Celebrated Theorem of Gauss

§1. Introduction. The *Celebrated Theorem of Gauss* (Theorema Egregium) says that the Gaussian curvature of a surface in the 3-dimensional Euclidean space, which is defined as the product of the two extremal curvature of curves cut out from a plane containing the normal, depends only on the metric of the surface and not on how the surface is embedded inside the 3-dimensional Euclidean space. Here we give the precise definition of the Gaussian curvature and its alternative geometric interpretations and a proof of the Celebrated Theorem of Gauss.

§2. First Fundamental Form. Suppose we have a surface S in $\mathbb{C}^3$ given by $\vec{r} = \vec{r}(u, v)$. We give S the metric induced from the Euclidean metric of $\mathbb{R}^3$. Then the metric in terms of the coordinates u and v is given by

$$d\vec{r} \cdot d\vec{r} = E\,du^2 + 2F\,dudv + G\,dv^2,$$

where $E = \vec{r}_u \cdot \vec{r}_u$, $F = \vec{r}_u \cdot \vec{r}_v$, $G = \vec{r}_v \cdot \vec{r}_v$ and the subscripts u and v mean partial differentiation with respect to u and v. The metric

$$E\,du^2 + 2F\,dudv + G\,dv^2$$

is known as the *first fundamental form.*

§3. Formula for Gaussian Curvature. Let $\vec{n}$ be the unit normal to the surface S. Choose a point P in the surface S and let C be a curve cut out from S by a plane Π through P containing $\vec{n}(P)$. Let $u = u(t)$, $v = v(t)$ be the equations defining C with the parameter t equal to the arc-length of C. We now compute the curvature of the curve C. Let $\vec{r}'$ and $\vec{r}''$ denote respectively the first and second order derivatives of $\vec{r}$ with respect to t. Since $\vec{r}'$ is a unit vector, the curvature of C is given by the length of $\vec{r}''$. Since C is a plane curve on the same plane as $\vec{n}(P)$, the vector $\vec{r}''(P)$ is parallel to $\vec{n}(P)$. Hence the curvature κ of C at P is given by $\vec{r}'' \cdot \vec{n}$ which by the chain rule is equal to

$$(\vec{r}_{uu} \cdot \vec{n}){u'}^2 + 2(\vec{r}_{uv} \cdot \vec{n})u'v' + (\vec{r}_{vv} \cdot \vec{n}){v'}^2,$$

where the primes for u and v mean differentiation with respect to t. Let $D = \vec{n} \cdot \vec{r}_{uu}$, $D' = \vec{n} \cdot \vec{r}_{uv}$, and $D'' = \vec{n} \cdot \vec{r}_{vv}$. Since t is the arc-length of C, we can write

$$\kappa = \frac{D\,du^2 + 2D'dudv + D''dv^2}{E\,du^2 + 2F\,dudv + F\,dv^2}.$$

The expression makes κ the quotient of two quadratic forms whose variables are the components of the tangent vector at P which belongs to Π. As we change this tangent vector or equivalently as we change Π, the value of κ changes and we have two extremal values. The product of these two extremal values is known as the Gaussian curvature. We let $\xi = \frac{du}{dt}$ and $\eta = \frac{dv}{dt}$. We have

$$\kappa(\xi, \eta)\left(E\,\xi^2 + 2F\,\xi\eta + G\,\eta^2\right) = D\,\xi^2 + 2D'\xi\eta + D''\eta^2.$$

By differentiating with respect to ξ and η, we conclude that the extremal values of κ satisfy the two equations

$$\begin{aligned}(D - \kappa E)\xi + (D' - \kappa F)\eta &= 0\\ (D' - \kappa F)\xi + (D'' - \kappa G)\eta &= 0.\end{aligned}$$

The vanishing of the determinant of the coefficients of ξ and η yields the following quadratic equation in κ.

