Connection and Curvature of Vector Bundle and Chern Classes

§1. Differentiation of Sections of Vector Bundle to Get Sections

We start out with a vector bundle V of rank r over a smooth manifold X, which is defined by transition functions $G_{j,k}$ with respect to a covering $\{U_j\}$ of X. A section σ over an open subset U is given by $\{\sigma_j\}$ so that σ_j is an r-tuple of functions over $U \cap U_j$ and $\sigma_j = G_{j,k}\sigma_k$. Here no summation is used and σ_j, σ_k are column vectors and $G_{j,k}$ is an $r \times r$ matrix. For our differentiation we do not specify the direction and consider the total derivative. From $\sigma_j = G_{j,k}\sigma_k$ we have

$$d\sigma_j = G_{j,k}d\sigma_k + (dG_{j,k})\sigma_k.$$

If the term $(dG_{j,k})\sigma_k$ is not present, we get a section of V from $d\sigma_j$ on U_j after we specify a direction. However, in general we have the term $(dG_{j,k})\sigma_k$ and it is not possible to use $d\sigma_j$ on U_j for differentiation and get a section. We have to use other ways to differentiate sections of a bundle.

§2. Definition of Connection of Vector Bundle

We want the procedure of differentiation to obey the Leibniz formula for differentiating products. So to define differentiation it suffices to define differentiation for a local basis e_α $(1 \leq \alpha \leq r)$ of V. We denote the (total) differential operator by D. Then De_α is a V-valued 1-form and we can express it in terms of our local basis e_α $(1 \leq \alpha \leq r)$ and get $De_\alpha = \omega_\alpha^\beta e_\beta$, where ω_α^β is a 1-form. To be able to differentiate sections is equivalent to having an $r \times r$ matrix valued 1-form (ω_α^β).

This matrix valued 1-form (ω_α^β) depends on the choice of e_α. Suppose we have another local basis $\tilde{e}_\alpha$ which is related to e_α by $\tilde{e}_\alpha = g_\alpha^\beta e_\beta$. Then

$$D\tilde{e}_\alpha = D(g_\alpha^\gamma e_\gamma) = g_\alpha^\gamma De_\gamma + (dg_\alpha^\gamma)e_\gamma = g_\alpha^\gamma \omega_\gamma^\beta e_\beta + (dg_\alpha^\beta)e_\beta.$$

In matrix notations we have $D\tilde{e} = g\,\omega\,e + dg\,e$, where $\tilde{e}$, e are column vectors with components $\tilde{e}_\alpha$, e_α and $g = (g_\alpha^\beta)$ and $\omega = (\omega_\alpha^\beta)$. Thus

$$D\tilde{e} = \left(g\,\omega\,g^{-1} + dg\,g^{-1}\right)\tilde{e}$$

and

$$\tilde{\omega} = g\,\omega\,g^{-1} + dg\,g^{-1}.$$

So to be able to differentiate sections is equivalent to having an $r \times r$ matrix valued 1-form ω for any local basis e so that the transformation rule

$$\tilde{\omega} = g\,\omega\,g^{-1} + dg\,g^{-1}$$

is satisfied. In that case we say that we have a connection.

We can express the connection $\omega = \left(\omega_\alpha^\beta\right)_{1 \leq \alpha, \beta \leq 2}$ in terms of local coordinates by

$$\omega_\alpha^\beta = \sum_{i=1}^r \Gamma_{\alpha i}^\beta dx^i$$

Each of the coefficient $\Gamma_{\alpha i}^\beta$ is called a Christoffel symbol.

When we have a connection for V, we have an induced connection on the dual bundle V^* of V. It is given as follows. Suppose s^* is a local section of V^* and s is a local section of V. Then Ds^* is defined so that $d\langle s^*, s\rangle = \langle Ds^*, s\rangle + \langle s^*, Ds\rangle$ is satisfied, where $\langle s^*, s\rangle$ means the evaluation of s^* at s. If e_*^α is the dual basis of e_α, then the connection $\omega_\alpha^{*\beta}$ is simply equal to $-\omega_\alpha^\beta$.

