Fundamental Theorem of Riemannian Geometry

Earlier we discussed connections for a smooth vector bundle V of rank over a smooth manifold X of real dimension n. Such connections are artificially introduced by specifying the total covariant differentiation of a local frame $e_1, \cdots, e_r$ to be

$$De_\alpha = \sum_{\beta=1}^{n} \omega_\alpha^\beta e_\beta,$$

where $\omega = (\omega_\alpha^\beta)_{1\leq\alpha,\beta\leq r}$ is an $r \times r$ matrix whose entries are differential 1-forms. When a vector field $\vec{\xi}$ is specified, the covariant derivative of e_α with respect to $\vec{\xi}$ is

$$\nabla_{\vec{\xi}} e_\alpha = \sum_{\beta=1}^{n} \left\langle \omega_\alpha^\beta, \vec{\xi} \right\rangle e_\beta.$$

Then the measure of the failure of commutativity of covariant differentiation is the curvature $\Omega = \left(\Omega_\alpha^\beta\right)_{1\leq\alpha,\beta\leq r}$ which is an $r \times r$ matrix whose entries are differential 2-form and is defined by

$$\Omega = d\omega - \omega \wedge \omega$$

in matrix notations.

The k-th Chern form γ_k is motivated by the k-th elementary symmetric function of the eigenvalues of a square matrix and is given as the sum of all the principal minors of the $r \times r$ matrix Ω of differential 2-forms. By using the Bianchi's identity $D\Omega = 0$ and the normal fiber coordinate of V at a point P where computation is done, one can show that the k-th Chern form γ_k is closed.

By using a homotopy argument of joining two connections by a real line segment and constructing a connection on $\mathbb{R} \times X$ whose k-th Chern form is closed, one can prove that the class, known as the k-th Chern class, defined by the k-th Chern form modulo exact differential $(2k-1)$-forms, is well-defined and independent of the choice of the connection.

We will later use the integration of the top degree Chern class over X (when n is even) to obtain the generalization of the Gauss-Bonnet theorem for a smooth closed surface S in $\mathbb{R}^3$ which states that the integral of the Gaussian curvature of S over S is equal to the Euler characteristic of S.

At this point, we would like to specialize to the case where the vector bundle V is the tangent bundle T_X of X, to discuss the fundamental theorem of Riemannian geometry.

§1. Statement of the Fundamental Theorem of Riemannian Geometry

Let X be a smooth manifold of real dimension n. Assume that a smooth metric

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i \, dx^j$$

(with g_{ij} symmetric in i, j) for the tangent bundle T_X, known as a Riemannian metric of X, is given. Then there exists uniquely a smooth connection $\omega = \left(\omega_k^j\right)_{1\leq i,j\leq n}$, known as the Levi-Civita connection, which is compatible with the given metric and is torsion-free in the following sense.

(i) (*Metric Compatibility*) For any smooth vector fields $\vec{\xi}$ and $\vec{\eta}$,

$$d\left\langle \vec{\xi}, \, \vec{\eta}\right\rangle = \left\langle D\vec{\xi}, \, \vec{\eta}\right\rangle + \left\langle \vec{\xi}, \, D\vec{\eta}\right\rangle.$$

Equivalently, $Dg = 0$, where Dg is the total covariant derivative of the tensor g of covariant rank 2.

(ii) (*Torsion-Freeness*) For any smooth differential 1-form σ, the skew-symmetrization of the tensor $D\sigma$ of covariant rank 2 is equal to the exterior derivative $d\sigma$. Equivalently, the Christoffel symbol Γ_{ij}^k defined by $\omega_i^k = \sum_{j=1}^n \Gamma_{ij}^k dx^j$ is symmetric is ij, *i.e.*, $\Gamma_{ij}^k = \Gamma_{ji}^k$. Also equivalently, the torsion tensor $T(\vec{\xi}, \vec{\eta}) := \nabla_{\vec{\xi}}\vec{\eta} - \nabla_{\vec{\xi}}\vec{\eta} - \left[\vec{\xi}, \vec{\eta}\right]$ of contravariant rank 1 and covariant rank 2 vanishes identically.

