)\Big|_{t=0} = b.$$

On the other hand,

$$t \mapsto= \varphi(t+s,a)$$

satisfies the differential equation

$$\frac{d\varphi(t+s,a)}{dt} = \vec{v}(\varphi(t+s,a))$$

from the ordinary differential equation

$$\frac{d\varphi(t,a)}{dt} = \vec{v}(\varphi(t,a))$$

and the chain rule. The initial condition

$$t \mapsto \varphi(t+s,a)$$

at $t=0$ is

$$\varphi(t+s,a)\Big|_{t=0} = \varphi(s,a) = b.$$

Thus, the uniqueness of the ordinary differential equation satisfied by the two functions $\psi(t)$ and $\varphi(t+s,a)$ with the same initial conditions at $t=0$ implies that

$$\psi(t) = \varphi(t+s,a),$$

which we can rewrite as

$$\varphi(t,\varphi(s,a)) = \varphi(t+s,a)$$

for $|s|<\eta,\ |t|<\eta,\ |a|<\eta$.

§3. Notation of Operator Representing Integral Function of Vector Field

To emphasize the dependence of $t \mapsto \varphi(t,a)$ on $\vec{v}$, we also denote $\varphi(t,a)$ by $\varphi_{\vec{v}}(t,a)$, sometimes referred to as the flow (function) of the vector field $\vec{v}$. Motivated by its analogy to the composition law of the exponential function $t \mapsto e^{\gamma t}$ (with $\gamma \in \mathbb{R}-\{0\}$), we also use the following notation $e^{\vec{v}t}x$ to denote $\varphi_{\vec{v}}(t,x)$ which is the solution of the ordinary differentiation

$$\frac{d\varphi_{\vec{v}}(t,x)}{dt} = \vec{v}(\varphi_{\vec{v}}(t,x))$$

satisfying the initial condition $\varphi_{\vec{v}}(t,x)\Big|_{t=0} = x$.

§4. Lie Derivative of a Function with Respect to a Vector Field

Let f be a smooth function on U. The *Lie derivative* $\mathcal{L}ie_{\vec{v}} f$ of f at $a \in U$ with respect to $\vec{v}$ is defined as

$$\lim_{t\to 0} \frac{f\left(\varphi_{\vec{v}}(t,a)\right) - f(a)}{t}.$$

By the chain rule, using the notations introduced above, we compute it to be

$$\begin{aligned} \frac{d}{dt} f\left(\varphi_{\vec{v}}(t,a)\right)\Big|_{t=0} &= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x^j}\Big|_{x=a}\right)\left(\frac{d\varphi^j(t,a)}{dt}\Big|_{t=0}\right) \\ &= \sum_{j=1}^{n} v^j(a) \frac{\partial f}{\partial x^j}\Big|_{x=a} \\ &= \vec{v}(f)\Big|_{x=a}, \end{aligned}$$

which is the result of the action of $\vec{v}$ on f evaluated at the point a. Thus,

$$\mathcal{L}ie_{\vec{v}} f = \vec{v}(f)$$

on U.

§5. Lie Derivative of a Vector Field with Respect to Another Vector Field

Suppose $\vec{u} = (u^1, \cdots, u^n)$ is another vector field. We would like to define and compute the Lie derivative $\mathcal{L}ie_{\vec{v}} \vec{u}$ of $\vec{u}$ with respect to $\vec{v}$ at a point x. Note that we **cannot** just define $\mathcal{L}ie_{\vec{v}} \vec{u}$ as

$$\lim_{t\to 0} \frac{\vec{u}\left(\varphi_{\vec{v}}(x,t)\right) - \vec{u}(x)}{t},$$

because we cannot directly compare a vector $\vec{u}\left(\varphi_{\vec{v}}(x,t)\right)$ at the point $\varphi_{\vec{v}}(x,t)$ with a vector $\vec{u}(x)$ at the point a, without any "connection" which tells us how to compare, especially we are in a smooth manifold X of dimension n instead of $\mathbb{R}^n$.