$$\begin{aligned}0 &= \begin{vmatrix} D - \kappa E & D' - \kappa F \\ D' - \kappa F & D'' - \kappa G \end{vmatrix}\\ &= (D - \kappa E)\,(D'' - \kappa G) - (D' - \kappa F)^2\\ &= DD' - D'^2 - (GD - ED'' + 2D'F)\,\kappa + \left(EG - F^2\right)\kappa^2\end{aligned}$$

The product of the two roots of κ in this equation can be read off from the coefficients of the equation. So we have the following *formula for the Gaussian curvature.*

$$\text{Gaussian Curvature} = \frac{DD'' - D'^2}{EG - F^2}.$$

Another way to see it is that the Gaussian curvature is the product of the eigenvalues of the matrix

$$\begin{pmatrix} D & D' \\ D' & D \end{pmatrix}$$

with respect to the matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and so is equal to the quotient of the determinants of the two matrices.

§4. Gaussian Curvature as Ratio of Volume Elements in Gauss Map. Let

$$\vec{n}_u \times \vec{n}_v = \lambda \left(\vec{r}_u \times \vec{r}_v \right).$$

Taking the dot product with $\vec{r}_u \times \vec{r}_v$, we get

$$\left(\vec{n}_u \cdot \vec{r}_u \right) \left(\vec{n}_v \cdot \vec{r}_v \right) - \left(\vec{n}_u \cdot \vec{r}_v \right) \left(\vec{n}_v \cdot \vec{r}_u \right) = \lambda \left| \vec{r}_u \times \vec{r}_v \right|^2 .$$

Now

$$\left| \vec{r}_u \times \vec{r}_v \right|^2 = EG - F^2$$

and

$$\begin{cases} \vec{n}_u \cdot \vec{r}_u = -\vec{r}_{uu} \cdot \vec{n} = -D \\ \\ \vec{n}_u \cdot \vec{r}_v = -\vec{r}_{uv} \cdot \vec{n} = -D' \\ \\ \vec{n}_v \cdot \vec{r}_u = -\vec{r}_{uv} \cdot \vec{n} = -D' \\ \\ \vec{n}_v \cdot \vec{r}_v = -\vec{r}_{vv} \cdot \vec{n} = -D'' \end{cases}$$

and we conclude that

$$D\,D'' - {D'}^2 = \left(\vec{n}_u \cdot \vec{r}_u \right) \left(\vec{n}_v \cdot \vec{r}_v \right) - \left(\vec{n}_u \cdot \vec{r}_v \right) \left(\vec{n}_v \cdot \vec{r}_u \right)$$

and

$$\lambda = \frac{D\,D'' - {D'}^2}{EG - F^2}.$$

The final conclusion is the the Gaussian curvature is the ratio of the volume elements in the map defined by the unit normal vector (which is known as the *Gauss map*).

§5. Gaussian Curvature from Non-Commutativity of Partial Covariant Differentiations. We define *covariant differentiation* of a vector field on S as the orthogonal projection of the usual differentiation onto the tangent planes of S. We consider the following four covariant derivatives:

(i) the covariant derivative $\nabla_u \vec{r}_u$ of the tangent vector field $\vec{r}_u$ with respect to u which is the projection $\vec{r}_{uu} - \left(\vec{r}_{uu} \cdot \vec{n} \right) \vec{n}$ of $\vec{r}_{uu}$ onto the tangent space of S,

(ii) the covariant derivative $\nabla_v \vec{r}_u$ of the tangent vector field $\vec{r}_u$ with respect to v which is the projection $\vec{r}_{uv} - \left(\vec{r}_{uv} \cdot \vec{n} \right) \vec{n}$ of $\vec{r}_{uv}$ onto the tangent space of S,

(iii) the covariant derivative $\nabla_u \vec{r}_v$ of the tangent vector field $\vec{r}_v$ with respect to u which is the projection $\vec{r}_{vu} - (\vec{r}_{vu} \cdot \vec{n})\,\vec{n}$ of $\vec{r}_{vu}$ onto the tangent space of S, and

(iv) the covariant derivative $\nabla_v \vec{r}_v$ of the tangent vector field $\vec{r}_v$ with respect to u which is the projection $\vec{r}_{vv} - (\vec{r}_{vv} \cdot \vec{n})\,\vec{n}$ of $\vec{r}_{vv}$ onto the tangent space of S.