Similarly, once we have a connection for V, we also know how to define the covariant derivative of sections of

$$\underbrace{(V \otimes V \otimes \cdots \otimes V)}_{p \text{ copies}} \wedge \underbrace{(V^* \otimes V^* \otimes \cdots \otimes V^*)}_{q \text{ copies}}.$$

by evaluation at sections of V and V^* and by using the Leibniz product rule.

Normal Fiber Coordinates for Vector Bundle at a Point. For computational convenience, it is useful to be able to do the computation in a good coordinate system or local frame only at one single point where the computation is carried out. When a local frame $e_1, \cdots, e_r$ of the vector bundle V is chosen, it means that a point v of V can be expressed as $v = \sum_{\alpha=1}^r w^\alpha e_\alpha$ so that $w^1, \cdots, w^r$ are coordinates along the fibers or fiber coordinates. Fix a point P of X. We want to show that there exists a frame $\tilde{e}_1, \cdots, \tilde{e}_r$ such that the given connection expressed in terms of $\tilde{e}_1, \cdots, \tilde{e}_r$ becomes 0 at the point P. In other words,

$$\tilde{\omega} = g\,\omega\,g^{-1} + dg\,g^{-1}$$

is zero at P, without any conclusion at points other than P. To construct $\tilde{e}_1, \cdots, \tilde{e}_r$, we simply set

$$(\tilde{\omega})_P = g(P)\,\omega_P\,g(P)^{-1} + dg_P\,g(P)^{-1} = 0,$$

or

$$dg_P = -g(P)\omega_P,$$

which simply means specifying the first-order derivative of g at P while keeping the given value $g(P)$ of g at P. Note that we can only specify the first-order derivative of g a one point and cannot do this at every point in a neighborhood of P, because solving such a partial differential equation requires the fulfilment of compatibility conditions. The particular frame $\tilde{e}_1, \cdots, \tilde{e}_r$ is called normal fiber coordinates at the point P for the vector bundle V.

§3 Curvature

Unlike the case of partial differentiation for functions, in general for sections of a vector bundle partial differentiations for different directions do not commute. The failure of the commutativity is measured by the curvature of the connection. The commutativity of partial differentiation of a function in two different directions is equivalent to the vanishing of the composite of two exterior differentiation applied to functions. We do this for a real manifold M of real dimension m and a smooth vector bundle V of rank r over M.

Take a local smooth basis e_α $(1 \leq \alpha \leq r)$ of V. We apply D twice to e_α and get a local section $D(D\,e_\alpha)$ of $V \otimes T^*_M \otimes T^*_M$, where T^*_M is the dual of the tangent bundle T_M of M. By skew-symmetrizing $D(D\,e_\alpha)$ in the two arguments for the tangent vectors of M, we get a local V-valued 2-form $D_\wedge D\,e_\alpha$. It is clumsy to use the notation $D_\wedge D\,e_\alpha$. We are going to simply use the notation $DD\,e_\alpha$, similar to the case of the composite $d\,d$ of two exterior differential operators. We can express $DD\,e_\alpha$ in terms of the connection ω and its exterior derivative $d\omega$ as follows. We use the column vector e with components e_α.

$$\begin{aligned} DD\,e &= D(\omega\,e) = d\omega\,e - \omega \wedge De \\ &= d\omega\,e - \omega\wedge\omega\,e = (d\omega - \omega\wedge\omega)e. \end{aligned}$$

We define the curvature to be $d\omega - \omega\wedge\omega$ and denote it by Ω. It is an $\mathrm{End}(V)$-valued 2-form on M and its pointwise value is well-defined, because if f is a local smooth matrix-valued function, then

$$\begin{aligned} &DD(fe) = D(df\, e + f\, De) \\ &= ddf\, e - df\wedge De + df\wedge De + fDDe \\ &= f\, DDe. \end{aligned}$$

To get the value of Ω at a point P, we take a matrix E and can a smooth matrix-valued function f whose value at P is E so that we can define $DD(fe)$ and take its value at P, which does not depend on the choice of f as we can see by taking the difference of the results obtained by another extension matrix-valued function g and ending up with 0.

