Implicit Function Theorem

In our discussion of the proof of Stokes's theorem for the case of a bounded domain D in $\mathbb{R}^2$ with piecewise continuously differentiable boundary, we put off the question the relation between a piece of the boundary defined by a continuously differentiable function with nonzero gradient and a curve parametrized by a continuously differentiable function. An answer to the question necessitates the introduction of the implicit function theorem which we now do. Another motivation for the implicit function theorem is the inverse mapping theorem which is a corollary to it. When we analytically define well-known functions such as the exponential function and the trigonometric sine and cosine functions by using definite integrals, we have to invert an indefinite integral. One key point of Abel's study of elliptic functions is to invert an indefinite integral. The inverse mapping theorem provides the theoretical foundation for the inversion of a map and relates the local invertibility of the map to the invertibility of its differential (which is its linear approximation at a point).

The implicit function theorem depends on the intermediate value theorem and the mean value theorem. Before we state and prove the implicit function theorem, let us recall the intermediate value theorem.

Theorem (Intermediate Value Theorem). Suppose $a < b$ are real numbers and $f(x)$ is a continuous function on $[a, b]$ with $f(a) < 0 < f(b)$. Then there exists some $c \in (a, b)$ such that $f(c) = 0$.

Earlier in the course we gave a proof for it by using the *least upper bound property* of the set of real numbers. The argument is as follows. Suppose the contrary. Let $A = \{x \in [a,b] \,\big|\, f(x) < 0\}$ and $\xi \in [a, b]$ be the supremum of A. Since $f(a) < 0 < f(b)$, the continuity of f at a and b implies that $a < \xi < b$. Since any number strictly less than ξ is not an upper bound, for $n \in \mathbb{N}$ exists some $x_n \in (\xi - \frac{1}{n}, \xi] \cap A$. It follows from the continuity of $f(x)$ at ξ and from $\lim_{n\to\infty} x_n = \xi$ that $f(\xi) = \lim_{n\to\infty} f(x_n) \leq 0$. Since 0 is not achieved by f on $[a, b]$, we must have $f(\xi) < 0$. By continuity of f at ξ, there exists $\varepsilon > 0$ such that $f(x) < 0$ for $x < \xi + \varepsilon$, contradicting that ξ is an upper bound of A.

The intermediate value theorem can also be proved by the following more intuitive technique of "divide and search".

Proof of Intermediate Value Theorem by Technique of "Divide and Search". Let $a_0 = a$ and $b_0 = b$. Let ξ be the mid-point of $[a_0, b_0]$. If $f(\xi) = 0$, we end our search. If $f(\xi) > 0$, we let $a_1 = a_0$ and $b_1 = \xi$ so that $b_1 - a_1 = \frac{b_0 - a_0}{2}$ and $f(a_1) < 0 < f(b_1)$. We now apply the argument with a_1 and b_1 replacing a_0 and b_0 respectively. So either we end our search at some point or we end up with a sequence of nested subintervals

$$[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$$

with $b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}$ and $f(a_n) < 0 < f(b_n)$. Since $b_n - a_n = \frac{b-a}{2^n}$ and

$$a = a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq a_n \leq b_n \leq b_{n-1} \leq \cdots \leq b_1 \leq b_0 = b,$$

it follows that

$$\sup_{n \in \mathbb{N}} a_n = \inf_{n \in \mathbb{N}} b_n$$

is the common limit c of the nondecreasing sequence a_n and the nonincreasing sequence b_n. From the continuity of f at c and $f(a_n) < 0 < f(b_n)$ it follows that

$$0 \geq \lim_{n \to \infty} f(a_n) = f(c) = \lim_{n \to \infty} f(b_n) \geq 0.$$

Hence $f(c) = 0$. Since $c \in [a, b]$ and $f(a) < 0 < f(b)$, the point c must be in (a, b). Q.E.D.

Now we state the implicit function theorem in the simplest setting of one equation, one dependent variable, and one independent variable.