Unlike the commutativity of partial differentiations for functions on $\mathbb{R}^2$, the covariant derivative $\nabla_u \nabla_v \vec{r}_u$ is in general not equal to $\nabla_v \nabla_u \vec{r}_u$, that is, the two partial covariant differentiations ∇_u, ∇_v do not commute when applied to the tangent vector field $\vec{r}_u$ of S. We now compute the difference of $\nabla_u \nabla_v \vec{r}_u$ and $\nabla_v \nabla_u \vec{r}_u$.

The covariant derivative $\nabla_u \nabla_v \vec{r}_u$ of $\nabla_v \vec{r}_u$ with respect to u is equal to

$$\begin{aligned} &\partial_u \left(\vec{r}_{uv} - (\vec{r}_{uv} \cdot \vec{n})\vec{n}\right) \mod \vec{n} \\ &= \vec{r}_{uvu} - (\vec{r}_{uv} \cdot \vec{n})\vec{n}_u \mod \vec{n}. \end{aligned}$$

It follows that

$$\nabla_u \nabla_v \vec{r}_u - \nabla_v \nabla_u \vec{r}_u = -(\vec{r}_{uv} \cdot \vec{n})\vec{n}_u + (\vec{r}_{uu} \cdot \vec{n})\vec{n}_v \mod \vec{n}.$$

Taking the inner product with $\vec{r}_v$ yields

$$\begin{aligned} &\left(\nabla_u \nabla_v \vec{r}_u - \nabla_v \nabla_u \vec{r}_u\right) \cdot \vec{r}_v \\ =&-(\vec{r}_{uv} \cdot \vec{n})\,(\vec{n}_u \cdot \vec{r}_v) + (\vec{r}_{uu} \cdot \vec{n})\,(\vec{n}_v \cdot \vec{r}_v) \\ =&D'^2 - D\,D''. \end{aligned}$$

Thus the Gaussian curvature can be expressed as

$$\frac{-1}{EG - F^2}\left(\nabla_u \nabla_v \vec{r}_u - \nabla_v \nabla_u \vec{r}_u\right) \cdot \vec{r}_v.$$

§6. Christoffel Symbols Expressing Covariant Differentiation in Terms of First Fundamental Form. The *Christoffel symbols* are the coefficients when $\nabla_u \vec{r}_u, \nabla_v \vec{r}_u, \nabla_u \vec{r}_v, \nabla_v \vec{r}_v$ are expressed as linear combinations of $\vec{r}_u$ and $\vec{r}_v$. The four equations which define the Christoffel symbols are

$$\begin{aligned} \vec{r}_{uu} - (\vec{r}_{uu} \cdot \vec{n})\,\vec{n} &= \Gamma^u_{uu} \vec{r}_u + \Gamma^v_{uu} \vec{r}_v, \\ \vec{r}_{uv} - (\vec{r}_{uv} \cdot \vec{n})\,\vec{n} &= \Gamma^u_{uv} \vec{r}_u + \Gamma^v_{uv} \vec{r}_v, \end{aligned}$$

$$\begin{aligned}\vec{r}_{vu} - (\vec{r}_{vu} \cdot \vec{n})\, \vec{n} &= \Gamma^u_{vu} \vec{r}_u + \Gamma^v_{vu} \vec{r}_v, \\ \vec{r}_{vv} - (\vec{r}_{vv} \cdot \vec{n})\, \vec{n} &= \Gamma^u_{vv} \vec{r}_u + \Gamma^v_{vv} \vec{r}_v.\end{aligned}$$

Note that the second and third equations are identical and $\Gamma^u_{uv} = \Gamma^u_{vu}$ and $\Gamma^v_{uv} = \Gamma^v_{vu}$.