§2. Proof of the Fundamental Theorem of Riemannian Geometry

We will use the characterization $Dg = 0$ for metric compatibility and the characterization of $\Gamma_{ij}^k = \Gamma_{ji}^k$ in our proof and deal with the question of equivalent characterizations for both properties later.

The condition $Dg = 0$ is the same as $\nabla_\ell g_{ij} = 0$, which means that

$$(*) \qquad \frac{\partial}{\partial x^\ell}\, g_{ij} = \sum_{k=1}^n \Gamma^k_{\ell i} g_{kj} + \sum_{k=1}^n \Gamma^k_{\ell j} g_{ik},$$

because

$$\begin{aligned}\nabla_\ell g_{ij} &= \frac{\partial}{\partial x^\ell}\, g_{ij} - \sum_{k=1}^n \omega_i^k \left(\frac{\partial}{\partial x^\ell}\right) g_{kj} - \sum_{k=1}^n \omega_j^k \left(\frac{\partial}{\partial x^\ell}\right) g_{ki} \\ &= \frac{\partial}{\partial x^\ell}\, g_{ij} - \sum_{k=1}^n \Gamma^k_{\ell i} g_{kj} - \sum_{k=1}^n \Gamma^k_{\ell j} g_{ik}.\end{aligned}$$

For the proof we need to solve the n^3 equations in $(*)$ in n^3 unknowns Γ^k_{ij} (for $1 \leq i, j, k \leq n$) uniquely. To facilitate solving for the n^3 unknowns in the n^3 equations, we transform the n^3 unknowns Γ^k_{ij} (for $1 \leq i, j, k \leq n$) to the n^3 unknowns

$$\Gamma_{k,ij} = \sum_{\ell=1}^n \Gamma^\ell_{ij} g_{\ell k}$$

for $1 \leq i, j, k \leq n$ so that the n^3 equations become

$$(\dagger)_{\ell,i,j} \qquad \frac{\partial}{\partial x^\ell}\, g_{ij} = \Gamma_{j,\ell i} + \Gamma_{i,\ell j}$$

for $1 \leq \ell, i, j \leq n$. Note that the symmetry property of $\Gamma^\ell_{ij} = \Gamma^\ell_{ji}$ goes over to the symmetry property $\Gamma_{\ell,ij} = \Gamma_{\ell,ji}$. From $(\dagger)_{\ell,i,j}$, we generate two more equations by cyclically permuting the three indices ℓ, i, j to get

$$(\dagger)_{i,j,\ell} \qquad \frac{\partial}{\partial x^i}\, g_{j\ell} = \Gamma_{\ell,ij} + \Gamma_{j,i\ell}$$

and

$$(\dagger)_{j,\ell,i} \qquad \frac{\partial}{\partial x^j}\, g_{\ell i} = \Gamma_{i,j\ell} + \Gamma_{\ell,ji}.$$

Now for fixed i, j, ℓ we have three linear equations and three unknowns, which we can easily solve as follows. Of the three equations we sum the last two and then subtract the first one to get the equation

$$(\dagger)_{i,j,\ell} + (\dagger)_{j,\ell,i} - (\dagger)_{i,j,\ell}.$$

Using the symmetry property $\Gamma_{\ell,ij} = \Gamma_{\ell,ji}$, we obtain

$$\frac{\partial}{\partial x^i}\, g_{j\ell} + \frac{\partial}{\partial x^j}\, g_{\ell i} - \frac{\partial}{\partial x^\ell}\, g_{ij} = 2\Gamma_{\ell,ij},$$

which means

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{\ell=1}^n g^{i\ell}\left(\frac{\partial}{\partial x^j}\, g_{k\ell} + \frac{\partial}{\partial x^k}\, g_{j\ell} - \frac{\partial}{\partial x^\ell}\, g_{jk}\right).$$

This gives the uniqueness of the Levi-Civita connection. By tracing the argument backwards, we can easily verify that the constructed connection with Chrstoffel symbol Γ^k_{ij} possesses both metric compatibility and torsion-freeness. Q.E.D.

