Differential Forms and Dual Formulation of Frobenius Integrability

We would like to discuss the dual formulation of Frobenius integrability in terms of differential forms. First of all, we want to introduce the notion of differential k-forms ω defined on an open subset G of $\mathbb{R}^n$ as an integrand on any local k-fold Y inside G. Because of the fundamental theorem of calculus

$$\int_{x\in(a,b)} f'(x)dx = f(b) - f(a),$$

this notion naturally leads to the notion of the exterior derivative $d\omega$ of ω, with ω generalizing the function f which is regarded as a differential 0-form and with $d\omega$ generalizing the integrand $f'(t)dt$ which is regarded as a differential 1-form.

§1. Definition of Differential k-Form as Integrand for Integration over Local Submanifold of Dimension k

In the integral

$$\int_{t\in(a,b)} f(t)dt$$

over an interval (a,b) in $\mathbb{R}$, we consider $f(x)dx$ as the integrand instead of the usual designation of $f(x)$ as the integrand. When there is an orientation-preserving reparametrization $t=\varphi(s)$ with $\varphi(\alpha)=a$ and $\varphi(\beta)=b$, the integral is equal to

$$\int_{s\in(\alpha,\beta)} \varphi^*\left(f(t)dt\right) = \int_{s\in(\alpha,\beta)} f(\varphi(s))\frac{dt}{ds}\,ds.$$

Suppose C is a local smooth curve in $\mathbb{R}^n$ which is projected diffeomorphically on (a,b) by the natural projection $\pi_1 : (x^1,\cdots,x^n)\to x^1 = t$. We have

$$\int_{t\in(a,b)} f(t)dt = \int_C \pi_1^*(f(x_1)dx_1).$$

We use f_1dx^1 to denote $\pi_1^*(f(x^1)dx^1)$. Then

$$\int_{t\in(a,b)} f(t)dt = \int_C f_1dx^1$$

and $f_1 dx_1$ can be regarded as an integrand on C. Instead of the projection π_1 we can use the natural projection $\pi_j : (x^1, \cdots, x^n) \to x^j$ for $2 \leq j \leq n$ to get an integrand of the form $f_j dx_j$ on C. The sum

$$\sum_{j=1}^{n} f_j dx^j$$

can be used as an integrand on any local curve C in $\mathbb{R}^n$. We can consider the more general situation where each f_j is a function $f_j(x^1, \cdots, x^n)$ depending on all the coordinates $x^1, \cdots, x^n$. The expression

$$\sum_{j=1}^{n} f_j dx^j$$

is an integrand which can be integrated on any local curve C in $\mathbb{R}^n$. We call

$$\sum_{j=1}^{n} f_j dx^j$$

a differential 1-form. We explain a bit about the name. A form means a homogeneous polynomial. In our case the variables are $dx^1, \cdots, dx^n$ and

$$\sum_{j=1}^{n} f_j dx^j$$

is a polynomial of homogeneous degree 1. That is why it is called a 1-form, indicating that the degree of this homogeneous polynomial is 1. Since the variables $dx^1, \cdots, dx^n$ are differentials, it is called a differential 1-form. The integral

$$\int_C \sum_{j=1}^{n} f_j dx^j$$

can be regarded as a usual a curve integral.

Similarly the notion of a differential k-form comes from replacing

$$\int_{t \in (a,b)} f(t) dt$$

by

$$\int_{(y^1,\cdots,y^k)\in G} f(y^1,\cdots,y^k)dy^1\wedge\cdots\wedge dy^k$$

(with G being an open subset of $\mathbb{R}^k$ with coordinates $y^1,\cdots,y^k$) in the above discussion. Here we use $dy^1\wedge\cdots\wedge dy^k$ instead of $dy^1\cdots dy^k$ to emphasize that the orientation (given by the order of $y^1,\cdots,y^k$) matters and a switch between two of $dy^1,\cdots,dy^k$ would change the orientation and as a consequence the sign of the integral. Instead of a local curve C in $\mathbb{R}^n$, we consider a local submanifold Y of dimension k. We consider an integrand

$$\sum_{1\leq j_1<\cdots<j_k\leq n} f_{j_1,\cdots,j_k}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}$$

(where each $f_{j_1,\cdots,j_k}$ is a function on an open subset Ω of $\mathbb{R}^n$) so that it can be integrated over any submanifold Y of dimension k in Ω to yield

$$\int_Y \sum_{1\leq j_1<\cdots<j_k\leq n} f_{j_1,\cdots,j_k}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}.$$

This integrand

$$\sum_{1\leq j_1<\cdots<j_k\leq n} f_{j_1,\cdots,j_k}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}$$

is called a differential k-form. It can be rewritten as

$$\frac{1}{k!}\sum_{j_1,\cdots,j_k=1}^{n} f_{j_1,\cdots,j_k}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}$$

with the condition that $f_{j_1,\cdots,j_k}$ is skew-symmetric (*i.e.,* alternating) in the indices $j_1,\cdots,j_k$.

§2. Coefficient of Differential k-Form from Shrinking a k-Dimensional Rectangle to a Point

For distinct $j_1,\cdots,j_k$ in $\{1,2,\cdots,n\}$, consider a k-dimensional "rectangle" $S_{j_1,\cdots,j_k}$ defined by x^ℓ being constant for $\ell\neq j_1,\cdots,j_k$, with the value of each of the other coordinates x^{j_ν} (for $1\leq j\leq k$) in a bounded interval in $\mathbb{R}$. The coefficient $\omega_{j_1,\cdots,j_k}$ of $dx^{j_1}\wedge\cdots\wedge dx^{j_k}$ in the differential k-form

$$\omega=\sum_{1\leq j_1<\cdots<j_k\leq n}\omega_{j_1,\cdots,j_k}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k},$$

at a point P is given by

$$\lim_{S_{j_1,\cdots,j_k}\to P}\frac{\int_{S_{j_1,\cdots,j_k}}\omega}{\int_{S_{j_1,\cdots,j_k}}dx^{j_1}\wedge\cdots\wedge dx^{j_k}}$$

as $S_{j_1,\cdots,j_k}$ shrinks down to a point P.