We now want to express the Christoffel symbols in terms of E, F, G and their partial derivatives by taking the inner products of the four equations with $\vec{r}_u$ and $vecr_v$ and then solve the linear equations so obtained for the Christoffol symbols. Taking the inner product of the first equation with $\vec{r}_v$ gives us

$$\begin{aligned}\vec{r}_{uu} \cdot \vec{r}_u &= \Gamma^u_{uu}\, (\vec{r}_u \cdot \vec{r}_u) + \Gamma^v_{uu}\, (\vec{r}_v \cdot \vec{r}_u) = \Gamma^u_{uu}\, E + \Gamma^v_{uu}\, F, \\ \vec{r}_{uu} \cdot \vec{r}_v &= \Gamma^u_{uu}\, (\vec{r}_u \cdot \vec{r}_v) + \Gamma^v_{uu}\, (\vec{r}_v \cdot \vec{r}_v) = \Gamma^u_{uu}\, F + \Gamma^v_{uu}\, G.\end{aligned}$$

The key point is to write

$$\begin{aligned}\vec{r}_{uu} \cdot \vec{r}_u &= \left(\frac{1}{2}\, \vec{r}_u \cdot \vec{r}_u\right)_u = \frac{1}{2}\, E_u, \\ \vec{r}_{uu} \cdot \vec{r}_v = (\vec{r}_u \cdot \vec{r}_v)_u - \vec{r}_u \cdot \vec{r}_{vu} &= (\vec{r}_u \cdot \vec{r}_v)_u - \left(\frac{1}{2}\, \vec{r}_u \cdot \vec{r}_u\right)_v = F_u - \frac{1}{2}\, E_v\end{aligned}$$

and to use

$$\det \begin{pmatrix} E & F \\ F & G \end{pmatrix} \neq 0$$

to solve for the unknowns Γ^u_{uu} and Γ^v_{uu} in the system of equations

$$\begin{cases} E\, \Gamma^u_{uu} + F\, \Gamma^v_{uu} = \frac{1}{2}\, E_u \\ \\ F\, \Gamma^u_{uu} + G\, \Gamma^v_{uu} = F_u - \frac{1}{2}\, E_v. \end{cases}$$

By Cramer's rule,

$$\Gamma^u_{uu} = \frac{\begin{vmatrix} \frac{1}{2}\, E_u & F \\ F_u - \frac{1}{2}\, E_v & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} \quad \text{and} \quad \Gamma^v_{uu} = \frac{\begin{vmatrix} E & \frac{1}{2}\, E_u \\ F & F_u - \frac{1}{2}\, E_v \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}.$$

Likewise, we write

$$\vec{r}_{uv} \cdot \vec{r}_u = \left(\frac{1}{2}\, \vec{r}_u \cdot \vec{r}_u\right)_v = \frac{1}{2}\, E_v,$$

$$\vec{r}_{uv} \cdot \vec{r}_v = \left(\frac{1}{2} \vec{r}_v \cdot \vec{r}_v \right)_u = \frac{1}{2} G_u$$

and

$$\vec{r}_{vv} \cdot \vec{r}_v = \left(\frac{1}{2} \vec{r}_v \cdot \vec{r}_v \right)_v = \frac{1}{2} G_v,$$

$$\vec{r}_{vv} \cdot \vec{r}_u = (\vec{r}_v \cdot \vec{r}_u)_v - \vec{r}_v \cdot \vec{r}_{uv} = (\vec{r}_v \cdot \vec{r}_u)_v - \left(\frac{1}{2} \vec{r}_v \cdot \vec{r}_v \right)_u = F_v - \frac{1}{2} G_u$$