To overcome this difficulty, we use the inverse of the differential (*i.e.,* the infinitesimal tangent map) of the map $x \mapsto \varphi_{\vec{v}}(x,t)$ to pull the vector $\vec{u}\left(\varphi_{\vec{v}}(x,t)\right)$ at the point $\varphi_{\vec{v}}(x,t)$ back to the point x before we take the difference quotient. First, we observe that if we have a function $A(t)$ of a real variable t with value in a nonsingular square matrix, from $A(t)A(t)^{-1} \equiv 1$ we get

$$\frac{dA(t)}{dt}A(t)^{-1} + A(t)\frac{d\left(A(t)^{-1}\right)}{dt} \equiv 0$$

and

$$\frac{d\left(A(t)^{-1}\right)}{dt} = -A(t)^{-1}\frac{dA(t)}{dt}A(t)^{-1}.$$

In particular, if $A(t)$ is the identity matrix at $t=0$, then

$$\left(\frac{d\left(A(t)^{-1}\right)}{dt}\right)_{t=0} = -\left(\frac{dA(t)}{dt}\right)_{t=0}.$$

Since the the differential of the map $x \mapsto \varphi_{\vec{v}}(x,t)$ is represented by the Jacobian matrix

$$A(x,t) := \left(\frac{\partial \varphi^j_{\vec{v}}(x,t)}{\partial x^k}\right)_{1\le j,k\le n},$$

and since $x \mapsto \varphi_{\vec{v}}(x,t)$ is the identity at $t=0$, we have

$$\begin{aligned}\left(\frac{d\left(A(x,t)^{-1}\right)}{dt}\right)_{t=0} &= -\left(\frac{dA(x,t)}{dt}\right)_{t=0}\\ &= -\left(\frac{\partial}{\partial x^k}\,\frac{d\varphi^j_{\vec{v}}(x,t)}{dt}\right)_{1\le j,k\le n}\\ &= -\left(\frac{\partial}{\partial x^k}\,v^j\left(\varphi_{\vec{v}}(t,x)\right)\right)_{1\le j,k\le n}.\end{aligned}$$

Thus, we should define $\mathcal{L}ie_{\vec{v}}\,\vec{u}$ at x as the derivative of

$$A(x,t)^{-1}\left(\vec{u}\left(\varphi^j_{\vec{v}}(x,t)\right)\right)$$

with respect to t at $t=0$, to end up with

$$\begin{aligned}\mathcal{L}ie_{\vec{v}}\,\vec{u} &= \left(\frac{d\left(A(x,t)^{-1}\right)}{dt}\right)_{t=0}\left(\vec{u}\left(\varphi^j_{\vec{v}}(x,t)\right)\right)_{t=0} + \left(A(x,t)^{-1}\right)_{t=0}\left(\frac{d\left(\vec{u}\left(\varphi^j_{\vec{v}}(x,t)\right)\right)}{dt}\right)_{t=0}\\ &= -\sum_{j,k=1}^{n} u_j\left(\frac{\partial v^k}{\partial x_j}\right)\frac{\partial}{\partial x^k} + \sum_{j,k=1}^{n}\left(\frac{\partial u^j}{\partial x_k}\right)v^k\frac{\partial}{\partial x^j},\end{aligned}$$

which is equal to the Lie bracket $[\vec{v}, \vec{u}]$. It is more informative to write it in the form

$$\mathcal{L}ie_{\vec{v}}\,\vec{u} = [\vec{v}, \vec{u}] = \sum_{j=1}^{n} \vec{v}(u^j)\frac{\partial}{\partial x_j} + \sum_{j=1}^{n}\left(\sum_{k=1}^{n}(\Gamma_{\vec{v}})^j_k u^k\right)\frac{\partial}{\partial x_j}$$

with

$$(\Gamma_{\vec{v}})^j_k = -\frac{\partial v^j}{\partial x^k},$$

where the first term involves differentiating the components of $\vec{u}$ with respect to $\vec{v}$ and the second term involves an algebraic operation of applying a matrix transformation to the components of $\vec{u}$ (with the entries of the matrix given by partial derivatives of the components of $\vec{v}$). The reason why this formulation is more informative is that it is in the form of differentiation with a "connection" and it can be readily generalized to the case of the Lie derivatives of tensor fields which we now discuss.

§6. Lie Derivative of a Tensor Field with Respect to Another Vector Field

We have an $\mathbb{R}$-basis $dx^1, \cdots, dx^n$ for the dual space of the tangent space spanned by $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$. This dual space $(T_{\mathbb{R}^n,P})^*$ of $T_{\mathbb{R}^n,P}$ is the space of 1-forms. When we consider a smooth manifold X of real dimension n instead of $\mathbb{R}^n$, we have the space of 1-forms $(T_{X,P})^*$ of X at P which is the dual of the tangent space $T_{X,P}$ of X at P. The totality of $(T_{X,P})^*$ as P varies in X constitutes a vector bundle $T^*_{\mathbb{R}^n}$ of rank n over X which is called the cotangent bundle of X. The cotangent bundle T^*_X of X is the dual of the tangent bundle T_X of X.