§3. Definition of Exterior Product of a Differential k-Form and a Differential ℓ-Form Motivated by Fubini's Theorem

Suppose

$$\eta=\sum_{1\leq i_1<\cdots<i_\ell\leq n}\eta_{i_1,\cdots,i_\ell}\,dx^{i_1}\wedge\cdots\wedge dx^{i_\ell}$$

is another differential ℓ-form. We define the exterior product $\omega\wedge\eta$ of ω and η as the differential $(k+\ell)$-form

$$\left(\sum_{1\leq j_1<\cdots<j_k\leq n}\omega_{j_1,\cdots,j_k}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}\right)\wedge\left(\sum_{1\leq i_1<\cdots<i_\ell\leq n}\eta_{i_1,\cdots,i_\ell}\,dx^{i_1}\wedge\cdots\wedge dx^{i_\ell}\right).$$

This is motivated by Fubini's theorem which gives

$$\begin{aligned}&\left(\int_{S_{j_1,\cdots,j_k}}\omega_{j_1,\cdots,j_k}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}\right)\left(\int_{S_{i_1,\cdots,i_\ell}}\eta_{i_1,\cdots,i_\ell}\,dx^{i_1}\wedge\cdots\wedge dx^{i_\ell}\right)\\&=\int_{S_{j_1,\cdots,j_k}\times S_{i_1,\cdots,i_\ell}}\omega_{j_1,\cdots,j_k}\,\eta_{i_1,\cdots,i_\ell}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_\ell}\\&=\int_{S_{j_1,\cdots,j_k,i_1,\cdots,i_\ell}}\omega_{j_1,\cdots,j_k}\,\eta_{i_1,\cdots,i_\ell}\,dx^{j_1}\wedge\cdots\wedge dx^{j_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_\ell}\end{aligned}$$

so that after we divide both sides by the volume of $S_{j_1,\cdots,j_k,i_1,\cdots,i_\ell}$ and letting $S_{j_1,\cdots,j_k,i_1,\cdots,i_\ell}$ shrink to a point P, we obtain the coefficient of $dx^{j_1}\wedge\cdots\wedge dx^{j_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_\ell}$ in the differential $(k+\ell)$-form $\omega\wedge\eta$.

§4. Definition of Exterior Derivative of Differential k-Form Motivated by Fundamental Theorem of Calculus for Rectangle of Dimension $k+1$

We now introduce the notion of the exterior derivative $d\omega$ of the differential k-form

$$\omega = \sum_{1\leq j_1<\cdots<j_k\leq n} \omega_{j_1,\cdots,j_k}\, dx^{j_1}\wedge\cdots\wedge dx^{j_k},$$

which is motivated by the fundamental theorem of calculus

$$\int_{t\in(a,b)} F'(t)dt = F(b)-F(a),$$

in which the function F is considered a differential 0-form on $\mathbb{R}$ and $F'(t)dt$ is considered its exterior derivative on $\mathbb{R}$.

Take a $(k+1)$-dimensional rectangle $S_{j_0,j_1,\cdots,j_k}$ defined by x^ℓ being constant for $\ell\neq j_0,j_1,\cdots,j_k$. The $(k+1)$-volume of $S_{j_0,j_1,\cdots,j_k}$ is equal to the k-volume of the k-face $S_{j_1,\cdots,j_k}$ times the length of the side of $S_{j_0,j_1,\cdots,j_k}$ along the direction of the x^{j_0}-axis. We now use a special case of Stokes's theorem where the domain is a $(k+1)$-dimensional rectangle. This special case is simply a consequence of Fubini's theorem on multiple and iterated integrals and the fundamental theorem of calculus applied separately to each of the variables along the $(k+1)$-dimensional rectangle. (See the Appendix at the end.) Apply Stokes's theorem (for a $(k+1)$-dimensional rectangle) to determine $d\omega$ from

$$\int_{S_{j_0,j_1,\cdots,j_k}} d\omega = \int_{\partial S_{j_0,j_1,\cdots,j_k}} \omega. \tag{‡}$$

The boundary $\partial S_{j_0,j_1,\cdots,j_k}$ is equal to

$$\sum_{\nu=0}^{k}(-1)^\nu\left(S_{j_0,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}-\tilde{S}_{j_0,j_1,\cdots,j_{\nu-1},j_{\nu+1},,\cdots,j_k}\right),$$

where $\tilde{S}_{j_0,j_1,\cdots,j_{\nu-1},j_{\nu+1},,\cdots,j_k}$ is the other k-dimensional rectangle which is parallel to $S_{j_0,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}$. By dividing both sides of (‡) by the $(k+1)$-volume of $S_{j_0,j_1,\cdots,j_k}$ and letting $S_{j_0,j_1,\cdots,j_k}$ shrink to a point P, we get

$$(d\omega)_{j_0,j_1,\cdots,j_k} = \sum_{\nu=0}^{k}(-1)^\nu\frac{\partial\omega_{j_0,j_1,\cdots,j_{\nu-1},j_{\nu+1},\cdots,j_k}}{\partial x^{j_\nu}}.$$

It means that

$$d\omega = \sum_{1 \le j_0 < j_1 < \cdots < j_k \le n} \sum_{\nu=0}^{k} (-1)^\nu \frac{\partial \omega_{j_0, j_1, \cdots, j_{\nu-1}, j_{\nu+1}, \cdots, j_k}}{\partial x^{j_\nu}} \, dx^{j_0} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

We can rewrite it as

$$\begin{aligned}
d\omega &= \frac{1}{(k+1)!} \sum_{j_0, j_1, \cdots, j_k = 1}^{n} \sum_{\nu=0}^{k} (-1)^\nu \frac{\partial \omega_{j_0, j_1, \cdots, j_{\nu-1}, j_{\nu+1}, \cdots, j_k}}{\partial x^{j_\nu}} \, dx^{j_0} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \\
&= \frac{1}{(k+1)!} \sum_{j_0, j_1, \cdots, j_k = 1}^{n} \sum_{\nu=0}^{k} \frac{\partial \omega_{j_1, \cdots, j_{\nu-1}, j_0, j_{\nu+1}, \cdots, j_k}}{\partial x^{j_\nu}} \, dx^{j_\nu} \wedge dx^{j_1} \wedge \cdots \\
&\qquad\qquad \cdots \wedge dx^{j_{\nu-1}} \wedge dx^{j_0} \wedge dx^{j_{\nu+1}} \wedge \cdots \wedge dx^{j_k} \\
&\qquad\qquad \text{(after transposing } dx^{j_0} \text{ and } dx^{j_\nu}\text{)} \\
&= \frac{1}{(k+1)!} \sum_{j_0, j_1, \cdots, j_k = 1}^{n} \sum_{\nu=0}^{k} \frac{\partial \omega_{j_1, \cdots, j_{\nu-1}, j_\nu, j_{\nu+1}, \cdots, j_k}}{\partial x^{j_0}} \, dx^{j_0} \wedge dx^{j_1} \wedge \cdots \\
&\qquad\qquad \cdots \wedge dx^{j_{\nu-1}} \wedge dx^{j_\nu} \wedge dx^{j_{\nu+1}} \wedge \cdots \wedge dx^{j_k} \\
&\qquad\qquad \text{(after interchanging the two dummy indices } j_0 \text{ and } j_\nu\text{)} \\
&= \sum_{1 \le j_1 < \cdots < j_k \le n} \left(\sum_{\ell=0}^{n} \frac{\partial \omega_{j_1, \cdots, j_k}}{\partial x^\ell} \, dx^\ell \right) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k},
\end{aligned}$$