§3. Invariant (Coordinate-Free) Formulation of Torsion-Free Condition

We would like to introduce the invariant formulation for the torsion-free condition. The Christoffel symbols are given by

$$D\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^n \omega_i^j \frac{\partial}{\partial x^j}$$

and

$$\omega_i^j = \sum_{k=1}^n \Gamma^j_{ik} dx^k.$$

In other words,

$$\nabla_k\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^n \Gamma^j_{ik} \frac{\partial}{\partial x^j}.$$

The torsion-free condition is that

$$\nabla_k\left(\frac{\partial}{\partial x^i}\right)$$

is symmetric in i and k, *i.e.* $\nabla_{\vec{\xi}}\vec{\eta}$ is symmetric in $\vec{\xi}$ and $\vec{\eta}$ when

$$\vec{\xi} = \frac{\partial}{\partial x^j} \quad \text{and} \quad \vec{\eta} = \frac{\partial}{\partial x^i}.$$

For general vector fields $\vec{\xi}$ and $\vec{\eta}$ we do not have the vanishing of $\nabla_{\vec{\xi}}\vec{\eta} - \nabla_{\vec{\eta}}\vec{\xi}$ from the torsion-free condition, because for smooth functions φ and ψ

$$\nabla_{\varphi\vec{\xi}}(\psi\vec{\eta}) - \nabla_{\psi\vec{\eta}}(\varphi\vec{\xi}) = \varphi\psi(\nabla_{\vec{\xi}}\vec{\eta} - \nabla_{\vec{\eta}}\vec{\xi}) + \varphi(\vec{\xi}\psi)\vec{\eta} - \psi(\vec{\eta}\varphi)\vec{\xi}$$

and we cannot expect to get the general case by multiplying the special case $\vec{\xi} = \frac{\partial}{\partial x^i}$, $\vec{\eta} = \frac{\partial}{\partial x^j}$ by smooth functions and then summing up. The trouble is the discrepancy terms $\varphi(\vec{\xi}\psi)\vec{\eta} - \psi(\vec{\eta}\varphi)\vec{\xi}$. The Lie bracket $[\vec{\xi}, \vec{\eta}]$ yields the same discrepancy terms when $\vec{\xi}$ and $\vec{\eta}$ are respectively multiplied by smooth functions φ and ψ, *viz.*

$$[\varphi\vec{\xi}, \psi\vec{\eta}] = \varphi\psi[\vec{\xi}, \vec{\eta}] + \varphi(\vec{\xi}\psi)\vec{\eta} - \psi(\vec{\eta}\varphi)\vec{\xi}.$$

So we conclude that the torsion-free condition is equivalent to the vanishing of $\nabla_{\vec{\xi}}\vec{\eta} - \nabla_{\vec{\eta}}\vec{\xi} - [\vec{\xi}, \vec{\eta}]$ for any pair of tangent vectors $\vec{\xi}$ and $\vec{\eta}$. Moreover, we have seen that

$$\nabla_{\varphi\vec{\xi}}(\psi\vec{\eta}) - \nabla_{\psi\vec{\eta}}(\varphi\vec{\xi}) - [\varphi\vec{\xi}, \psi\vec{\eta}] = \varphi\psi(\nabla_{\vec{\xi}}\vec{\eta} - \nabla_{\vec{\eta}}\vec{\xi} - [\vec{\xi}, \vec{\eta}]).$$

When the torsion-free condition is not satisfied, we define $T(\vec{\xi}, \vec{\eta}) = \nabla_{\vec{\xi}}\vec{\eta} - \nabla_{\vec{\eta}}\vec{\xi} - [\vec{\xi}, \vec{\eta}]$. Then $T(\varphi\vec{\xi}, \psi\vec{\eta}) = \varphi\psi T(\vec{\xi}, \vec{\eta})$ and T is a tensor and is called the *torsion tensor*. In local coordinates we write

$$T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{k=1}^{n} T_{ij}^k \frac{\partial}{\partial x^k}$$

with $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$.