so that we can solve the systems of linear equations

$$\begin{cases} E\,\Gamma^u_{uv} + F\,\Gamma^v_{uv} = \frac{1}{2} E_v \\ \\ F\,\Gamma^u_{uv} + G\,\Gamma^v_{uv} = \frac{1}{2} G_u \end{cases}$$

and

$$\begin{cases} E\,\Gamma^u_{vv} + F\,\Gamma^v_{vv} = \frac{1}{2} G_v \\ \\ F\,\Gamma^u_{vv} + G\,\Gamma^v_{vv} = F_v - \frac{1}{2} G_u \end{cases}$$

for the pair of unknowns $(\Gamma^u_{uv}, \Gamma^v_{uv})$ and the pair of unknowns $(\Gamma^u_{vv}, \Gamma^v_{vv})$ respectively. By Cramer's rule,

$$\Gamma^u_{uv} = \frac{\begin{vmatrix} \frac{1}{2} E_v & F \\ \\ \frac{1}{2} G_u & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}, \qquad \Gamma^v_{uv} = \frac{\begin{vmatrix} E & \frac{1}{2} E_v \\ \\ F & \frac{1}{2} G_u \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

$$\Gamma^u_{vv} = \frac{\begin{vmatrix} \frac{1}{2} G_v & F \\ \\ F_v - \frac{1}{2} G_u & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}, \quad \text{and} \quad \Gamma^v_{vv} = \frac{\begin{vmatrix} E & \frac{1}{2} G_v \\ \\ F & F_v - \frac{1}{2} G_u \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}.$$

§7. Gaussian Curvature in Terms of First Fundamental Form. To finish the proof of the Celebrated Theorem of Gauss it suffices to express the

$$(\nabla_u \nabla_v \vec{r}_u - \nabla_v \nabla_u \vec{r}_u) \cdot \vec{r}_v$$

in terms of the Christoffel symbols and their partial derivatives, because the Gaussian curvature is equal to

$$\frac{-1}{EG-F^2}\left(\nabla_u\nabla_v\vec{r}_u-\nabla_v\nabla_u\vec{r}_u\right)\cdot\vec{r}_v$$

and the Christoffel symbols can be expressed in terms of E, F, G and their partial derivatives. Taking the covariant derivative of

$$\nabla_v\vec{r}_u=\vec{r}_{uv}-\left(\vec{r}_{uv}\cdot\vec{n}\right)\vec{n}=\Gamma^u_{uv}\vec{r}_u+\Gamma^v_{uv}\vec{r}_v$$

with respect to u yields

$$\begin{gathered}\nabla_u\nabla_v\vec{r}_u=\left(\partial_u\Gamma^u_{uv}\right)\vec{r}_u+\Gamma^u_{uv}\nabla_u\vec{r}_u+\left(\partial_u\Gamma^v_{uv}\right)\vec{r}_v+\Gamma^v_{uv}\nabla_u\vec{r}_v\\ =\left(\partial_u\Gamma^u_{uv}\right)\vec{r}_u+\Gamma^u_{uv}\left(\Gamma^u_{uu}\vec{r}_u+\Gamma^v_{uu}\vec{r}_v\right)+\left(\partial_u\Gamma^v_{uv}\right)\vec{r}_v+\Gamma^v_{uv}\left(\Gamma^u_{uv}\vec{r}_u+\Gamma^v_{uv}\vec{r}_v\right)\end{gathered}$$