where the final expression on the right-hand side is obtained by changing j_0 to ℓ and imposing the monotone condition $j_1 < \cdots < j_k$. The monotone condition $j_0 < \cdots < j_k$ removes $k!$ from the denominator $(k+1)!$. The remaining factor $k+1$ in $(k+1)!$ is removed because of the removal of the summation $\sum_{\nu=0}^{k}$. Thus, we arrive at the formula

$$d\omega = \sum_{1 \le j_1 < \cdots < j_k \le n} (d\omega_{j_1, \cdots, j_k}) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

By using this formula, we get

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

when the degree of ω is k. The sign $(-1)^k$ of the second term on the right-hand side comes from

$$
\begin{aligned}
&\left(\omega_{j_1,\cdots,j_k}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) \left(\frac{\partial \eta_{i_1,\cdots,i_\ell}}{\partial x^{j_0}}\, dx^{j_0} \wedge dx^{i_1} \wedge \cdots dx^{i_\ell}\right) \\
&= (-1)^k \omega_{j_1,\cdots,j_k} \frac{\partial \eta_{i_1,\cdots,i_\ell}}{\partial x^{j_0}}\, dx^{j_0} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \wedge dx^{i_1} \wedge \cdots dx^{i_\ell}
\end{aligned}
$$

in the transposition of dx^{j_0} and $dx^{j_1} \wedge \cdots \wedge dx^{j_k}$.

§5. Composite of Two Exterior Differentiations Results in Zero Map

For any differential k-form

$$\omega = \frac{1}{k!} \sum_{j_1,\cdots,j_k=1}^{n} \omega_{j_1,\cdots,j_k}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k},$$

we have $d(d\omega) = 0$, because of the commutativity of partial differentiations and the skew-symmetric property of exterior product in

$$
\begin{aligned}
d(d\omega) &= d\left(\sum_{p=1}^{n} \frac{1}{k!} \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial \omega_{j_1,\cdots,j_k}}{\partial x^p}\, dx^p \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) \\
&= \sum_{q=1}^{n} \sum_{p=1}^{n} \frac{1}{k!} \sum_{j_1,\cdots,j_k=1}^{n} \frac{\partial^2 \omega_{j_1,\cdots,j_k}}{\partial x^q \partial x^p}\, dx^q \wedge dx^p \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k},
\end{aligned}
$$

which is 0 when the two dummy indices p and q are interchanged to yield

$$\frac{\partial^2 \omega_{j_1,\cdots,j_k}}{\partial x^q \partial x^p} = \frac{\partial^2 \omega_{j_1,\cdots,j_k}}{\partial x^p \partial x^q}$$

and

$$dx^q \wedge dx^p = -\, dx^p \wedge dx^q.$$

Thus, $d \circ d = 0$.

§6. Differential 1-Forms $dx^1, \cdots, dx^n$ as Dual Basis of $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$

For an open neighborhood U of P in $\mathbb{R}^n$ (with coordinates $x^1, \cdots, x^n$) and an open neighborhood W of Q in $\mathbb{R}^n$ (with coordinates $y^1, \cdots, y^n$) and for a smooth map $\varphi : U \to W$ with $\varphi(P) = Q$, the tangent map $d\varphi$ from the tangent space $T_{U,P}$ to the tangent space $T_{W,Q}$ is given by the Jacobian matrix

$$\left(\frac{\partial y^j}{\partial x^k}\right)_{1 \leq j,k \leq n}$$

(evaluated at P).

Now fix $1 \leq \nu \leq n$ and consider the special case where $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by the coordinate function x^ν. Let t be the coordinate for the target space $\mathbb{R}$. Then dx^ν (as $d\varphi$) maps any tangent vector $\vec{v}$ of $\mathbb{R}^n$ at P to a vector $a\frac{d}{dt}$. After we identify $a\frac{d}{dt}$ with the real number a, we can regard dx^ν (which maps $\vec{v}$ to a) as an $\mathbb{R}$-linear map from the vector space $\mathbb{R}^n$ to $\mathbb{R}$. Using the Jacobian matrix for the map $\varphi = x^\nu$, we conclude that dx^ν maps the vector $\frac{\partial}{\partial x^\lambda}$ to the Kronecker delta $\delta_{\nu,\lambda}$. In this interpretation the differential 1-forms $dx^1, \cdots, dx^n$ is the dual basis of $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^1}$.

§7. Coordinate Change for Differential Form and Value of Differential Form at Multi-Vector

When we have another coordinate system $(y^1, \cdots, y^n)$ of $\mathbb{R}^n$, the differential k-form

$$\omega = \sum_{1 \leq j_1 < \cdots < j_k \leq n} f_{j_1, \cdots, j_k}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k},$$

given in the coordinate system $(x^1, \cdots, x^n)$ of $\mathbb{R}^n$, becomes

$$\omega = \sum_{1 \leq \ell_1 < \cdots < \ell_k \leq n} g_{\ell_1, \cdots, \ell_k}\, dy^{\ell_1} \wedge \cdots \wedge dy^{\ell_k}.$$

From

$$\begin{aligned}
&\sum_{1 \leq \ell_1 < \cdots < \ell_k \leq n} g_{\ell_1, \cdots, \ell_k}\, dy^{\ell_1} \wedge \cdots \wedge dy^{\ell_k} \\
&= \sum_{1 \leq \ell_1 < \cdots < \ell_k \leq n} g_{\ell_1, \cdots, \ell_k} \left(\sum_{\nu_1=1}^n \frac{\partial y^{\ell_1}}{\partial x^{\nu_1}}\, dx^{\nu_1}\right) \wedge \cdots\cdots \wedge \left(\sum_{\nu_k=1}^n \frac{\partial y^{\ell_k}}{\partial x^{\nu_k}}\, dx^{\nu_k}\right) \\
&= \sum_{1 \leq \ell_1 < \cdots < \ell_k \leq n} g_{\ell_1, \cdots, \ell_k} \sum_{1 \leq \nu_1 < \cdots < \nu_k \leq n} \frac{\partial\left(y^{\ell_1}, \cdots, y^{\ell_k}\right)}{\partial\left(x^{\nu_1}, \cdots, x^{\nu_k}\right)}\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_k}
\end{aligned}$$

and

$$\omega = \sum_{1 \leq \nu_1 < \cdots < \nu_k \leq n} f_{\nu_1, \cdots, \nu_k} \, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_k}$$

we obtain the transformation rule

$$f_{\nu_1, \cdots, \nu_k} = \sum_{1 \leq \ell_1 < \cdots < \ell_k \leq n} g_{\ell_1, \cdots, \ell_k} \frac{\partial \left(y^{\ell_1}, \cdots, y^{\ell_k}\right)}{\partial \left(x^{\nu_1}, \cdots, x^{\nu_k}\right)}$$

for $1 \leq \nu_1 < \cdots < \nu_k \leq n$.

