Math 230a Homework # 3 Assigned September 23, 2025
due September 30, 2025

Problem 1. *(Frobenius Integrabiltiy Proved by Integrating One Vector Field at a Time and Using Uniqueness of Ordinary Differential Equations)*

Let U be a nonempty connected open neighborhood of the origin 0 in $\mathbb{R}^n$ and let $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_q$ be C^∞ vector fields on U which are $\mathbb{R}$-linearly independent at every point of U. Assume that Frobenius integrability condition

$$[\boldsymbol{\xi}_j, \boldsymbol{\xi}_k] \in \langle \boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_q \rangle$$

is satisfied at every point of U, where $[\boldsymbol{\xi}_j, \boldsymbol{\xi}_k]$ means the Lie bracket of $\boldsymbol{\xi}_j, \boldsymbol{\xi}_k$ and $\langle \boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_q \rangle$ means the $\mathbb{R}$-linear subspace spanned by $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_q$ at the point under consideration.

In the posted lecture notes a proof is given for the Frobenius integrability theorem which states that (if the Frobenius integrability condition is satisfied) there exists locally a C^∞ submanifold Y of real dimension q in U such that the tangent space $T_{Y,x}$ of Y at $x \in Y$ is equal to $\langle \boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_q \rangle$ at x.

We now ask for an alternative method of proof which starts with one integral curve C_1 for $\boldsymbol{\xi}_1$ through the origin 0, then constructs the integral curve $C_{2,x}$ for $\boldsymbol{\xi}_2$ with $x \in C_1$ near 0, then constructs the integral curve $C_{3,y}$ for $\boldsymbol{\xi}_3$ with y in the parameter space $\bigcup_{x \in C_1} C_{2,x}$ of real dimension 2, then constructs the integral curve $C_{4,z}$ with z in the parameter space $\bigcup_{x \in C_1, y \in C_{2,x}} C_{3,y}$ of real dimension 3, and continues with the procedure until one ends with a q-fold Y. The problem now is to prove that the $q-1$ vector fields $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_{q-1}$ are tangential to Y at every point of Y. (Note that the tangency of $\boldsymbol{\xi}_q$ to Y is automatic, because $\boldsymbol{\xi}_q$ is tangential to $C_{q,w}$ and Y is the union of $C_{q,w}$ as w varies in a parameter space of real dimension $q-1$.) This tangency statement will come from the uniqueness part of the fundamental theorem of ordinary differential equations.

In this problem you are asked to provide a rigorous proof of this alternative method by following the steps given below. In the steps below, the presentation of integrating q vector fields successively is simplified by an induction process.

(1) *Step One.* By relabeling the given vector fields $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_q$ and by using a new coordinate system $x_1, \cdots, x_n$ of $\mathbb{R}^N$ if necessary, we can assume without loss of generality (after shrinking U if necessary) that we have $\boldsymbol{\xi}_1 = \frac{\partial}{\partial x_1}$ (from the result of integrating a single nonzero vector field). Let H be the hyperplane defined by $x_1 = 0$. For each point P in H, we integrate the vector field $\boldsymbol{\xi}_1$ to obtain a curve $C_{1,P}$ through P, parametrized by x_1 such that $\boldsymbol{\xi}_1$ is tangential to $C_{1,P}$. By replacing $\boldsymbol{\xi}_j$ by a C^∞ vector field in $\langle \boldsymbol{\xi}_j, \boldsymbol{\xi}_1 \rangle$ for $2 \leq j \leq q$ if necessary, we can assume without loss of generality that $\boldsymbol{\xi}_j(x_1) \equiv 0$ for $2 \leq j \leq q$ so that $\boldsymbol{\xi}_j$ is tangential to H for $2 \leq j \leq q$.

(2) *Step Two.* By induction hypothesis, on H (which is locally diffeomorphic to $\mathbb{R}^{n-1}$) the $q-1$ vector fields $\boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_q$ satisfy the Frobenius integrability condition. By integrating the $q-1$ vector fields $\boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_q$ successively, we can find a coordinate system $(y_2, \cdots, y_n)$ of H such that $\langle \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_q \rangle = \left\langle \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_q} \right\rangle$.

(3) *Step Three.* Define a coordinate system $(y_1, \cdots, y_n)$ of U (after shrinking U if necessary) as follows. For $Q \in U$, there is a unique integral curve $C_{1,P}$ with $P \in H$ such that $C_{1,P}$ contains Q. We define $y_1(Q)$ to be $x_1(Q)$ which is the parameter on $C_{1,P}$ giving the position of Q on $C_{1,P}$. We then define $y_j(Q) = y_j(P)$ for $2 \leq j \leq q$. Verify that on U we have

$$\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_q \rangle = \left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_q} \right\rangle$$

by checking $\boldsymbol{\xi}_j(y_k) \equiv 0$ for $1 \leq j \leq q$ and $q+1 \leq k \leq n$ and using the uniqueness of the solution of an ordinary differential equation with prescribed initial data.