Theorem (Implicit Function Theorem). Suppose U is an open subset of $\mathbb{R}^2$ containing 0 and $f(x, y)$ is a real-valued function on U with continuous first-order partial derivatives. Assume that $f(0, 0) = 0$ and $\frac{\partial f}{\partial y}(0, 0) \neq 0$. Then there exist real numbers $a < 0 < b$ and $c < 0 < d$ with $[a, b] \times [c, d] \subset U$ and a function $y = g(x)$ on (a, b) such that $c < g(x) < d$ for $a < x < b$ and $f(x, g(x)) \equiv 0$ for $a < x < b$ and $g'(x)$ exists for $a < x < b$ with

$$g'(x) = -\frac{\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))}$$

for $a < x < b$. In particular, $g'(x)$ is a continuous function of $x \in (a, b)$.

Proof. Without loss of generality we can assume that $\frac{\partial f}{\partial y}(0,0) > 0$, otherwise we replace $f(x,y)$ by $-f(x,y)$ in our argument. By replacing U by a smaller open neighborhood of 0, we can assume without loss of generality that

(i) $\frac{\partial f}{\partial y} \geq \kappa$ on U for some *positive* number $\kappa \leq 1$,

(ii) $\left|\frac{\partial f}{\partial x}\right| \leq A$ and $\left|\frac{\partial f}{\partial y}\right| \leq A$ on U for some positive number $A \geq 1$,

(iii) both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are uniformly continuous on U so that for any $\varepsilon > 0$ there exists some $\delta_\varepsilon > 0$ such that

$$\left|\frac{\partial f}{\partial x}(x_1,y_1) - \frac{\partial f}{\partial x}(x_2,y_2)\right| < \varepsilon$$

and

$$\left|\frac{\partial f}{\partial y}(x_1,y_1) - \frac{\partial f}{\partial y}(x_2,y_2)\right| < \varepsilon$$

for $(x_1,y_1),(x_2,y_2)$ in U with distance between (x_1,y_1) and (x_2,y_2) strictly less than δ_ε.

Choose $c < 0 < d$ such that $[(0,c),(0,d)] \subset U$. Since $y \mapsto f(0,y)$ is strictly increasing for $c \leq y \leq d$ from $\frac{\partial f}{\partial y} > 0$ on U and the mean-value theorem, it follows from $f(0,0) = 0$ that $f(0,c) < 0 < f(0,d)$. By continuity of $f(x,y)$ on U that for some $a < 0 < b$ we have $f(x,c) < 0$ for $a \leq x \leq b$ and $f(x,d) > 0$ for $a \leq x \leq b$.

By applying the intermediate value theorem to the function $y \mapsto f(x,y)$ on $[c,d]$ for fixed $a \leq x \leq b$, we conclude that for $a \leq x \leq b$ there exists $c < g(x) < d$ such that $f(x,g(x)) = 0$. Since for fixed $a \leq x \leq b$ the function $y \mapsto f(0,y)$ is strictly increasing for $c \leq y \leq d$ due to $\frac{\partial f}{\partial y} > 0$ on U and the mean-value theorem, it follows that the value $c < g(x) < d$ with $f(x,g(x)) = 0$ is unique for $a \leq x \leq b$.

To prepare for the proof of the differentiability of $g(x)$ at $x = 0$, we now verify the weaker conclusion that the difference quotient $\frac{g(x)}{x}$ of $g(x)$ is bounded in absolute value by $\frac{A}{\kappa}$ for $x \in [a,b] - \{0\}$. Fix $x \in [a,b] - \{0\}$. By applying the mean value theorem to

$$t \mapsto f(tx, tg(x))$$

for $t \in [0,1]$, we obtain $0 < \tau < 1$ such that

$$(*) \qquad f(x,g(x)) - f(0,0) = \left.\frac{d}{dt} f(tx, tg(x))\right|_{t=\tau} = \frac{\partial f}{\partial x}(\tau x, \tau g(x))x + \frac{\partial f}{\partial y}(\tau x, \tau g(x))g(x).$$