In the interpretation of a differential k-form as an integrand for integration over any local submanifold Y of dimension k in $\mathbb{R}^n$ as described earlier, we use a k-dimensional rectangle $T_{\nu_1, \cdots, \nu_k}$ (in the local coordinate system $(y^1, \cdots, y^n)$) defined by y^j being constant for $j \neq \nu_1, \cdots, \nu_k$ to get

$$g_{\nu_1, \cdots, \nu_k}(P) = \lim_{T_{\nu_1, \cdots, \nu_k} \to P} \frac{\int_{T_{\nu_1, \cdots, \nu_k}} \omega}{\int_{T_{\nu_1, \cdots, \nu_k}} dy^{\nu_1} \wedge \cdots \wedge dy^{\nu_k}}$$

as $T_{\nu_1, \cdots, \nu_k}$ shrinks down to a point P in $\mathbb{R}^n$. This value $g_{\nu_1, \cdots, \nu_k}(P)$ depends only on the tangent space of the k-dimensional rectangle $T_{\nu_1, \cdots, \nu_k}$ at P when $T_{\nu_1, \cdots, \nu_k}$ is regarded as local a submanifold of dimension k in $\mathbb{R}^n$ with coordinates $(x^1, \cdots, x^n)$. This tangent space is spanned by the k vector fields

$$\frac{\partial}{\partial y^{\nu_1}}, \cdots, \frac{\partial}{\partial y^{\nu_k}}$$

at P. We use the notation

$$\frac{\partial}{\partial y^{\nu_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{\nu_k}}$$

to denote this tangent space with the orientation given by the order $\nu_1, \cdots, \nu_k$. We call

$$\frac{\partial}{\partial y^{\nu_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{\nu_k}}$$

a k-vector or a multi-vector (when its dependence on k is suppressed). This k-vector is skew-symmetric in the indices $\nu_1, \cdots, \nu_k$ because of the orientation given to the tangent space. The value $g_{\nu_1, \cdots, \nu_k}(P)$ can be considered as the value of ω at the k-vector

$$\frac{\partial}{\partial y^{\nu_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{\nu_k}}$$

at P and we use the notation

$$\omega\left(\frac{\partial}{\partial y^{\nu_1}}\wedge\cdots\wedge\frac{\partial}{\partial y^{\nu_k}}\right)$$

or

$$\left\langle\omega,\,\frac{\partial}{\partial y^{\nu_1}}\wedge\cdots\wedge\frac{\partial}{\partial y^{\nu_k}}\right\rangle$$

or

$$\left\langle\frac{\partial}{\partial y^{\nu_1}}\wedge\cdots\wedge\frac{\partial}{\partial y^{\nu_k}}\,\omega\right\rangle$$

(to denote $g_{\nu_1,\cdots,\nu_k}$). In particular, from this notation we have

$$\left\langle dx^{j_1}\wedge\cdots\wedge dx^{j_k},\,\frac{\partial}{\partial y^{\nu_1}}\wedge\cdots\wedge\frac{\partial}{\partial y^{\nu_k}}\right\rangle=\frac{\partial\left(x^{j_1},\cdots,x^{j_k}\right)}{\partial\left(y^{\nu_1},\cdots,y^{\nu_k}\right)}.$$

Remark. Usually we avoid using the notation

$$\omega\left(\frac{\partial}{\partial y^{\nu_1}},\cdots,\frac{\partial}{\partial y^{\nu_k}}\right),$$

because, for example, when $\omega=dx^{j_1}\wedge\cdots\wedge dx^{j_k}$, this might give the impression that

$$\left(dx^{j_1}\wedge\cdots\wedge dx^{j_k}\right)\left(\frac{\partial}{\partial y^{\nu_1}},\cdots,\frac{\partial}{\partial y^{\nu_k}}\right)$$

means the skew-symmetrization

$$\frac{1}{k!}\sum_\sigma\mathrm{sgn}(\sigma)\left\langle dx^{j_1},\,\frac{\partial}{\partial y^{\sigma\nu_1}}\right\rangle\cdots\left\langle dx^{j_1},\,\frac{\partial}{\partial y^{\sigma\nu_k}}\right\rangle=\frac{1}{k!}\sum_\sigma\mathrm{sgn}(\sigma)\frac{\partial x^{j_1}}{\partial y^{\sigma(\nu_1)}}\cdots\frac{\partial x^{j_k}}{\partial y^{\sigma(\nu_k)}}$$

of

$$\left(dx^{j_1}\otimes\cdots\otimes dx^{j_k}\right)\left(\frac{\partial}{\partial y^{\nu_1}},\cdots,\frac{\partial}{\partial y^{\nu_k}}\right)$$

(where σ runs over the set of all $k!$ permutations of $\{\nu_1,\cdots,\nu_k\}$ with signature $\mathrm{sgn}(\sigma)$) instead of the correct expression

$$\frac{\partial\left(x^{j_1},\cdots,x^{j_k}\right)}{\partial\left(y^{\nu_1},\cdots,y^{\nu_k}\right)}=\sum_\sigma\mathrm{sgn}(\sigma)\frac{\partial x^{j_1}}{\partial y^{\sigma(\nu_1)}}\cdots\frac{\partial x^{j_k}}{\partial y^{\sigma(\nu_k)}},$$

which differs by a factor of $k!$. If one insists on using the notation

$$\omega\left(\frac{\partial}{\partial y^{\nu_1}}, \cdots, \frac{\partial}{\partial y^{\nu_k}}\right)$$

despite the possibility of a confusion about a factor of $k!$, one can define

$$dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

as $k!$ times the skew-symmetrization of

$$dx^{j_1} \otimes \cdots \otimes dx^{j_k}$$

(instead of just the usual skew-symmetrization without the additional factor).