Another way to check that Ω is a tensor is to directly get the transformation rule for it explicitly as follows.

$$\begin{aligned} \tilde\Omega &= d\tilde\omega - \tilde\omega \wedge \tilde\omega \\ &= d\left(g\omega g^{-1} + dgg^{-1}\right) - \left(g\omega g^{-1} + dgg^{-1}\right) \wedge \left(g\omega g^{-1} + dgg^{-1}\right) \\ &= \left(dg \wedge \omega g^{-1} + gd\omega g^{-1} + g\omega \wedge g^{-1}dgg^{-1} - dgg^{-1}dgg^{-1}\right) \\ &\qquad - \left(g\omega g^{-1} \wedge g\omega g^{-1} + dgg^{-1} \wedge g\omega g^{-1} + g\omega g^{-1} \wedge dgg^{-1} + dgg^{-1} \wedge dgg^{-1}\right) \\ &= \left(dg \wedge \omega g^{-1} + gd\omega g^{-1} + g\omega \wedge g^{-1}dgg^{-1} - dgg^{-1}dgg^{-1}\right) \\ &\qquad - \left(g\omega \wedge \omega g^{-1} + dg \wedge \omega g^{-1} + g\omega g^{-1} \wedge dgg^{-1} + dgg^{-1} \wedge dgg^{-1}\right) \\ &= g\Omega g^{-1}. \end{aligned}$$

Let us look at the curvature tensor Ω in local coordinates. With respect to a local frame e_α $(1 \le \alpha \le r)$ we have $\Omega_\alpha^\beta = \Omega_{\alpha ij}^\beta dx^i\wedge dx^j$. Using the Christoffel symbol $\Gamma_{\alpha i}^\beta$ of the connection ω_α^β, we have

$$\Omega_{\alpha ij}^\beta dx^i\wedge dx^j = d(\Gamma_{\alpha i}^\beta dx^i) - (\Gamma_{\alpha i}^\gamma dx^i)\wedge(\Gamma_{\gamma j}^\beta dx^j)$$

and

$$\Omega_{\alpha ij}^\beta = \partial_i\Gamma_{\alpha j}^\beta - \partial_j\Gamma_{\alpha i}^\beta + \Gamma_{\alpha i}^\gamma\Gamma_{\gamma j}^\beta - \Gamma_{\alpha j}^\gamma\Gamma_{\gamma i}^\beta.$$

§4. The (Second) Bianchi Identity

The curvature tensor Ω satisfies a Bianchi identity $D\,\Omega = 0$. In local coordinates the Bianchi identiy $D\,\Omega = 0$ says that

$$\nabla_i \Omega^\beta_{\alpha jk} + \nabla_j \Omega^\beta_{\alpha ki} + \nabla_k \Omega^\beta_{\alpha ij} = 0.$$

Here we use the notation ∇_i instead of D, because it is a covariant derivative with respect to a given vector field and not the total covariant derivative. The result of applying the total covariant operator D is a differential form (with the skew-symmetric property).

The proof of the Bianchi identity is as follows

$$\begin{aligned}
&D\,\Omega = d\Omega + \Omega\wedge\omega - \omega\wedge\Omega \\
&= d(d\omega - \omega\wedge\omega) + (d\omega - \omega\wedge\omega)\wedge\omega - \omega\wedge(d\omega - \omega\wedge\omega) \\
&= -d\omega\wedge\omega + \omega\wedge d\omega + d\omega\wedge\omega - \omega\wedge\omega\wedge\omega - \omega\wedge d\omega + \omega\wedge\omega\wedge\omega = 0.
\end{aligned}$$