From $f(x,g(x)) = f(0,0) = 0$ it follows that

$$\left|\frac{g(x)}{x}\right| = \left|\frac{\frac{\partial f}{\partial x}(\tau x, \tau g(x))}{\frac{\partial f}{\partial y}(\tau x, \tau g(x))}\right| \leq \frac{A}{\kappa}.$$

We are now ready to prove the differentiability of $g(x)$ at $x = 0$. Take any $\varepsilon > 0$. Choose $0 < |x| < \frac{\kappa}{2A}\,\delta_\varepsilon$. Then $|g(x)| < \frac{\delta_\varepsilon}{2}$ and $|x| < \frac{\delta_\varepsilon}{2}$ and the distance between $(x, g(x))$ and $(0,0)$ is strictly less than δ_ε and in particular the distnace between $(\tau x, \tau g(x))$ and $(0,0)$ is strictly less than δ_ε. We now use $(*)$. From $f(x,g(x)) = f(0,0) = 0$ it follows that

$$\frac{g(x)}{x} + \frac{\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial y}(0,0)} = -\frac{\frac{\partial f}{\partial x}(\tau x, \tau g(x))}{\frac{\partial f}{\partial y}(\tau x, \tau g(x))} + \frac{\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial y}(0,0)}$$

$$= \frac{-\frac{\partial f}{\partial x}(\tau x, \tau g(x))\frac{\partial f}{\partial y}(0,0) + \frac{\partial f}{\partial x}(0,0)\frac{\partial f}{\partial y}(\tau x, \tau g(x))}{\frac{\partial f}{\partial y}(\tau x, \tau g(x))\frac{\partial f}{\partial y}(0,0)}$$

$$= \frac{\left(\frac{\partial f}{\partial x}(0,0) - \frac{\partial f}{\partial x}(\tau x, \tau g(x))\right)\frac{\partial f}{\partial y}(0,0) + \frac{\partial f}{\partial x}(0,0)\left(\frac{\partial f}{\partial y}(\tau x, \tau g(x)) - \frac{\partial f}{\partial y}(0,0)\right)}{\frac{\partial f}{\partial y}(\tau x, \tau g(x))\frac{\partial f}{\partial y}(0,0)}$$

so that by (ii) and (iii)

$$\left|\frac{g(x)}{x} + \frac{\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial y}(0,0)}\right| \leq \frac{2A}{\kappa^2}\,\varepsilon$$

for any $0 < |x| < \frac{\kappa}{2A}\,\delta_\varepsilon$. This finishes the proof that

$$g'(0) = -\frac{\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial y}(0,0)}.$$

Since we can apply the same argument to any point x in (a,b) instead of at $x = 0$ with $(x, g(x))$ replacing $(0,0)$, we conclude that

$$g'(x) = -\frac{\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))}.$$

Q.E.D.

One Equation and Several Independent Variables. The above argument still holds when the variable x is replaced by a vector variable $\vec{x} = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$ to yield the following implicit function theorem for one equation and several independent variables.

Theorem (Implicit Function Theorem for One Equation and Several Independent Variables). Let $n \geq 1$. Suppose U is an open subset of $\mathbb{R}^{n+1}$ containing 0 and $f(x_1, \cdots, x_n, y)$ is a real-valued function on U with continuous first-order partial derivatives. Assume that $f(0, \cdots, 0) = 0$ and $\frac{\partial f}{\partial y}(0, \cdots, 0) \neq 0$. Then there exist real numbers $a_j < 0 < b_j$ for $1 \leq j \leq n$ and $c < 0 < d$ with $[a_1, b_1] \times \cdots \times [a_n, b_n] \times [c, d] \subset U$ and a function $y = g(x_1, \cdots, x_n)$ on $(a_1, b_1) \times \cdots \times (a_n, b_n)$ such that $c < g(x_1, \cdots, x_n) < d$ for $a_j < x_j < b_j$ (for $1 \leq j \leq n$) and $f(x_1, \cdots, x_n, g(x_1, \cdots, x_n)) \equiv 0$ for $a_j < x_j < b_j$ for $1 \leq j \leq n$ and $\frac{\partial g}{\partial x_j}(x_1, \cdots, x_n)$ exists for $a_j < x_j < b_j$ (for $1 \leq j \leq n$) with