This problem of the factor $k!$ occurs already in the exterior product of an abstract vector space V of real dimension n with basis vectors $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$. Let V^* with be the dual vector space of V with dual basis $\boldsymbol{f}_1, \cdots, \boldsymbol{f}_n$. For $1 \leq k \leq n$ the exterior product $\bigwedge^k V$ of k copies of V is inside the tensor product $V^{\otimes k}$ of k copies of V. If we define

$$\boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_k}$$

(with $1 \leq j_1 < \cdots < j_k \leq n$) as the skew-symmetrization

$$\frac{1}{k!}\sum_{\sigma} \operatorname{sgn}(\sigma)\boldsymbol{e}_{\sigma(j_1)} \otimes \cdots \otimes \boldsymbol{e}_{\sigma(j_k)}$$

of

$$\boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_k}$$

as σ runs through all the $k!$ permutations of $\{j_1, j_2, \cdots, j_k\}$ with signature $\operatorname{sgn}(\sigma)$, then we have to define

$$\boldsymbol{f}_{j_1} \wedge \cdots \wedge \boldsymbol{f}_{j_k}$$

as $k!$ times the skew-symmetrization of

$$\boldsymbol{f}_{j_1} \otimes \cdots \otimes \boldsymbol{f}_{j_k}$$

to make the basis

$$\{\boldsymbol{f}_{j_1} \wedge \cdots \wedge \boldsymbol{f}_{j_k}\}_{1 \leq j_1 < \cdots < j_k \leq n}$$

of $\bigwedge^k V^*$ dual to the basis

$$\{\boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_k}\}_{1 \leq j_1 < \cdots < j_k \leq n}$$

of $\bigwedge^k V$.

§8. Relation between Lie Bracket of Vector Fields and Exterior Derivative of Differential Forms

First, we would like to explore what the relationship between the Lie brackets of vector fields and the exterior derivatives of differential forms would look like by first looking at the special case of differential 1-forms. Let $\omega = f dg$ and vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Since $d\omega = df \wedge dg$, after we form

$$\langle d\omega,\, \boldsymbol{\xi} \wedge \boldsymbol{\eta}\rangle = \langle df \wedge dg,\, \boldsymbol{\xi} \wedge \boldsymbol{\eta}\rangle\,,$$

we would like to express the right-hand side in terms of ω and the Lie bracket $[\boldsymbol{\xi}, \boldsymbol{\eta}]$ by getting rid of the differentiation d in df to change to the Lie bracket through the use of identities like $\langle df, \boldsymbol{\xi}\rangle = \boldsymbol{\xi}(f)$. In carrying out the string of identities below, we have in mind such a goal. Then

$$\begin{aligned}
\langle d\omega,\, \boldsymbol{\xi} \wedge \boldsymbol{\eta}\rangle = \langle df \wedge dg,\, \boldsymbol{\xi} \wedge \boldsymbol{\eta}\rangle &= \begin{vmatrix} \langle df, \boldsymbol{\xi}\rangle & \langle df, \boldsymbol{\eta}\rangle \\ \langle dg, \boldsymbol{\xi}\rangle & \langle dg, \boldsymbol{\eta}\rangle \end{vmatrix} \\
&= \langle df, \boldsymbol{\xi}\rangle \langle dg, \boldsymbol{\eta}\rangle - \langle df, \boldsymbol{\eta}\rangle \langle dg, \boldsymbol{\xi}\rangle \\
&= \boldsymbol{\xi}(f)\boldsymbol{\eta}(g) - \boldsymbol{\eta}(f)\boldsymbol{\xi}(g) \\
&= \boldsymbol{\xi}\left(f\boldsymbol{\eta}(g)\right) - f\boldsymbol{\xi}\boldsymbol{\eta}(g) - \boldsymbol{\eta}\left(f\boldsymbol{\xi}(g)\right) + f\boldsymbol{\eta}\boldsymbol{\xi}(g) \\
&\quad \text{(add and subtract a term to prepare to get to the Lie bracket of } \boldsymbol{\xi} \text{and } \boldsymbol{\eta}) \\
&= \boldsymbol{\xi}\langle \omega, \boldsymbol{\eta}\rangle - f\boldsymbol{\xi}\boldsymbol{\eta}(g) - \boldsymbol{\eta}\langle \omega, \boldsymbol{\xi}\rangle + f\boldsymbol{\eta}\boldsymbol{\xi}(g) \\
&\qquad\qquad \text{(replace } f \text{ and } g \text{ in two terms by } \omega) \\
&= \boldsymbol{\xi}\langle \omega, \boldsymbol{\eta}\rangle - \boldsymbol{\eta}\langle \omega, \boldsymbol{\xi}\rangle + f\left[\boldsymbol{\eta}, \boldsymbol{\xi}\right](g) \\
&= \boldsymbol{\xi}\langle \omega, \boldsymbol{\eta}\rangle - \boldsymbol{\eta}\langle \omega, \boldsymbol{\xi}\rangle - \langle \omega,\, [\boldsymbol{\xi}, \boldsymbol{\eta}]\rangle \\
&\qquad\qquad \text{(replace } f \text{ and } g \text{ in the remaining term by } \omega).
\end{aligned}$$

Since a general differential 1-form is an $\mathbb{R}$-linear combination of $f dg$, we get the formula

$$\langle d\omega,\, \boldsymbol{\xi} \wedge \boldsymbol{\eta}\rangle = \boldsymbol{\xi}\langle \omega, \boldsymbol{\eta}\rangle - \boldsymbol{\eta}\langle \omega, \boldsymbol{\xi}\rangle - \langle \omega,\, [\boldsymbol{\xi}, \boldsymbol{\eta}]\rangle$$

for a general differential 1-form and vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