§4. Geometric Interpretation of Torsion

We give here a geometric interpretation of torsion. Take a point P of M and two tangent vectors $\vec{\xi}$ and $\vec{\eta}$ at P. We take a curve $\gamma_{\vec{\xi}}$ going through P whose tangent at P is $\vec{\xi}$. We transport $\vec{\eta}$ for a distance s along $\gamma_{\vec{\xi}}$ and get a tangent vector $\vec{\eta}'$ of M at $\gamma_{\vec{\xi}}(s)$. Take a curve $\gamma_{\vec{\eta}}$ through $\gamma_{\vec{\xi}}(s)$ whose tangent at $\gamma_{\vec{\xi}}(s)$ is $\vec{\eta}'$. Let Q be the point $\gamma_{\vec{\eta}}(t)$.

Now reverse the roles of $\vec{\xi}$ and $\vec{\eta}$. We take a curve $\sigma_{\vec{\eta}}$ going through P whose tangent at P is $\vec{\eta}$. We transport $\vec{\xi}$ for a distance t along $\sigma_{\vec{\eta}}$ and get a tangent vector $\vec{\xi}'$ of M at $\sigma_{\vec{\eta}}(t)$. Take a curve $\sigma_{\vec{\xi}}$ through $\sigma_{\vec{\eta}}(t)$ whose tangent at $\sigma_{\vec{\eta}}(t)$ is $\vec{\xi}'$. Let R be the point $\sigma_{\vec{\xi}}(s)$.

The limit of the vector $\frac{1}{st}\vec{RQ}$ (in any coordinate system) as s and t approach zero defines a tangent vector of M which is independent of the choice of the curves by order considerations. We claim that this tangent vector is $T(\vec{\xi},\vec{\eta})$.

Let us now verify. We choose a local coordinate system x^i $(1 \leq i \leq m)$ at P so that P corresponds to the origin. Let $\vec{\xi}^i$ and $\vec{\eta}^i$ be respectively the components of $\vec{\xi}$ and $\vec{\eta}$ in terms of this local coordinate system. We choose as our curve $\gamma_{\vec{\xi}}$ the curve defined by the equations $\gamma^i_{\vec{\xi}}(s) = \vec{\xi}^i s$. The equation for parallel transport of $\vec{\eta}$ along $\gamma_{\vec{\xi}}$ is

$$\frac{d}{ds}\vec{\eta}^i + \sum_{j,k=1}^{n} \Gamma^i_{jk}\vec{\xi}^j\vec{\eta}^k = 0.$$

Hence

$$\left(\vec{\eta}'\right)^i \approx \vec{\eta}^i - \left(\sum_{j,k=1}^{n} \Gamma^i_{jk}\vec{\xi}^j\vec{\eta}^k\right) s.$$

Here higher orders are ignored in the approximation. We can take as $\gamma_{\vec{\eta}}$ the curve defined by the equations $\gamma^i_{\vec{\eta}}(t) = \vec{\xi}^i s + \left(\vec{\eta}'\right)^i t$. Hence

$$x^i(Q) \approx \vec{\xi}^i s + \vec{\eta}^i t - \left(\sum_{j,k=1}^{n} \Gamma^i_{jk}\vec{\xi}^j\vec{\eta}^k\right) st.$$

Likewise we get the expression for R by changing the roles of $\vec{\xi}, s$ and $\vec{\eta}, t$. Thus

$$\frac{1}{st}(x^i(R) - x^i(Q)) \approx \sum_{j,k}^{n}(\Gamma^i_{jk} - \Gamma^i_{jk})\vec{\xi}^j\vec{\eta}^k = T(\vec{\xi},\vec{\eta})^i.$$

§5. Curvature of Levi-Civita Connection (*i.e.,* Curvature of Riemannian Metric)

Again, unlike the case of partial differentiation for functions, in general covariant differentiation for sections of a vector bundle like T_X and T^*_X with respect to the Levi-Civita connection do not commute. The failure of the commutativity is measured by the curvature of the Levi-Civita connection, also called the curvature of the Riemannian metric. Take a local smooth basis e_α $(1 \leq \alpha \leq r)$ of T_X. We apply D twice to $e_\wedge$ and, after skew-symmetrization we get a local T_X-valued 2-form $DD\,e_\alpha$. We can express $DD\,e_\alpha$ in terms of the connection ω and its exterior derivative $d\omega$ as follows. We use the column vector e with components e_α.