$$\frac{\partial g}{\partial x_j}(x_1, \cdots, x_n) = -\frac{\frac{\partial f}{\partial x_j}(x_1, \cdots, x_n, g(x_1, \cdots, x_n))}{\frac{\partial f}{\partial y}(x_1, \cdots, x_n, g(x_1, \cdots, x_n))}$$

for $a_j < x_j < b_j$ (for $1 \leq j \leq n$). In particular, $\frac{\partial g}{\partial x_j}(x_1, \cdots, x_n)$ is a continuous function of $(x_1, \cdots, x_n) \in [a_1, b_1] \times \cdots \times [a_n, b_n]$.

Several Equations and Several Independent Variables. The implicit function theorem works also for the case of several equations and several independent variables. The precise statement is as follows.

Theorem (Implicit Function Theorem for Several Equations and Several Independent Variables). Let $m \geq 1$ and $n \geq 1$. Suppose U is an open subset of $\mathbb{R}^m \times \mathbb{R}^n$ containing the origin 0 and $f_k(x_1, \cdots, x_n, y_1, \cdots, y_m)$ is a real-valued function on U with continuous first-order partial derivatives for $1 \leq k \leq m$. Assume that $f_k(0, \cdots, 0) = 0$ for $1 \leq k \leq m$ and the Jacobian determinant

$$\frac{\partial(f_1, \cdots, f_m)}{\partial(y_1, \cdots, y_m)}$$

is nonzero at the origin of $\mathbb{R}^m \times \mathbb{R}^n$. Then there exist real numbers $a_j < 0 < b_j$ for $1 \leq j \leq n$ and $c_k < 0 < d_k$ for $1 \leq k \leq m$ with

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \times [c_1, d_1] \times \cdots \times [c_m, d_m] \subset U$$

and function $y_k = g_k(x_1, \cdots, x_n)$ on $(a_1, b_1) \times \cdots \times (a_n, b_n)$ such that $c_k < g_k(x_1, \cdots, x_n) < d_k$ on $(a_1, b_1) \times \cdots \times (a_n, b_n)$ for $1 \leq k \leq m$ and

$$f_k(x_1, \cdots, x_n, g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n)) \equiv 0$$

on $(a_1, b_1) \times \cdots \times (a_n, b_n)$ for $1 \leq k \leq m$ and $\frac{\partial g_k}{\partial x_j}(x_1, \cdots, x_n)$ exists on $(a_1, b_1) \times \cdots \times (a_n, b_n)$ for $1 \leq k \leq m$ and $1 \leq j \leq n$ whose value $\frac{\partial g_k}{\partial x_j}(x_1, \cdots, x_n)$ for fixed $1 \leq j \leq n$ is obtained by applying Cramer's rule to the system

$$\frac{\partial f_k}{\partial x_j} + \sum_{\ell=1}^{m} \frac{\partial f_k}{\partial y_\ell} \frac{\partial g_\ell}{\partial x_j} = 0 \quad (\text{for } 1 \leq k \leq m)$$

of m linear equations in the m unknowns

$$\frac{\partial g_\ell}{\partial x_j} \quad \text{for} \quad 1 \leq \ell \leq m$$

which is obtained by applying the chain rule of differentiation with respect to x_j to the m identities

$$f_k(x_1, \cdots, x_m, g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n)) \equiv 0$$

on $(a_1, b_1) \times \cdots \times (a_n, b_n)$ for $1 \leq k \leq m$. In particular, $\frac{\partial g_k}{\partial x_j}(x_1, \cdots, x_n)$ is a continuous function on $(a_1, b_1) \times \cdots \times (a_n, b_n)$ for $1 \leq k \leq m$ and $1 \leq j \leq n$.