The tensor bundle of contrariant rank r and covariant rank s on X is the bundle

$$\underbrace{T_X \otimes \cdots \otimes T_X}_{r \text{ copies}} \otimes \underbrace{T^*_X \otimes \cdots \otimes T^*_X}_{s \text{ copies}}$$

whose elements at a point P of X are tensors of contrariant rank r and covariant rank s on X at P. A (smooth) tensor field T of contrariant rank r and covariant rank s on an open subset G of X is a (smoothly) defined assignment of a tensor of contrariant rank r and covariant rank s to every point of G. In terms of local coordinates $x^1, \cdots, x^n$ of X, we can write

$$T = \sum_{1 \leq j_1, \cdots, j_r, k_1, \cdots, k_s \leq n} T^{j_1, \cdots, j_r}_{k_1, \cdots, k_s} \left(\frac{\partial}{\partial x_{j_1}}\right) \otimes \cdots \otimes \left(\frac{\partial}{\partial x_{j_r}}\right) \otimes (dx^{k_1}) \otimes \cdots \otimes (dx^{k_s}),$$

when each $\frac{\partial}{\partial x^{j_\nu}}$ is regarded as a vector field and each dx^{k_λ} is regarded as a field of 1-forms.

By pairing a 1-form field $\omega = \sum_{j=1}^n \omega_j \, dx^j$ with an arbitrary vector field $\vec{u}$ to form a function $\langle \omega, \, \vec{u} \rangle$ and using the Leibniz product rule, we derive the Lie derivative $\mathcal{L}ie_{\vec{v}}\omega$ of ω with respect to a vector field v by using the equation

$$\vec{v}\left(\langle \omega, \, \vec{u} \rangle\right) = \langle \mathcal{L}ie_{\vec{v}}\omega, \, \vec{u} \rangle + \langle \omega, \, \mathcal{L}ie_{\vec{v}}\vec{u} \rangle$$

to get

$$\mathcal{L}ie_{\vec{v}}\omega = \sum_{j=1}^n \vec{v}(\omega_j) \, dx^j - \sum_{k=1}^n \left(\sum_{j=1}^n (\Gamma_{\vec{v}})^j_k \omega_j \right) dx^k.$$

By pairing a tensor field

$$T = \sum_{1 \leq j_1, \cdots, j_r, k_1, \cdots, k_s \leq n} T^{j_1, \cdots, j_r}_{k_1, \cdots, k_s} \left(\frac{\partial}{\partial x_{j_1}}\right) \otimes \cdots \otimes \left(\frac{\partial}{\partial x_{j_r}}\right) \otimes (dx^{k_1}) \otimes \cdots \otimes (dx^{k_s})$$

of contravariant rank r and covariant rank s with r arbitrary 1-form fields $\omega^{(1)}, \cdots, \omega^{(r)}$ and s arbitrary vector fields $\vec{u}_1, \cdots, \vec{u}_s$ and using the Leibniz product rule, we derive the Lie derivative $\mathcal{L}ie_{\vec{v}}\, T$ of T with respect to a vector field v by using the equation

$$\begin{aligned}
&\vec{v}\left(\left\langle T, \, \omega^{(1)} \otimes \cdots \otimes \omega^{(r)} \otimes \vec{u}_1 \otimes \cdots \otimes \vec{u}_s \right\rangle\right) \\
&= \left\langle \mathcal{L}ie_{\vec{v}}\, T, \, \omega^{(1)} \otimes \cdots \otimes \omega^{(r)} \otimes \vec{u}_1 \otimes \cdots \otimes \vec{u}_s \right\rangle \\
&+ \sum_{\mu=1}^r \left\langle T, \, \omega^{(1)} \otimes \cdots \otimes \omega^{(\mu-1)} \otimes \left(\mathcal{L}ie_{\vec{v}}\, \omega^{(\mu)}\right) \otimes \omega^{(\mu+1)} \otimes \cdots \otimes \omega^{(r)} \otimes \vec{u}_1 \otimes \cdots \otimes \vec{u}_s \right\rangle \\
&+ \sum_{\nu=1}^s \left\langle T, \, \omega^{(1)} \otimes \cdots \otimes \omega^{(r)} \otimes \vec{u}_1 \otimes \cdots \otimes \vec{u}_{\nu-1} \otimes \left(\mathcal{L}ie_{\vec{v}}\, \vec{u}_\nu\right) \otimes \vec{u}_{\nu+1} \cdots \otimes \vec{u}_s \right\rangle
\end{aligned}$$