§5. Frame-Free (Invariant) Definition of Curvature

We can also define the curvature in invariant formulation. The curvature is defined to measure the failure of the commutativity of partial covariant differentiation. So for tangent vector fields X, Y and a smooth local section s of V, we should consider $\nabla_X \nabla_Y s - \nabla_Y \nabla_X s$. However, when X and Y are arbitrary tangent vector fields instead of tangent vector fields defined by coordinate functions, we do not expect to have commutativity of differentiations along X and Y even in the case of functions, because for a smooth function f in general $X(Yf) - Y(Xf)$ is nonzero and is equal to the derivative $[X,Y]f$ of f along the direction of the Lie bracket $[X,Y]$ of X and Y. So we have to make accommodation for that and consider $R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$.

We would like to check that $R(X,Y)s$ is pointwise well-defined. We need to check that for smooth function φ, ψ, θ we have

$$R(\varphi X, \psi Y)(\theta s) = \varphi\psi\theta\, R(X,Y)s.$$

The verification is straightforwardly done by the following computation. From the four equations

$$\nabla_{\psi Y}(\theta s) = \psi(Y\theta)s + \psi\theta \nabla_Y s$$

and

$$\begin{aligned}\nabla_{\varphi X}(\nabla_{\psi Y}(\theta s)) &= \varphi\nabla_X(\psi(Y\theta)s+\psi\theta\nabla_Y s)\\ &= \varphi(X\psi)(Y\theta)s+\varphi\psi(XY\theta)s+\varphi\psi(Y\theta)\nabla_X s\\ &\quad+\varphi(X\psi)\theta\nabla_Y s+\varphi\psi(X\theta)\nabla_Y s+\varphi\psi\theta\nabla_X\nabla_Y s\end{aligned}$$

and

$$[\varphi X,\psi Y]=\varphi(X\psi)Y-\psi(Y\varphi)X+\varphi\psi[X,Y]$$

and

$$\begin{aligned}\nabla_{[\varphi X,\psi Y]}(\theta s) &= \varphi(X\psi)(Y\theta)s-\psi(Y\varphi)(X\theta)s+\varphi\psi([X,Y]\theta)s\\ &\quad+\varphi(X\psi)\theta\nabla_Y s-\psi(Y\varphi)\theta\nabla_X s+\varphi\psi\theta\nabla_{[X,Y]}s,\end{aligned}$$

we get

$$\nabla_{\varphi X}(\nabla_{\psi Y}(\theta s))-\nabla_{\psi Y}(\nabla_{\varphi X}(\theta s))-\nabla_{[\varphi X,\psi Y]}(\theta s)=\varphi\psi\theta(\nabla_X\nabla_Y s-\nabla_Y\nabla_X s-\nabla_{[X,Y]}s).$$

So the value of $R(X,Y)s$ at a point P depends only on the values of X, Y, s at the point P. For $X=\frac{\partial}{\partial x^i}$ and $Y=\frac{\partial}{\partial x^j}$ and $s=e_\alpha$ it follows from definitions that

$$R(X,Y)e_\alpha=\sum_{\beta=1}^r\Omega_\alpha^\beta e_\beta.$$

So the curvature can be defined in an invariant way by

$$R(X,Y)s=\nabla_X\nabla_Y s-\nabla_Y\nabla_X s-\nabla_{[X,Y]}s.$$

As an operator

$$R(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$$

and

$$\langle\Omega,\,X\wedge Y\rangle=R(X,Y).$$

§6. Chern Forms of Connection of Vector Bundle

The curvature Ω of the connection ω of the vector bundle V is an $\mathrm{End}(V)$-valued differential 2-form on X. For $1 \leq k \leq n$, we are going to construct a differential $(2k)$-form, called the k-th Chern form, on X. The construction is motivated by the elementary symmetric functions of the eigenvalues $\lambda_1, \cdots, \lambda_r$ of an $r \times r$ matrix