$$\begin{aligned} DD\,e = D(\omega\, e) &= d\omega\, e - \omega\wedge De \\ &= d\omega\, e - \omega\wedge\omega\, e = (d\omega - \omega\wedge\omega)e. \end{aligned}$$

We define the *curvature* of the Riemannian metric to be $d\omega - \omega\wedge\omega$ and denote it by Ω. It is an $\mathrm{End}(T_X)$-valued differential 2-form on X, because if f is a local smooth matrix-valued function, then

$$\begin{aligned} DD(fe) &= D(df\, e + f\, De) \\ &= ddf\, e - df\wedge De + df\wedge De + f DDe \\ &= f\, DDe. \end{aligned}$$

§5. (First) Bianchi Identity for Curvature of Riemannian Metric

Let us look at the curvature tensor Ω in local coordinates. With respect to a local frame e_α $(1 \leq \alpha \leq r)$ we have

$$\Omega^\ell_k = \sum_{i,j=1}^n \Omega^\ell_{kij} dx^i\wedge dx^j.$$

Using the Christoffel symbol Γ^k_{ij} of the connection ω^k_i, we have

$$\sum_{i,j=1}^n \Omega^\ell_{kij} dx^i\wedge dx^j = \sum_{i,j=1}^n d(\Gamma^\ell_{ki} dx^i) - \sum_{i,j,p=1}^n (\Gamma^p_{ki} dx^i)\wedge(\Gamma^\ell_{pj} dx^j)$$

and

$$\Omega^\ell_{kij} = \partial_i \Gamma^\ell_{kj} - \partial_j \Gamma^\ell_{ki} + \sum_{p=1}^n \Gamma^p_{ki}\Gamma^\ell_{pj} - \sum_{p=1}^n \Gamma^p_{ij}\Gamma^\ell_{pi}.$$

For the curvature of the Riemannian metric we have another Bianchi identity, called the first Bianchi identity, which is obtained as follows. At the one point P under consideration, we can choose a coordinate system so that the first derivatives of the coefficients of the Riemannian metric tensor all vanish at that point P. This coordinate system is known as a normal coordinate system at P. The construction of a normal coordinate system (known as the geodesic normal coordinate system) will be given in the Appendix at the end of this set of lecture notes. When a normal coordinate system is used at that point P, we simply have

$$\Omega^{\ell}_{kij} = \partial_i \Gamma^{\ell}_{kj} - \partial_j \Gamma^{\ell}_{ki},$$

from which it follows that

$$\Omega^{\ell}_{kij} + \Omega^{\ell}_{ijk} + \Omega^{\ell}_{jki} = 0,$$

because of the symmetry of Γ^{ℓ}_{kj} in k and j. This is the first Bianchi identity.

The first Bianchi identity can also be derived as a consequence of the torsion-free condition. We can see this more clearly in the following way. Suppose e_i $(1 \leq i \leq m)$ is a local frame for T_M and ω^i $(1 \leq i \leq m)$ is its dual frame. Let ω_i^j with

$$(\flat) \qquad De_i = \sum_{j=1}^{n} \omega_i^j e_j$$

be the matrix valued 1-form of the connection. The torion-free condition is

$$d\omega^i + \sum_{j=1}^{n} \omega^i_j \wedge \omega^j = 0.$$

We leave the verification of this alternative formulation of the torsion-free condition to a homework assignment.

Taking exterior derivative of both sides of $(\flat)$ and then using $(\flat)$ to get rid of $d\omega^i$, we get

$$\begin{aligned} 0 &= \sum_{j=1}^{n} d\omega^i_j \wedge \omega^j - \sum_{j=1}^{n} \omega^i_j \wedge d\omega^j \\ &= \sum_{j=1}^{n} d\omega^i_j \wedge \omega^j + \sum_{j,k=1}^{n} \omega^i_j \wedge \omega^j_k \wedge \omega^k \\ &= \sum_{j=1}^{n} \Omega^i_j \wedge \omega^j. \end{aligned}$$

When we write
$$\Omega^i_j = \sum_{k,\ell=1}^n \Omega^i_{j\,k\ell}\omega^k\wedge\omega^\ell,$$
we have
$$\Omega^i_{j\,k\ell} + \Omega^i_{k\,\ell j} + \Omega^i_{\ell\,jk} = 0,$$
which is the first Bianchi identity.