to get

$$\mathcal{L}ie_{\vec{v}}\, T = \sum_{1 \leq j_1, \cdots, j_r, k_1, \cdots, k_s \leq n} (\mathcal{L}ie_{\vec{v}}\, T)^{j_1, \cdots, j_r}_{k_1, \cdots, k_s} \left(\frac{\partial}{\partial x_{j_1}}\right) \otimes \cdots \otimes \left(\frac{\partial}{\partial x_{j_r}}\right) \otimes (dx^{k_1}) \otimes \cdots \otimes (dx^{k_s})$$

with

$$\begin{aligned}(\mathcal{L}ie_{\vec{v}}\,T)^{j_1,\cdots,j_r}_{k_1,\cdots,k_s} &= \vec{v}\left(T^{j_1,\cdots,j_r}_{k_1,\cdots,k_s}\right) + \sum_{\mu=1}^{r}\left(\sum_{\ell=1}^{n}(\Gamma_{\vec{v}})^{j_\mu}_{\ell}\,T^{j_1,\cdots,j_{\mu-1},\ell,j_{\mu+1},\cdots,j_r}_{k_1,\cdots,k_s}\right)\\ &\quad - \sum_{\nu=1}^{s}\left(\sum_{\ell=1}^{n}(\Gamma_{\vec{v}})^{\ell}_{k_\nu}\,T^{j_1,\cdots,j_r}_{k_1,\cdots,k_{\nu-1},\ell,k_{\nu+1},\cdots,k_s}\right).\end{aligned}$$

§7. Lie Differentiation of Differential Form as Anticommutator of Exterior Differentiation and Interior Product

After we introduced Lie derivative as differentiation with respect to the Lie group action by the additive group $\mathbb{R}$, we saw how the Lie derivative of a vector field with respect to another vector field is equated to the Lie bracket of the two vector fields. Now we would like to link the Lie derivative of a differential k-form to the exterior derivative of the differential k-form. The relation was established by Elie Cartan, as a Cartan formula known also as Cartan's magic formula.

To state Cartan's magic formula, we need to introduce the notion of an interior product between a vector field $\vec{v}$ and a differential k-form ω. The interior product $\iota_{\vec{v}}\omega$ is defined by

$$(\iota_{\vec{v}}\omega)\left(\vec{u}_1\wedge\cdots\wedge\vec{u}_{k-1}\right) = \omega\left(\vec{v}\wedge\vec{u}_1\wedge\cdots\wedge\vec{u}_{k-1}\right)$$

for arbitrary vector fields $\vec{u}_1,\cdots,\vec{u}_{k-1}$. In terms of coordinates $x^1,\cdots,x^n$, the components of $\iota_{\vec{v}}\omega$ are given by

$$(\iota_{\vec{v}}\omega)_{j_1,\cdots,j_{k-1}} = \sum_{\ell=1}^{n} v^\ell\,\omega_{\ell,j_1,\cdots,j_{k-1}}$$

when the components of $\vec{v}$ are $v^1,\cdots,v^n$ and

$$\omega = \frac{1}{k!}\sum_{j_1,\cdots,j_k=1}^{n}\omega_{j_1,\cdots,j_k}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}.$$

Cartan's magic formula states that equation of operators

$$\mathcal{L}ie_{\vec{v}} = d\circ\iota_{\vec{v}} + \iota_{\vec{v}}\circ d$$

holds when it is applied to any differential k-form ω. In other words, Lie differentiation $\mathcal{L}ie_{\vec{v}}$ of a differential form (with respect to a vector field $\vec{v}$) is equal to the anticommutator

$$\{d, \iota_{\vec{v}}\} = d \circ \iota_{\vec{v}} + \iota_{\vec{v}} \circ d$$

of exterior differentiation and interior product applied to the differential form. Cartan's formula can verified straightforwardly as follows. We use the skew-symmetric components of both sides of the equation for verifcation. On the one hand, we have