We continue our exploration by carrying out the same kind of computation for for a 2-form ω. It suffices to look at the special case of $\omega = f\,dg\wedge dh$ and three vector fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$. Then

$$\begin{aligned}
\langle d\omega,\, \boldsymbol{\xi}\wedge\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle &= \langle df\wedge dg\wedge dh,\, \boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle = \begin{vmatrix} \langle df,\boldsymbol{\xi}\rangle & \langle df,\boldsymbol{\eta}\rangle & \langle df,\boldsymbol{\zeta}\rangle \\ \langle dg,\boldsymbol{\xi}\rangle & \langle dg,\boldsymbol{\eta}\rangle & \langle dg,\boldsymbol{\zeta}\rangle \\ \langle dh,\boldsymbol{\xi}\rangle & \langle dh,\boldsymbol{\eta}\rangle & \langle dh,\boldsymbol{\zeta}\rangle \end{vmatrix} \\
&= \langle df,\boldsymbol{\xi}\rangle \begin{vmatrix} \langle dg,\boldsymbol{\eta}\rangle & \langle dg,\boldsymbol{\zeta}\rangle \\ \langle dh,\boldsymbol{\eta}\rangle & \langle dh,\boldsymbol{\zeta}\rangle \end{vmatrix} - \langle df,\boldsymbol{\eta}\rangle \begin{vmatrix} \langle dg,\boldsymbol{\xi}\rangle & \langle dg,\boldsymbol{\zeta}\rangle \\ \langle dh,\boldsymbol{\xi}\rangle & \langle dh,\boldsymbol{\zeta}\rangle \end{vmatrix} - \langle df,\boldsymbol{\zeta}\rangle \begin{vmatrix} \langle dg,\boldsymbol{\xi}\rangle & \langle dg,\boldsymbol{\eta}\rangle \\ \langle dh,\boldsymbol{\xi}\rangle & \langle dh,\boldsymbol{\eta}\rangle \end{vmatrix} \\
&= \boldsymbol{\xi}(f)\,\langle dg\wedge dh,\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle - \boldsymbol{\eta}(f)\,\langle dg\wedge dh,\boldsymbol{\xi}\wedge\boldsymbol{\zeta}\rangle + \boldsymbol{\zeta}(f)\,\langle dg\wedge dh,\boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle \\
&= \boldsymbol{\xi}\left(f\,\langle dg\wedge dh,\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle\right) - f\boldsymbol{\xi}\left(\langle dg\wedge dh,\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle\right) \\
&\qquad - \boldsymbol{\eta}\left(f\,\langle dg\wedge dh,\boldsymbol{\xi}\wedge\boldsymbol{\zeta}\rangle\right) + f\boldsymbol{\eta}\,\langle dg\wedge dh,\boldsymbol{\xi}\wedge\boldsymbol{\zeta}\rangle \\
&\qquad + \boldsymbol{\zeta}\left(f\,\langle dg\wedge dh,\boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle\right) - f\boldsymbol{\zeta}\,\langle dg\wedge dh,\boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle \\
&= \boldsymbol{\xi}\,\langle\omega,\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle - f\boldsymbol{\xi}\left(\boldsymbol{\eta}(g)\boldsymbol{\zeta}(h) - \boldsymbol{\eta}(h)\boldsymbol{\zeta}(g)\right) \\
&\qquad - \boldsymbol{\eta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\zeta}\rangle + f\boldsymbol{\eta}\left(\boldsymbol{\xi}(g)\boldsymbol{\zeta}(h) - \boldsymbol{\xi}(h)\boldsymbol{\zeta}(g)\right) \\
&\qquad + \boldsymbol{\zeta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle - f\boldsymbol{\zeta}\left(\boldsymbol{\xi}(g)\boldsymbol{\eta}(h) - \boldsymbol{\xi}(h)\boldsymbol{\eta}(g)\right) \\
&= \boldsymbol{\xi}\,\langle\omega,\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle - f\boldsymbol{\xi}\boldsymbol{\eta}(g)\boldsymbol{\zeta}(h) - f\boldsymbol{\eta}(g)\boldsymbol{\xi}\boldsymbol{\zeta}(h) + f\boldsymbol{\xi}\boldsymbol{\eta}(h)\boldsymbol{\zeta}(g) + f\boldsymbol{\eta}(h)\boldsymbol{\xi}\boldsymbol{\zeta}(g) \\
&\qquad - \boldsymbol{\eta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\zeta}\rangle + f\boldsymbol{\eta}\boldsymbol{\xi}(g)\boldsymbol{\zeta}(h) + f\boldsymbol{\xi}(g)\boldsymbol{\eta}\boldsymbol{\zeta}(h) - f\boldsymbol{\eta}\boldsymbol{\xi}(h)\boldsymbol{\zeta}(g) - f\boldsymbol{\xi}(h)\boldsymbol{\eta}\boldsymbol{\zeta}(g) \\
&\qquad + \boldsymbol{\zeta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle - f\boldsymbol{\zeta}\boldsymbol{\xi}(g)\boldsymbol{\eta}(h) - f\boldsymbol{\xi}(g)\boldsymbol{\zeta}\boldsymbol{\eta}(h) + f\boldsymbol{\zeta}\boldsymbol{\xi}(g)\boldsymbol{\eta}(h) + f\boldsymbol{\xi}(g)\boldsymbol{\zeta}\boldsymbol{\eta}(h) \\
&= \boldsymbol{\xi}\,\langle\omega,\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle - \boldsymbol{\eta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\zeta}\rangle + \boldsymbol{\zeta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle \\
&\qquad - f\boldsymbol{\xi}\boldsymbol{\eta}(g)\boldsymbol{\zeta}(h) + f\boldsymbol{\eta}\boldsymbol{\xi}(g)\boldsymbol{\zeta}(h) + f\boldsymbol{\xi}\boldsymbol{\eta}(h)\boldsymbol{\zeta}(g) - f\boldsymbol{\eta}\boldsymbol{\xi}(h)\boldsymbol{\zeta}(g) \\
&\qquad - f\boldsymbol{\xi}\boldsymbol{\zeta}(g)\boldsymbol{\eta}(h) + f\boldsymbol{\zeta}\boldsymbol{\xi}(g)\boldsymbol{\eta}(h) + f\boldsymbol{\xi}\boldsymbol{\zeta}(h)\boldsymbol{\eta}(g) - f\boldsymbol{\zeta}\boldsymbol{\xi}(h)\boldsymbol{\eta}(g) \\
&\qquad - f\boldsymbol{\eta}\boldsymbol{\zeta}(g)\boldsymbol{\xi}(h) + f\boldsymbol{\eta}\boldsymbol{\zeta}(g)\boldsymbol{\xi}(h) + f\boldsymbol{\eta}\boldsymbol{\zeta}(h)\boldsymbol{\xi}(g) - f\boldsymbol{\eta}\boldsymbol{\zeta}(h)\boldsymbol{\xi}(g) \\
&= \boldsymbol{\xi}\,\langle\omega,\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle - \boldsymbol{\eta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\zeta}\rangle + \boldsymbol{\zeta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle \\
&\qquad - \langle\omega,[\boldsymbol{\xi},\boldsymbol{\eta}]\wedge\boldsymbol{\zeta}\rangle + \langle\omega,[\boldsymbol{\xi},\boldsymbol{\zeta}]\wedge\boldsymbol{\eta}\rangle - \langle\omega,[\boldsymbol{\eta},\boldsymbol{\zeta}]\wedge\boldsymbol{\xi}\rangle\,.
\end{aligned}$$