Hint: $\boldsymbol{\xi}_1(y_1) \equiv \boldsymbol{\xi}_1(x_1) \equiv 1$ and $\boldsymbol{\xi}_j(y_1) \equiv \boldsymbol{\xi}_j(x_1) \equiv 0$ for $2 \leq j \leq q$ imply that (after evaluation at y_1)

$$[\boldsymbol{\xi}_k, \boldsymbol{\xi}_\ell] = \sum_{j=2}^{n} c_{k,\ell,j} \boldsymbol{\xi}_j \quad \text{for} \quad 1 \leq k, \ell \leq q$$

for some scalar functions $c_{k,\ell,j}$. By applying the uniqueness statement on ordinary differential equations to

$$\frac{\partial}{\partial y_1} (\boldsymbol{\xi}_j y_k) = \boldsymbol{\xi}_1 \boldsymbol{\xi}_j y_k = \boldsymbol{\xi}_j \boldsymbol{\xi}_1 y_k + [\boldsymbol{\xi}_1, \boldsymbol{\xi}_j]\, y_k = \sum_{\ell=2}^{\infty} c_{1,j,\ell} (\boldsymbol{\xi}_\ell y_k)$$

for $1 \leq j \leq q$ and $q+1 \leq k \leq n$ to conclude $\boldsymbol{\xi}_j y_k \equiv 0$ from $\left(\boldsymbol{\xi}_j y_k\right)_{y_1=0} = 0$.

Problem 2. *(Verification of Cartan's Coordinate-Free Definition of Exterior Derivative in Terms of Lie Brackets by Special Induction Process)*

Cartan's formula for a differential k-form ω and vector fields $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_{k+1}$ is

$$\begin{aligned}(d\omega)\left(\boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_{k+1}\right) &= \sum_{j=1}^{k+1} (-1)^{j+1} \boldsymbol{\xi}_j \left(\omega\left(\boldsymbol{\xi}_1 \wedge \cdots \wedge \widehat{\ \boldsymbol{\xi}_j\ } \wedge \cdots \wedge \boldsymbol{\xi}_{k+1}\right)\right) \\ &\quad + \sum_{1 \leq j < \ell \leq k+1} (-1)^{j+\ell} \omega\left(\left[\boldsymbol{\xi}_j, \boldsymbol{\xi}_\ell\right] \wedge \cdots \wedge \widehat{\ \boldsymbol{\xi}_j\ } \wedge \cdots \wedge \widehat{\ \boldsymbol{\xi}_\ell\ } \wedge \cdots \wedge \boldsymbol{\xi}_{k+1}\right),\end{aligned}$$

where the notation $\widehat{\ \boldsymbol{\xi}_j\ }$ means that $\boldsymbol{\xi}_j$ is removed.

In this problem you are asked to provide the rigorous details of the following special induction argument introduced specifically to prove Cartan's formula.

(1) For the initial step, we verify Cartan's formula directly for

$$\omega = f dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

for some $1 \leq j_1 < \cdots < j_k \leq n$, where f is a function, and for $\boldsymbol{\xi}_\nu = \frac{\partial}{\partial x^{\ell_\nu}}$ for $1 \leq \nu \leq k+1$ with $1 \leq \ell_\nu \leq n$. (Note that this implies that Cartan's formula holds for any differential k-form ω and for $\boldsymbol{\xi}_\nu = \frac{\partial}{\partial x^{\ell_\nu}}$ for $1 \leq \nu \leq k+1$, because any differential k-form is a sum of special differential k-forms $\omega = f dx^{j_1} \wedge \cdots \wedge dx^{j_k}$.)

(2) For the induction step, we verify that if Cartan's formula holds for some ω and $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_{k+1}$, then Cartan's formula holds for the case with $\boldsymbol{\xi}_\nu$ replaced by $g\boldsymbol{\xi}_\nu$ for some arbitrary function g and some ν from $\{1, 2, \cdots, k+1\}$.