$$\begin{aligned}\left(\mathcal{L}ie_{\vec{v}}\,\omega\right)_{j_1,\cdots,j_k} &= \sum_{\ell=1}^n v^\ell \frac{\partial \omega_{j_1,\cdots,j_k}}{\partial x^\ell} - \sum_{\mu=1}^k \left(\sum_{\ell=1}^n (\Gamma_{\vec{v}})^\ell_{j_\mu} \omega_{j_1,\cdots,j_{\mu-1},\ell,j_{\mu+1},\cdots,j_k}\right) \\ &= \sum_{\ell=1}^n v^\ell \frac{\partial \omega_{j_1,\cdots,j_k}}{\partial x^\ell} + \sum_{\mu=1}^k \left(\sum_{\ell=1}^n \left(\frac{\partial v^\ell}{\partial x^{j_\mu}}\right) \omega_{j_1,\cdots,j_{\mu-1},\ell,j_{\mu+1},\cdots,j_k}\right).\end{aligned}$$

On the other hand, since

$$\left(d\omega\right)_{j_0,j_1,\cdots,j_k} = \sum_{\nu=0}^k (-1)^\nu \frac{\partial \omega_{j_0,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}}{\partial x^{j_\nu}},$$

we have

$$\begin{aligned}\left((\iota_{\vec{v}} \circ d)\omega\right)_{j_1,\cdots,j_k} &= \left(\iota_{\vec{v}}\,(d\omega)\right)_{j_1,\cdots,j_k} \\ &= \sum_{\ell=1}^n v^\ell (d\omega)_{\ell,j_1,j_2,\cdots,j_k} \\ &= \sum_{\ell=1}^n v^\ell \left(\frac{\partial \omega_{\ell,j_1,\cdots,j_k}}{\partial x^\ell} + \sum_{\nu=1}^k (-1)^\nu \frac{\partial \omega_{\ell,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}}{\partial x^{j_\nu}}\right)\end{aligned}$$

For the other term, we have

$$
\begin{aligned}
&\left((d\circ\iota_{\vec v})\,\omega\right)_{j_1,\cdots,j_k} = \left(d\,(\iota_{\vec v}\omega)\right)_{j_1,\cdots,j_k}\\
&=\sum_{\nu=1}^{k}(-1)^{\nu+1}\frac{\partial}{\partial x^{j_\nu}}\left(\iota_{\vec v}\omega\right)_{j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}\\
&=\sum_{\nu=1}^{k}(-1)^{\nu+1}\frac{\partial}{\partial x^{j_\nu}}\left(\sum_{\ell=1}^{n}v^\ell\omega_{\ell,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}\right)\\
&=\sum_{\nu=1}^{k}(-1)^{\nu+1}\left(\sum_{\ell=1}^{n}\frac{\partial v^\ell}{\partial x^{j_\nu}}\,\omega_{\ell,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}\right)+\sum_{\nu=1}^{k}(-1)^{\nu+1}\left(\sum_{\ell=1}^{n}v^\ell\frac{\partial}{\partial x^{j_\nu}}\,\omega_{\ell,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}\right)\\
&=\sum_{\nu=1}^{k}\left(\sum_{\ell=1}^{n}\frac{\partial v^\ell}{\partial x^{j_\nu}}\,\omega_{j_1,\cdots,j_{\nu-1},\ell,j_{\nu+1},\cdots,j_k}\right)+\sum_{\nu=1}^{k}(-1)^{\nu+1}\left(\sum_{\ell=1}^{n}v^\ell\frac{\partial}{\partial x^{j_\nu}}\,\omega_{\ell,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}\right)
\end{aligned}
$$

Adding up the two equations, we get

$$
\begin{aligned}
&\left((\iota_{\vec v}\circ d)\omega+(d\circ\iota_{\vec v})\,\omega\right)_{j_1,\cdots,j_k}\\
&=\sum_{\ell=1}^{n}v^\ell\frac{\partial\omega_{\ell,j_1,\cdots,j_k}}{\partial x^\ell}+\sum_{\nu=1}^{k}\left(\sum_{\ell=1}^{n}\frac{\partial v^\ell}{\partial x^{j_\nu}}\,\omega_{j_1,\cdots,j_{\nu-1},\ell,j_{\nu+1},\cdots,j_k}\right)\\
&=\left(\mathcal{L}ie_{\vec v}\,\omega\right)_{j_1,\cdots,j_k}.
\end{aligned}
$$

This completes the proof of Cartan's magic formula.