To avoid the use of complicated notations, we will prove here only the simplest typical case of $m = 2$ and $n = 1$. Suppose U is an open neighborhood of the origin in $\mathbb{R}^3$ with coordinates x, y, z and suppose we are given two functions $f(x, y, z)$ and $g(x, y, z)$ on U with continuous partial derivatives such that the Jacobian determinant

$$\frac{\partial(f, g)}{\partial(y, z)} = \begin{vmatrix} D_2 f & D_3 f \\ D_2 g & D_3 g \end{vmatrix}$$

is nonzero at the origin, where $D_j f$ denotes the j-th partial derivative of f and $D_j g$ denotes the j-th partial derivative of g (with x, y, z being respectively the first, second, and third variables). The vanishing of

$$\frac{\partial(f, g)}{\partial(y, z)} = \begin{vmatrix} D_2 f & D_3 f \\ D_2 g & D_3 g \end{vmatrix}$$

at the origin means that one of $D_3 f$ and $D_3 g$ is nonzero at the origin. Without loss of generality we assume that $D_3 g \neq 0$ at the origin.

We apply the implicit function theorem to the single equation $g(x, y, z) = 0$ with one dependent variable z and two independent variables x, y to get a continuously differentiable function $z = h(x, y)$ so that $g(x, y, h(x, y)) \equiv 0$ and

$$D_j h = -\frac{D_j g}{D_3 g}$$

for $j = 1, 2$.

Now we substitute z by $z = h(x, y)$ in the function $f(x, y, z)$ to define the function $F(x, y) = f(x, y, h(x, y))$. We would like to apply the implicit function theorem to the single equation $F(x, y) = f(x, y, h(x, y)) = 0$ with one dependent variable y and one independent variable x. We need to verify that the partial derivative of the defining function $F(x, y)$ with respect to the dependent variable y is nonzero at the origin $(x, y) = (0, 0)$. By the chain rule

$$\begin{aligned} D_2 F &= D_2 f + (D_3 f)(D_2 h) \\ &= D_2 f + (D_3 f)\left(-\frac{D_2 g}{D_3 g}\right) \\ &= \frac{(D_2 f)(D_3 f) - (D_3 f)(D_2 g)}{D_3 g} \\ &= \frac{\frac{\partial(f,g)}{\partial(y,z)}}{D_3 g} \end{aligned}$$

which is nonzero. From the implicit function theorem for one equation, one dependent variable and one independent variable applied to the equation $F(x, y) = 0$ and the dependent variable y and independent variable x at $(x, y) = (0, 0)$, we get a continuously differentiable function $y = \sigma(x)$ such that $F(x, \sigma(x)) \equiv 0$. Let $\tau(x) = h(x, \sigma(x))$. Then

$$f(x, \sigma(x), \tau(x)) = f(x, \sigma(x), h(x, \sigma(x)) = F(x, \sigma(x)) \equiv 0.$$

From $g(x, y, h(x, y)) \equiv 0$ it follows that

$$g(x, \sigma(x), \tau(x)) = g(x, \sigma(x), h(x, \sigma(x)) \equiv 0.$$

The first-order derivatives of $\sigma(x)$ and $\tau(x)$ are given by applying Cramer's rule to the two equations

$$\begin{aligned} D_1 f + (D_2 f)\sigma' + (D_3 f)\tau' &\equiv 0 \\ D_1 g + (D_2 g)\sigma' + (D_3 g)\tau' &\equiv 0 \end{aligned}$$

which are obtained by applying the chain rule of differentiation with respect to x to the two equatsions

$$f(x,\sigma(x),\tau(x))\equiv 0 \quad \text{and} \quad g(x,\sigma(x),\tau(x))\equiv 0.$$

Q.E.D.

Inverse Mapping Theorem. We now discuss how to derive the following inverse mapping theorem is obtained from the implicit function theorem.