$$A = (a_{\alpha,\beta})_{1 \leq \alpha,\beta \leq r}$$

whose entries are constants. The eigenvalues $\lambda_1, \cdots, \lambda_r$ of A are the roots of the characteristic equation

$$\det(A - \lambda I_r) = 0,$$

where λ is the unknown in the equation and I_r is the identity matrix of order r. The k-th elementary symmetric function σ_k of the eigenvalues are given by $(-1)^{r-k}$ times the coefficient of λ^k in the characteristic equation. It means that σ_k is the sum of all the principal minors of A of order k. Recall that a principal minor of a square matrix means the determinant of a submatrix formed by deleting the same set of rows and columns from the original matrix.

We apply the computation to Ω instead of A to define the k-th Chern form γ_k as the sum of all the principal minors of order k of the $r \times r$ matrix Ω. (Note that this definition differs from the correct definition of Chern forms by a factor $\left(\frac{\sqrt{-1}}{2\pi}\right)^k$, which is introduced to make sure that certain integrals of Chern forms which are expected to have integer values actually do have integer values. At this point, we drop this factor to work with the Chern forms we just defined.)

In the case of Ω instead of a matrix A whose entries are constants, we have to worry about the commutativity of entries of Ω. Fortunately, each entry of the $r \times r$ matrix Ω is a 2-form and in our computation of the principal minors from the entries. For any change of local frames of V given by an $r \times r$ matrix g, we have

$$\det(g\Omega g^{-1} - \lambda I_r) = \det(g(\Omega - \lambda I_r)g^{-1}) = \det(\Omega - \lambda I_r),$$

from which we conclude that the k-th Chern form γ_k is independent of the choice of the local frame.

We claim that the k-th Chern form γ_k is d-closed on X. We need only check this at any given point P. Once P is given, we can choose a local frame, which is a normal fiber coordinates system for V, such that the connection with respect to this new local frame is 0 at P. With respect to this new local frame, at P the covariant differentiation $D\Omega$ of Ω agrees with the usual exterior differentiation $d\Omega$ of Ω. By (the second) Bianchi identity, $d\Omega_\alpha^\beta$ at P for $1 \leq \alpha, \beta \leq r$ with respect to this new local frame. This implies that $d\gamma_k = 0$ at P from the computation of γ_k as the sum of all the principal minors of order k of the $r \times r$ matrix Ω. Since γ_k is independent of the choice of the local frame, the claim that $d\gamma_k$ is proved.

§7. Chern Classes of Vector Bundle

The k-th Chern form γ_k still depends on the choice of the connection ω of the vector bundle V. We are going to prove that the k-th Chern form γ_k is independent of the choice of the connection ω modulo an exact $(2k)$-form on X.

Suppose a different connection ω' is chosen for the vector bundle V. Let γ'_k be the Chern form from the connection ω' of the vector bundle V. We define a family of connections $\omega_t = t\omega + (1-t)\omega'$ for $t \in \mathbb{R}$. The connection ω_t is well-defined, because when there is a change of frames given by $\tilde{e} = ge$, the transformation rule

$$\tilde{\omega}_t = g\,\omega_t\,g^{-1} + dg\,g^{-1}$$

for ω_t is verified by adding the two transformation rules

$$\tilde{\omega} = g\,\omega\,g^{-1} + dg\,g^{-1}$$

and

$$\tilde{\omega}' = g\,\omega'\,g^{-1} + dg\,g^{-1}$$

for the connections ω and ω' respectively.