The final formula for a general differential 2-form and three vector fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ is

$$\begin{aligned}
\langle d\omega,\, \boldsymbol{\xi}\wedge\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle &= \boldsymbol{\xi}\,\langle\omega,\boldsymbol{\eta}\wedge\boldsymbol{\zeta}\rangle - \boldsymbol{\eta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\zeta}\rangle + \boldsymbol{\zeta}\,\langle\omega,\boldsymbol{\xi}\wedge\boldsymbol{\eta}\rangle \\
&\qquad - \langle\omega,[\boldsymbol{\xi},\boldsymbol{\eta}]\wedge\boldsymbol{\zeta}\rangle + \langle\omega,[\boldsymbol{\xi},\boldsymbol{\zeta}]\wedge\boldsymbol{\eta}\rangle - \langle\omega,[\boldsymbol{\eta},\boldsymbol{\zeta}]\wedge\boldsymbol{\xi}\rangle\,.
\end{aligned}$$

The formulas for the cases of general differential 1-forms and differential

2-forms suggest that the following Cartan's formula

$$(d\omega)\left(\boldsymbol{\xi}_1\wedge\cdots\wedge\boldsymbol{\xi}_{k+1}\right)=\sum_{j=1}^{k+1}(-1)^{j+1}\boldsymbol{\xi}_j\left(\omega\left(\boldsymbol{\xi}_1\wedge\cdots\wedge\widehat{\boldsymbol{\xi}_j}\wedge\cdots\wedge\boldsymbol{\xi}_{k+1}\right)\right)$$
$$+\sum_{i\leq j<\ell\leq k+1}(-1)^{j+\ell}\omega\left([\boldsymbol{\xi}_j,\,\boldsymbol{\xi}_\ell]\wedge\cdots\wedge\widehat{\boldsymbol{\xi}_j}\wedge\cdots\wedge\widehat{\boldsymbol{\xi}_k}\wedge\cdots\wedge\boldsymbol{\xi}_{k+1}\right)$$

for a differential k-form ω and vector fields $\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_{k+1}$. Here the notation $\widehat{\boldsymbol{\xi}_j}$ means that $\boldsymbol{\xi}_j$ is removed.

If we would like to follow the same path as for the case of $k=1,2$ to get a proof for the case of a general k, the computation would be too tedious. A more elegant way of proving Cartan's formula is to use the following induction argument. For the initial step, we check Cartan's formula directly for $\omega = f dx^{j_1}\wedge\cdots\wedge dx^{j_k}$ for some $1\leq j_1<\cdots<j_k\leq n$, where f is a function, and for $\boldsymbol{\xi}_\nu=\frac{\partial}{\partial x^{\ell_\nu}}$ for $1\leq\nu\leq k+1$. This implies that Cartan's formula holds for any differential k-form ω and for $\boldsymbol{\xi}_\nu=\frac{\partial}{\partial x^{\ell_\nu}}$ for $1\leq\nu\leq k+1$, because any differential k-form is a sum of special differential k-forms $\omega = f dx^{j_1}\wedge\cdots\wedge dx^{j_k}$. For the induction step, we need only assume that Cartan's formula is true for some ω and $\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_{k+1}$ and then verify it for the case with $\boldsymbol{\xi}_\nu$ replaced by $g\boldsymbol{\xi}_\nu$ for some arbitrary function g and some ν from $\{1,2,\cdots,k+1\}$. We will leave the details of the verification by induction arguments to a homework problem.

When we defined the exterior derivative of a differential k-form, we used local coordinates and did not verify that the definition is independent of the choice of local coordinates, though the description invoking Fubini's theorem did give a geometric definition, in which the transformation rule for different coordinates systems is explicitly given by Jacobian determinants. Cartan's formula offers a coordinate-free definition of the exterior derivative of a differential k-form.

§9. Dual Formulation of Frobenius Integrability Condition

Suppose $\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_m$ define a distribution of tangent subspace of dimension m in $\mathbb{R}^n$. Dually such a distribution of tangent subspaces can be defined by the vanishing of $n-m$ differential 1-forms $\omega_{m+1},\cdots,\omega_n$. The integrability of the distribution $\langle \boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_m \rangle$ of tangent subspaces can be dually formulated as the existence of differential 1-forms $\theta_{j,k}$ (for $m+1 \leq j, k \leq n$) such that

$$d\omega_j = \sum_{k=m+1}^{n} \theta_{j,k} \wedge \omega_k$$

for $m+1 \leq j, k \leq n$. An alternative way of writing this condition is that

$$(\flat) \qquad d\omega_j \equiv 0 \bmod \langle \omega_{m+1}, \cdots, \omega_n \rangle \text{ for } m+1 \leq j \leq n.$$

We now use Cartan's formula to verify that this condition $(\flat)$ on $\omega_{m+1}, \cdots, \omega_n$ is equivalent to the Frobenius integrability condition

$$(\flat\flat) \qquad [\boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\nu] \equiv 0 \bmod \langle \boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_m \rangle \text{ for } 1 \leq \mu < \nu \leq m.$$

First of all, we observe that condition $(\flat)$ remains the same when we replace $\omega_{m+1}, \cdots, \omega_n$ by $n-m$ linear combinations defined by a nonsingular $(n-m) \times (n-m)$ matrix whose entries are functions. Thus, we can assume without loss of generality that we can choose vector fields $\boldsymbol{\xi}_{m+1}, \cdots, \boldsymbol{\xi}_n$ to complete $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_m$ to a basis $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_n$ and also assume that $\omega_{m+1}, \cdots, \omega_n$ is part of the basis $\omega_1, \cdots, \omega_n$ which is dual to the basis $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_n$.

Cartan's formula states that

$$\langle d\omega_j, \boldsymbol{\xi}_\mu \wedge \boldsymbol{\xi}_\nu \rangle = \boldsymbol{\xi}_\mu \langle \omega_j, \boldsymbol{\xi}_\nu \rangle - \boldsymbol{\xi}_\nu \langle \omega_j, \boldsymbol{\xi}_\mu \rangle - \langle \omega_j, [\boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\nu] \rangle .$$

Since

$$\langle \omega_j, \boldsymbol{\xi}_\lambda \rangle \equiv 0$$

for any $m+1 \leq j \leq n$ and any $1 \leq \lambda \leq m$, it follows from Cartan's formula that

$$(\#) \qquad \langle d\omega_j, \boldsymbol{\xi}_\mu \wedge \boldsymbol{\xi}_\nu \rangle \equiv 0 \text{ for } m+1 \leq j \leq n \text{ and } 1 \leq \mu < \nu \leq k$$

if and only if

$$(\#\#) \qquad \langle \omega_j, [\boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\mu] \rangle \equiv 0 \text{ for } m+1 \leq j \leq n \text{ and } 1 \leq \mu < \nu \leq k.$$

Since the basis $\omega_1, \cdots, \omega_n$ is dual to the basis $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_n$, the condition $(\flat)$ is clearly equivalent to the condition $(\#)$ and the condition $(\flat\flat)$ is clearly equivalent to the condition $(\#\#)$.