§5. Another Symmetric Property of Curvature of Riemannian Metric

There is another symmetry of the curvature of the Riemannian metric, which we are going to derive. Let
$$R_{k\ell ij} = \sum_{p=1}^n g_{\ell p}\Omega^p_{kij}.$$
From
$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell}\left(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}\right),$$
by using the normal coordinates of X at the point under consideration, we have
$$\begin{aligned} R_{k\ell ij} &= \partial_i \Gamma_{\ell,kj} - \partial_j \Gamma_{\ell,ki} \\ &= \frac{1}{2}\,\partial_i\left(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}\right) - \frac{1}{2}\,\partial_j\left(\partial_i g_{k\ell} + \partial_k g_{i\ell} - \partial_\ell g_{ik}\right) \\ &= \frac{1}{2}\left(\partial_i\partial_k g_{j\ell} + \partial_j\partial_\ell g_{ik} - \partial_i\partial_\ell g_{jk} - \partial_j\partial_k g_{i\ell}\right), \end{aligned}$$
which means that we have the symmetry property
$$R_{k\ell ij} = R_{ijk\ell}$$
for $1 \leq i, j, k, \ell$.

To summarize, for the curvature of the Riemannian metric we have the following symmetric properties.
$$\begin{aligned} &R_{ijk\ell} = -R_{ij\ell k}. \\ &R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0. \\ &R_{k\ell ij} = R_{ijk\ell}. \end{aligned}$$

In words, they are

(i) the skew-symmetry in the last two indices,

(ii) the vanishing of the sum from cyclic permutation of the last three indices, and

(iii) the symmetry in the two pairs of indices.

APPENDIX: Geodesic Normal Coordinate System

Fix a point P of X and a local coordinate system $x^1, \cdots, x^n$ centered at P (*i.e.,* $x(P) = 0$). By a geodesic we mean a curve $x^i(t)$ which satisfies the second-order ordinary differential equation

$$\frac{d^2x^i}{dt^2} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0,$$

equivalently the covariant derivative its tangent vector along the curve with respect to t vanishes (*i.e.,* its tangent vector is parallel along the curve with respect to the Levi-Civita connection. Choose an indeterminate vector $\xi = (\xi^1, \cdots, \xi^n) \in \mathbb{R}^n$. Let $x^i = \varphi^i(t, \xi)$ be the solution of this second-order ordinary differential equation with the initial condition

$$\varphi^i(0, \xi) = 0 \quad \text{and} \quad \frac{d\varphi^i}{dt}(0, \xi) = \xi^i.$$

For $c \in \mathbb{R}$, we have

$$(**) \qquad \varphi^i(ct, \xi) = \varphi^i(t, c\xi)$$

by uniqueness of the solution of a second-order ordinary differential equation satisfying the same initial conditions for its value and first-order derivative. Setting $t = 1$ in $(**)$ and differentiating both sides with respect to c and evaluating at $c = 1$, we get

$$\xi^i = \sum_{j=1}^{n} \frac{\partial \varphi^i}{\partial \xi^j}(1, \xi)\xi^j$$

for any indeterminate ξ. Thus,

$$\frac{\partial \varphi^i}{\partial \xi^j}(1, \xi) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. We now introduce $y^1, \cdots, y^n$ as the new coordinate system defined by

$$x^i = \varphi^i(1, y),$$

that is,

$$x^i = \varphi^i(1, \xi)\Big|_{\xi = y}.$$

With respect to the coordinate system, the geodesic is now described by $t \mapsto y^i = \xi^i t$. Thus, the Christoffel symbols vanish with respect to the new system y at P. It is equivalent to the vanishing of all first-order derivatives of the metric tensor at P. The new coordinate system $y^1, \cdots, y^n$ is the geodesic normal coordinate system at P. Geometrically, the map from $T_{X,P}$ to the X defined by using the geodesic emanating at P with tangent vector $\xi \in T_{X,P}$ defines the new geodesic normal coordinate system at P with ξ as the new coordinates.