Theorem (Inverse Mapping Theorem). Let $n\geq 1$ and W be an open neighborhood of 0 in $\mathbb{R}^n$ and $\Phi: W\to\mathbb{R}^n$ be a continuously differentiable map which maps the origin to the origin. Suppose the differential $d\Phi$ of Φ as an $\mathbb{R}$-linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is nonsingular (that is, the determinant of the $n\times n$ matrix representing the $\mathbb{R}$-linear map $d\Phi:\mathbb{R}^n\to\mathbb{R}^n$ is nonzero). Then there exist open neighborhoods Ω and G of the origin in $\mathbb{R}^n$ such that $\Omega\subset W$ and Φ maps Ω bijectively onto G and the inverse map Φ^{-1} from G to Ω is continuously differentiable with the differential $d\left(\Phi^{-1}\right)$ of Φ^{-1} equal to the inverse $\left(d\Phi\right)^{-1}$ of the differential $d\Phi$ of Φ.

Proof. Let Φ be represented by functions

$$y_j=f_j(x_1,\cdots,x_n) \ \text{ for } \ 1\leq j\leq n$$

with continuous partial derivatives. Let

$$F_j(y_1,\cdots,y_n,x_1,\cdots,x_n)=f_j(x_1,\cdots,x_n)-y_j=0 \ \text{ for } \ 1\leq j\leq n$$

We apply the implicit function theorem to the n equations

$$f_j(x_1,\cdots,x_n)-y_j=0 \ \text{ for } \ 1\leq j\leq n$$

for the n independent variables $y_1,\cdots,y_n$ and the n dependent variables $x_1,\cdots,x_n$. Since the $n\times n$ determinant

$$\frac{\partial(F_1,\cdots,F_n)}{\partial(x_1,\cdots,x_n)}=\frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}$$

is equal to the determinant of the $n\times n$ matrix which represents the differential $d\Phi$ at the origin as an $\mathbb{R}$-linear map, it follows from the invertibility of $d\Phi$ that the $n\times n$ determinant

$$\frac{\partial(F_1,\cdots,F_n)}{\partial(x_1,\cdots,x_n)}$$

is nonzero at the origin of $\mathbb{R}^n \times \mathbb{R}^n$. By the implicit function theorem, we have n continuously differentiable functions

$$x_j = g_j(y_1, \cdots, y_n) \quad \text{for} \quad 1 \leq j \leq n$$

on $(a_1, b_1) \times \cdots \times (a_n, b_n)$ for some $a_1 < 0 < b_1, \cdots, a_n < 0 < b_n$ such that

$$F_j(g_1(y_1, \cdots, y_n), \cdots, g_n(y_1, \cdots, y_n), y_1, \cdots, y_n) \equiv 0 \quad \text{for} \quad 1 \leq j \leq n$$

on $(a_1, b_1) \times \cdots \times (a_n, b_n)$, which means that

$$f_j(g_1(y_1, \cdots, y_n), \cdots, g_n(y_1, \cdots, y_n)) - y_j \equiv 0 \quad \text{for} \quad 1 \leq j \leq n$$

on $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Let $G = (a_1, b_1) \times \cdots \times (a_n, b_n)$ and define the map $\Psi : G \to \mathbb{R}^n$ by

$$(y_1, \cdots, y_n) \mapsto (x_1, \cdots, x_n) = (g_1(y_1, \cdots, y_n), \cdots, g_n(y_1, \cdots, y_n))$$

for $(y_1, \cdots, y_n) \in G$. Let Ω be the image of G under Ψ. Then Ψ is the inverse of the map $\Phi : \Omega \to G$. The last statement that $d\left(\Phi^{-1}\right) = (d\Phi)^{-1}$ follows from the chain rule applied to the equation that $\Phi \circ \Phi^{-1}$ equals the identity on G and to the equation that $\Phi^{-1} \circ \Phi$ equals the identity on Ω. Q.E.D.

Remark. The inverse mapping theorem tells us that a continuously differentiable map is locally invertible at a point if and only if its differential as a linear map is invertible at that point.