Taking the covariant derivative of

$$\nabla_u\vec{r}_u=\vec{r}_{uu}-\left(\vec{r}_{uu}\cdot\vec{n}\right)\vec{n}=\Gamma^u_{uu}\vec{r}_u+\Gamma^v_{uu}\vec{r}_v$$

with respect to v yields

$$\begin{gathered}\nabla_v\nabla_u\vec{r}_u=\left(\partial_v\Gamma^u_{uu}\right)\vec{r}_u+\Gamma^u_{uu}\nabla_v\vec{r}_u+\left(\partial_v\Gamma^v_{uu}\right)\vec{r}_v+\Gamma^v_{uu}\nabla_v\vec{r}_v\\ =\left(\partial_v\Gamma^u_{uu}\right)\vec{r}_u+\Gamma^u_{uu}\left(\Gamma^u_{uv}\vec{r}_u+\Gamma^v_{uv}\vec{r}_v\right)+\left(\partial_v\Gamma^v_{uu}\right)\vec{r}_v+\Gamma^v_{uu}\left(\Gamma^u_{vv}\vec{r}_u+\Gamma^v_{vv}\vec{r}_v\right).\end{gathered}$$

Taking the difference of the two expressions and then taking the inner product with $\vec{r}_v$, we obtain

$$\begin{aligned}&\left(\nabla_u\nabla_v\vec{r}_u-\nabla_v\nabla_u\vec{r}_u\right)\cdot\vec{r}_v\\ =\ &\left(\partial_u\Gamma^u_{uv}\right)F+\Gamma^u_{uv}\left(\Gamma^u_{uu}F+\Gamma^v_{uu}G\right)+\left(\partial_u\Gamma^v_{uv}\right)G+\Gamma^v_{uv}\left(\Gamma^u_{uv}F+\Gamma^v_{uv}G\right)\\ &-\left[\left(\partial_v\Gamma^u_{uu}\right)F+\Gamma^u_{uu}\left(\Gamma^u_{uv}F+\Gamma^v_{uv}G\right)+\left(\partial_v\Gamma^v_{uu}\right)G+\Gamma^v_{uu}\left(\Gamma^u_{vv}F+\Gamma^v_{vv}G\right)\right],\end{aligned}$$

which depends only on E, F, G and their partial derivatives. This finishes the proof of the Celebrated Theorem of Gauss.

§7. Gauss Curvature in Terms of Riemann Curvature Tensor. Gauss curvature is defined for a surface in $\mathbb{R}^3$ or an abstract surface with the first fundamental form

$$ds^2=E\,du^2+2F\,dudv+G\,dv^2.$$

Later in the course, we will discuss Riemann's generalization of the Gauss curvature to the Riemann curvature tensor. The Riemann curvature tensor is defined by

$$R(\xi, \eta) = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}$$

(where ξ, η are vector fields and $[\xi, \eta] = \xi\eta - \eta\xi$ and ∇_ξ (respectively ∇_η and $\nabla_{[\xi,\eta]}$) means covariant derivative with respect to ξ (respectively η and $[\xi, \eta]$) so that $R(\xi, \eta)$ maps a vector field ζ to the vector field $R(\xi, \eta)\zeta$. In local coordinates, if $\xi = (\xi^k)$, $\eta = (\eta^\ell)$, and $\zeta = (\zeta^i)$, then

$$\big(R(\xi, \eta)\zeta\big)^j = \sum_{i,j,k} R_i{}^j{}_{k\ell}\, \zeta^i \xi^k \eta^\ell.$$

The sectional curvature for the two directions ξ and η is

$$-\frac{\langle R(\xi, \eta)\xi, \eta\rangle}{\langle \xi, \xi\rangle \langle \eta, \eta\rangle - \langle \xi, \eta\rangle^2}.$$

For the local coordinates $(u, v) = (u^1, u^2)$ in the surface under consideration

$$-\left(\nabla_u \nabla_v \vec{r}_u - \nabla_v \nabla_u \vec{r}_u\right) \cdot \vec{r}_v = -\left\langle R\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right)\left(\frac{\partial}{\partial u^1}\right), \frac{\partial}{\partial u^2}\right\rangle = -R_{1212}$$

The Gauss curvature is the sectional curvature

$$-\frac{R_{1212}}{EG - F^2}$$

for the two directions $\frac{\partial}{\partial u^1} = \vec{r}_u$ and $\frac{\partial}{\partial u^2} = \vec{r}_v$.