Let $\pi : \mathbb{R} \times X \to X$ be the natural projection onto the second factor. Denote by π^*V the pullback of the vector bundle V to $\mathbb{R} \times X$ by the projection π. It means that the transition functions for π^*V with respect to the open cover $\{\mathbb{R} \times U_j\}$ of $\mathbb{R} \times X$ are the pullbacks of the transition functions $G_{j,\ell}$

for V with respect to the open cover $\{U_j\}$ of X. The family of connections $\{\omega_t\}_{t\in\mathbb{R}}$ of V defines a connection ω^* of π^*V such that the restriction of the $r\times r$ matrix ω^* of differential 1-forms to $\{t\}\times X$ is the same as the $r\times r$ matrix ω_t of differential 1-forms on X when $\{t\}\times X$ is naturally identified with X. Let γ_k^* be the k-th Chern form on $\mathbb{R}\times X$ from the connection ω^* of the vector bundle π^*V on $\mathbb{R}\times X$. Then the pullback of γ_k^* to $\{1\}\times X$ agrees with γ_k and the pullback of γ_k^* to $\{0\}\times X$ agrees with γ_k'. We are going to use a homotopy argument for closed differential forms to prove that $\gamma_k-\gamma_k'=d\sigma$ for some differential $(2k-1)$-form σ on X.

We write $\gamma_k^*=\varphi+dt\wedge\psi$ such that φ and ψ are respectively a differential $(2k)$-form and a differential $(2k-1)$-form on $\mathbb{R}\times X$ without any dt (whose coefficients, of course, still depend on t). From the d-closedness of γ_k^* on $\mathbb{R}\times C$ it follows that

$$(\&*)\qquad 0=d\omega_k=d_X\varphi+dt\wedge\frac{\partial\varphi}{\partial t}-dt\wedge d_X\psi,$$

where d_X means the exterior derivative computed only with respect to the coordinates of X and $\frac{\partial\varphi}{\partial t}$ means the result of applying partial differentiation, with respect to t, to each coefficient of the differential $(2k)$-form φ. Equating the coefficients of dt on both sides of $(\&*)$, we conclude that

$$\frac{\partial\varphi}{\partial t}=d_X\psi$$

and integrating with respect to t from $t=0$ to $t=1$ yields

$$\varphi\Big|_{t=1}-\varphi\Big|_{t=0}=d_X\left(\int_{t=0}^1\psi\,dt\right),$$

which means that $\gamma_k-\gamma_k'=d\sigma$ on X when we set

$$\sigma=\int_{t=0}^1\psi\,dt.$$

This finishes the proof that the k-th Chern form γ_k is independent of the choice of the connection ω modulo an exact $(2k)$-form on X. The equivalent class defined by the k-th Chern form γ_k modulo exact $(2k-1)$-forms is well-defined and is independent of the choice of the connection of V. This equivalence class which depends only on the vector bundle is called the k-Chern class of the vector bundle V.

APPENDIX: Poincaré Lemma

Statement of the Poincaré Lemma. Let $1 \leq q \leq n$. If α is a C^∞ differential q-form on some open neighborhood U of 0 in $\mathbb{R}^n$ such that $d\alpha = 0$, then there exist a C^∞ differential $(q-1)$-form β on some open neighborhood U' of 0 in U such that $d\beta = \alpha$ on U'.

Argument by the Television Method. Suppose $q \geq 1$ and α is a smooth q-form on some open neighborhood U of 0 in $\mathbb{R}^n$ with coordinates $x^1, \cdots, x^n$ so that $d\alpha = 0$. We are going to use induction on n to find a smooth $(q-1)$-form β on some open neighborhood U' of 0 in U such that $d\beta = \alpha$.

To start the induction process, we write

$$\alpha = \sigma + dx^n \wedge \tau,$$

where σ and τ are respectively a q-form and a $(q-1)$-form not containing dx^n, but the coefficients of σ and τ depend on $x^1, \cdots, x^n$.