Problem 3 *(Star Operator)*

For a point P of $\mathbb{R}^n$ (with coordinates $x_1, \cdots, x_n$) and a differential k-form

$$\omega_P = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \omega_{P, j_1, \cdots, j_k} \left(dx_{j_1}\right)_P \wedge \cdots \wedge \left(dx_{j_k}\right)_P$$

at P (with $\omega_{P,j_1,\cdots,j_k} \in \mathbb{R}$), define a differential $(n-k)$-form $*\omega_P$ at P given by

$$*\omega_P = \sum_{1 \leq j_{k+1} < \cdots < j_n \leq n} a_{j_{k+1},\cdots,j_n} \left(dx_{j_{k+1}}\right)_P \wedge \cdots \wedge \left(dx_{j_n}\right)_P .$$

with

$$a_{j_{k+1},\cdots,j_n} = \operatorname{sgn} \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ j_1 & \cdots & j_k & j_{k+1} & \cdots & j_n \end{pmatrix} \omega_{P,j_1,\cdots,j_k},$$

where $(j_1, \cdots, j_k, j_{k+1}, \cdots, j_n)$ is a permutation of $(1, \cdots, k, k+1, \cdots, n)$ and sgn means the *signature*. The operator

$$* : \omega_P \mapsto *\omega_P$$

is known as the *star operator*. For example,

$$\begin{gathered} *\left((dx_1)_P \wedge \cdots \wedge (dx_k)_P\right) = (dx_{k+1})_P \wedge \cdots \wedge (dx_n)_P , \\ *\left((dx_{k+1})_P \wedge \cdots \wedge (dx_n)_P\right) = (-1)^{k(n-k)} (dx_1)_P \wedge \cdots \wedge (dx_k)_P . \end{gathered}$$

with the convention

$$\begin{gathered} *1 = (dx_1)_P \wedge \cdots \wedge (dx_n)_P , \\ *\left((dx_1)_P \wedge \cdots \wedge (dx_n)_P\right) = 1. \end{gathered}$$

For a differential k-form ω_P on $\mathbb{R}^n$ at a point P of $\mathbb{R}^n$, the expression $|*\omega_P|$ means the absolute-value when it is evaluated at a k-vector at P.

(a) Let C be a curve in $\mathbb{R}^2$ which is the image of a continuously differentiable map $\varphi : [a,b] \to \mathbb{R}^2$. Assume that $f(x,y)$ is a function (with continuously first-order partial derivatives) on an open subset of $\mathbb{R}^2$ containing C such that every point (x_0, y_0) of C satisfies $f(x_0, y_0) = 0$ with one of the two numbers $\frac{\partial f}{\partial x}(x_0, y_0)$, $\frac{\partial f}{\partial y}(x_0, y_0)$ nonzero. Prove that the arc-length of C is equal to

$$\int_C \frac{|*(df)|}{\sqrt{\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2}}.$$

In other words,

$$\int_C \sqrt{(dx)^2 + (dy)^2} = \int_C \frac{|*(df)|}{\sqrt{\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2}}.$$

(b) Let C be a curve in $\mathbb{R}^3$ which is the image of a continuously differentiable map $\varphi : [a,b] \to \mathbb{R}^3$. Assume that $f(x,y,z)$ and $g(x,y,z)$ are functions (with continuously first-order partial derivatives) on an open subset of $\mathbb{R}^3$ containing C such that every point (x_0,y_0,z_0) of C satisfies $f\left(x_0,y_0,z_0\right)=0$ and $g\left(x_0,y_0,z_0\right)=0$ with one of three Jacobians

$$\left(\frac{\partial\left(f,g\right)}{\partial\left(x,y\right)}\right)_{(x_0,y_0,z_0)}, \quad \left(\frac{\partial\left(f,g\right)}{\partial\left(y,z\right)}\right)_{(x_0,y_0,z_0)}, \quad \left(\frac{\partial\left(f,g\right)}{\partial\left(z,x\right)}\right)_{(x_0,y_0,z_0)}$$

nonzero. Prove that the arc-length of C is equal to

$$\int_C \frac{\left|*\left(df\wedge dg\right)\right|}{\sqrt{\left|\frac{\partial(f,g)}{\partial(x,y)}\right|^2+\left|\frac{\partial(f,g)}{\partial(y,z)}\right|^2+\left|\frac{\partial(f,g)}{\partial(z,x)}\right|^2}}.$$

In other words,

$$\int_C \sqrt{(dx)^2+(dy)^2+(dz)^2} = \int_C \frac{\left|*\left(df\wedge dg\right)\right|}{\sqrt{\left|\frac{\partial(f,g)}{\partial(x,y)}\right|^2+\left|\frac{\partial(f,g)}{\partial(y,z)}\right|^2+\left|\frac{\partial(f,g)}{\partial(z,x)}\right|^2}}.$$

(c) Let S be a surface in $\mathbb{R}^3$ which is the image of a map

$$\varphi : [a,b]\times[c,d] \to \mathbb{R}^3$$

whose three components have continuous first-order derivatives. Assume that $f(x,y,z)$ is a functions (with continuously first-order partial derivatives) on an open subset of $\mathbb{R}^3$ containing S such that every point (x_0,y_0,z_0) of S satisfies $f\left(x_0,y_0,z_0\right)=0$ with one of three numbers

$$\frac{\partial f}{\partial x}\left(x_0,y_0,z_0\right), \quad \frac{\partial f}{\partial y}\left(x_0,y_0,z_0\right), \quad \frac{\partial f}{\partial z}\left(x_0,y_0,z_0\right)$$

nonzero. Prove that the surface area of S is equal to

$$\int_S \frac{\left|*\left(df\right)\right|}{\sqrt{\left|\frac{\partial f}{\partial x}\right|^2+\left|\frac{\partial f}{\partial y}\right|^2+\left|\frac{\partial f}{\partial z}\right|^2}}.$$