APPENDIX: Stokes's Theorem

Stokes's theorem is a generalization of the following part of the fundamental theorem of calculus

$$\int_{x\in(a,b)} f'(x)dx = f(b) - f(a)$$

with the right-hand side interpreted as the evaluation of the function f (as a differential 0-form) on the boundary of (a,b) and with the left-hand side interpreted as the evaluation of the exterior derivative df of the differential 0-form f.

Fubini's theorem on double and iterated integration, together with the fundamental theorem of calculus applied to each variable separately, yields

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial x}\right) dx \wedge dy = \int_{\partial R} (P(x,y)dx + Q(x,y))$$

for a bounded open rectangle R in $\mathbb{R}^2$ with boundary ∂R, where $P(x,y)$ and $Q(x,y)$ are C^1 functions on an open neighborhood of the topological closure of R. The statement is known as Green's theorem, which is a special case of Stokes's theorem (and also a special case of the divergence theorem).

For a bounded open n-dimensional rectangle Ω in $\mathbb{R}^n$ and a differential $(n-1)$-form

$$\omega = \sum_{j=1}^n f_j(x^1, \cdots, x^n) dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n,$$

Fubini's theorem on n-fold and iterated integration, together with the fundamental theorem of calculus applied separately to each of the variables $x^1, \cdots, x^n$ yields

$$\begin{aligned} &\int_R \left(\sum_{j=1}^n (-1)^{j-1} \frac{\partial f_j}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n \\ &= \int_{\partial R} \left(\sum_{j=1}^n f_j(x^1, \cdots, x^n) dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n\right), \end{aligned}$$

which is a special case of Stokes's theorem and also a special case of the divergence theorem. In terms of the differential $(n-1)$-form ω the equation is simply

$$\int_R d\omega = \int_{\partial R} \omega. \tag{\$}$$

Suppose there is a smooth local submanifold X in $\mathbb{R}^{n+m}$ which is diffeomrophic to an open neighborhood U of the topological closure of a bounded open n-dimensional rectangle R in $\mathbb{R}^n$ under the natural projection $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ given by $(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{n+m}) \mapsto (x^1, \cdots, x^n)$. Let ω be a smooth differential $(n-1)$-form defined on an open neighborhood of the topological closure of X in $\mathbb{R}^{n+m}$. Let $\Omega = X \cap \pi^{-1}(R)$. Let $\partial\Omega$ be the boundary of Ω in X. Then

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega.$$

This is a consequence of applying, to the preceding result ($), the reparametrizing U by X by the diffeomorphism π.

By using a C^∞ partition of unity, we can formulate the result in the following general setting. Let X be smooth a local manifold of dimension n in $\mathbb{R}^N$ and Ω be a relatively compact domain in X. Let ω be a smooth differential $(n-1)$-form on some open neighborhood G of the topological closure $\bar{\Omega}$ of Ω in X. Suppose $\bar{\Omega}$ is covered by a finite number of relatively compact open subsets U_j (for $1 \leq j \leq J$) of X with the following property. There exist some open bounded rectangle R_j in $\mathbb{R}^n$ and some diffeomorphism φ_j from U_j onto an open neighborhood of the topological closure $\bar{R}_j$ in $\mathbb{R}^n$ such that φ_j maps $U_j \cap \Omega$ onto R_j (for $1 \leq j \leq J$). The statement of Stokes's theorem

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

holds in this general setting, because we can use the preceding case to show that

$$\int_{U_j \cap \Omega} d\left(\rho_j \omega\right) = \int_{U_j \cap \partial\Omega} \rho_j \omega$$

and sum over $1 \leq j \leq J$ after we introduce a C^∞ partition of unity $\{\rho_j\}_{1 \leq j \leq J}$ of C^∞ functions with $0 \leq \rho_j \leq 1$ (subordinate to the open cover $\{U_j\}_{1 \leq j \leq J}$

of the compact set $\bar{\Omega}$) such that the support of ρ_j is a compact subset of U_j and the sum $\sum_{j=1}^J \rho_j$ is identically 1 on some open neighborhood of $\bar{\Omega}$ in $\bigcup_{j=1}^J U_j$.

Recall that the C^∞ partition of unity is constructed as follows. Define $f(t)$ on $\mathbb{R}$ with variable t by setting $f(t) = e^{-\frac{1}{x}}$ for $x > 0$ and $f(t) = 0$ for $x \leq 0$. Then $f(t)$ is C^∞ on $\mathbb{R}$ with value 0 for $x \leq 0$ and with value in $(0, 1)$ for $x > 0$. For $0 < a < b$ in $\mathbb{R}$, define $g(x)$ on $\mathbb{R}^n$ by

$$g(x) = \frac{f(b - |x|)}{f(b - |x|) + f(|x| - a)}.$$

Then the support of $g(x)$ is contained in the closed disk $\bar{\mathbb{D}}_b = \{|x| \leq b\}$ of radius b and the value of $g(x)$ is identically 1 on the closed disk $\bar{\mathbb{D}}_a = \{|x| \leq a\}$ of radius a. In the above setting, choose a refinement cover $\{W_\lambda\}_{1 \leq \lambda \leq \Lambda}$ of $\bar{\Omega}$ with the following property. Each W_λ is contained in U_{j_λ} for some $1 \leq j_\lambda \leq J$ and there is a diffeomorphism ψ_λ between W_λ and the open disk $\mathbb{D}_c$ of radius c for some $c > b$ such that $\bigcup_{\lambda=1}^\Lambda \psi_\lambda^{-1}(\mathbb{D}_a)$ contains $\bar{\Omega}$ and each $\psi_\lambda^{-1}(\bar{\mathbb{D}}_b)$ is contained in G. The C^∞ partition of unity $\{\rho_j\}_{1 \leq j \leq J}$ (subordinate to the open cover $\{U_j\}_{1 \leq j \leq J}$ of the compact set $\bar{\Omega}$) is given by the formula

$$\rho_\ell = \sum_{1 \leq \lambda \leq \Lambda,\, j_\lambda = \ell} g \circ \psi_\lambda$$

for $1 \leq \ell \leq J$.