By applying the fundamental theorem of calculus to the coefficients of the $(q-1)$-form τ which does not contain dx^n, we obtain a $(q-1)$-form γ not containing dx^n such that $\frac{\partial}{\partial x^n}\gamma = \tau$, where $\frac{\partial}{\partial x^n}\gamma$ means applying $\frac{\partial}{\partial x^n}$ to every coefficient of γ. Denote by $d'\gamma$ the result of applying the exterior differentiation only in the variables $x^1, \cdots, x^{n-1}$ with the variable x^n being regarded as a constant. Then

$$d\gamma = d'\gamma + dx^n \wedge \frac{\partial}{\partial x^n}\gamma$$

and the q-form

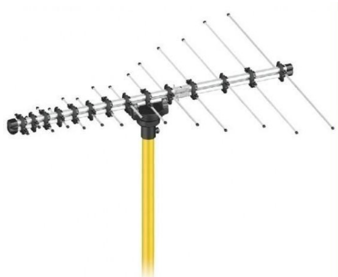
$$\alpha - d\gamma = \sigma + d'\gamma$$

does not contain dx^n, because $\frac{\partial}{\partial x^n}\gamma = \tau$.

Since $d(\alpha - d\gamma)$ vanishes and since $\alpha - d\gamma$ does not contain dx^n, it follows that the coefficients of $\alpha - d\gamma$ are all independent of x^n so that we can now regard $\alpha - d\gamma$ as a q-form on some open neighborhood of the origin of $\mathbb{R}^{n-1}$.

By induction hypothesis we can find a smooth $(q-1)$-form θ on some open neighborhood of the origin in $\mathbb{R}^{n-1}$ such that $d\theta = \alpha - d\gamma$. We can now regard θ as a $(q-1)$-form on some open neighborhood of the origin in $\mathbb{R}^n$ which is independent of x^n and set $\beta = \theta + \gamma$. Then $d\beta = \alpha$ on some open neighborhood of the origin in $\mathbb{R}^n$. This concludes the proof of the Poincaré lemma by the television antenna method.

The above method of proof is sometimes jokingly referred to as the "television antenna method", because the path of integrating along one coordinate and then another and then another reminds one of an outdoor television antenna. The alternative proof given below using the homotopy of contracting to a point is sometimes jokingly referred to as the "satellite dish method".




Argument by the Satellite Dish Method. We start out with a star-like open neighborhood Ω of the origin 0 in $\mathbb{R}^n$ so that the map $\Phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^b$ defined by $(x,t) \mapsto tx$ for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ maps $\Omega \times [0,1]$ to Ω. Let Ω' be a relatively compact open neighborhood of 0 in Ω so that for some $\varepsilon > 0$, the map Φ maps $\Omega' \times (-\varepsilon, 1+\varepsilon)$ to Ω.

Let α be a closed smooth q-form on Ω for some $q \geq 1$. We want to show that α is the exterior differential of some smooth $(q-1)$-form β on some open neighborhood of 0 in Ω. Write $\Phi^*\alpha = \sigma + dt \wedge \tau$, where σ is a q-form on G in which dt does not occur and τ is a $(q-1)$-form on on G in which dt does not occur. Let d_G be the exterior differentiation with respect to only the coordinates of G and with t regarded as constant.

From $d\left(\Phi^*\alpha\right) = 0$ it follows that

$$d_G\sigma + dt\wedge\frac{\partial}{\partial t}\sigma - dt\wedge\frac{\partial}{\partial t}d_G\tau = 0,$$

where $\frac{\partial}{\partial t}\tau$ means differentiating the coefficients of τ respect to the variable t. Hence $\frac{\partial}{\partial t}\sigma = d_G\tau$ when we equate terms in which dt occurs. Integrating $\frac{\partial}{\partial t}\sigma = d_G\tau$ with respect t from $t = 0$ to $t = 1$ yields

$$\sigma\Big|_{t=1} - \sigma\Big|_{t=0} = d_G\left(\int_{t=0}^{1} \tau\, dt\right).$$

Since $\sigma\Big|_{t=1} = \alpha$ and $\sigma\Big|_{t=0} = 0$, it follows that α is equal to $d_G\left(\int_{t=0}^{1} \tau\, dt\right)$. This finishes the proof of the Poincaré lemma by the satellite dish method.

Earlier we have already seen the use of the homotopy argument in the satellite dish method when we proved that the k-Chern form γ_k of a connection of a vector bundle changes only by an exact $(2k-1)$-form with the choice of another connection.