In other words,

$$\int_S \sqrt{\left|dy\wedge dz\right|^2+\left|dz\wedge dx\right|^2+\left|dx\wedge dy\right|^2} = \int_S \frac{\left|*\left(df\wedge dg\right)\right|}{\sqrt{\left|\frac{\partial(f,g)}{\partial(x,y)}\right|^2+\left|\frac{\partial(f,g)}{\partial(y,z)}\right|^2+\left|\frac{\partial(f,g)}{\partial(z,x)}\right|^2}}.$$

(d) *(Formulation of Maxwell's Equations in Terms of Modified Star Operator)*

Denote the three coordinates x, y, z of $\mathbb{R}^3$ by x_1, x_2, x_3 and denote the time variable t by x_0 and use x_0, x_1, x_2, x_3 as the coordinates of $\mathbb{R}^4$. Choose the convention of normalizing constants and units so that the four Maxwell's equations simply read as follows.

$$\begin{aligned}
\operatorname{div} \vec{E} &= \rho. \\
\operatorname{div} \vec{B} &= 0. \\
-\operatorname{curl} \vec{E} &= \frac{\partial \vec{B}}{\partial t}. \\
\operatorname{curl} \vec{B} &= \frac{\partial \vec{E}}{\partial t} + \vec{J}.
\end{aligned}$$

Here $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ρ is the charge density, and $\vec{J}$ is the current density. Let

$$\vec{E} = (E_1, E_2, E_3)\,, \quad \vec{B} = (B_1, B_2, B_3)\,, \quad \vec{J} = (J_1, J_2, J_3)\,.$$

Introduce the differential 2-form F on $\mathbb{R}^4$ defined by

$$F = (E_1 dx_1 + E_2 dx_2 + E_3 dx_3) \wedge dx_0 + B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2$$

and the differential 1-form J on $\mathbb{R}^4$ defined by

$$J = -\rho dx_0 + J_1 dx_1 + J_2 dx_2 + J_3 dx_3.$$

Because of the special rôle of x_0 among the four coordinates x_0, x_1, x_2, x_3, for the space $\mathbb{R}^4$ the star operator is modified as follows. For $0 \le k \le 3$,

$$*\,(dx_{j_0} \wedge \cdots \wedge dx_{j_k}) = \epsilon\, dx_{j_{k+1}} \wedge \cdots \wedge dx_{j_3},$$

where

$$\epsilon = \begin{cases} \operatorname{sgn} \begin{pmatrix} 0 & \cdots & k & k+1 & \cdots & 3 \\ j_0 & \cdots & j_k & j_{k+1} & \cdots & j_3 \end{pmatrix} & \text{if } j_1, \cdots, j_k \text{ are all nonzero} \\[2ex] -\operatorname{sgn} \begin{pmatrix} 0 & \cdots & k & k+1 & \cdots & 3 \\ j_0 & \cdots & j_k & j_{k+1} & \cdots & j_3 \end{pmatrix} & \text{if one of } j_1, \cdots, j_k \text{ is zero.} \end{cases}$$

With this modified definition of the star operator, verify that the four Maxwell's equations are equivalent to the two equations

$$\begin{cases} dF = 0 \\[2ex] *\,(d\,(*F)) = J \end{cases}$$

in terms of exterior derivatives and the modified star operator.

Here the divergence $\operatorname{div}\vec{E}$ of the vector $\vec{E}=(E_1, E-2, E_3)$ in $\mathbb{R}^3$ means $\sum_{j=1}^3 \frac{\partial E_j}{\partial x_j}$. The curl $\operatorname{curl}\vec{B}$ of the vector $\vec{B}=(B_1,B_2,B_3)$ in $\mathbb{R}^3$ means

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ B_1 & B_2 & B_3 \end{vmatrix} = \left(\frac{\partial B_3}{\partial x_2}-\frac{\partial B_2}{\partial x_3}\right)\vec{i}+\left(\frac{\partial B_1}{\partial x_3}-\frac{\partial B_3}{\partial x_1}\right)\vec{j}+\left(\frac{\partial B_1}{\partial x_2}-\frac{\partial B_2}{\partial x_1}\right)\vec{k}$$

with $\vec{i},\vec{j},\vec{k}$ being the three unit vectors along the positive directions of the three coordinate axes of $\mathbb{R}^3$.