TAKE-HOME FINAL EXAMINATION OF MATH 213a
(due December 11, 2024)
Total Number of Problems = 12

Please submit the PDF file of your final exam paper to the CANVAS website for Math 213a

Instructions. YOU ARE ALLOWED TO CONSULT THE TEXTBOOK *COMPLEX ANALYSIS* BY STEIN-SHAKARCHI, THE MATERIAL POSTED ON THE COURSE WEBSITE, AND YOUR OWN SOLUTIONS OF THE PROBLEMS IN THE HOMEWORK ASSIGNMENTS. YOU ARE NOT ALLOWED TO CONSULT OTHER BOOKS, LOOK UP ANY MATERIAL ON THE INTERNET, OR CONSULT OTHER PEOPLE.

N.B. SOLUTIONS WITH RIGOROUS DETAILS ARE EXPECTED. FOR PROBLEMS WHICH CAN BE SOLVED BY TECHNIQUES IN SOME HOMEWORK PROBLEMS, COMPLETE SELF-CONTAINED SOLUTIONS ARE REQUIRED AND HOMEWORK PROBLEMS CANNOT BE QUOTED SIMPLY AS KNOWN FACTS IN THE SOLUTIONS.

Problem 1 *(Computation of Integrals by Using Branches of Holomorphic Functions).* Let $0 < a < 1$. Evaluate

$$\int_{x=0}^{\infty} \frac{x^a dx}{(x+1)^4}$$

and give your answer in the form of

$$\frac{\pi P(a)}{\sin \pi a}$$

for some polynomial $P(a)$ with rational coefficients.

Hint: Consider a contour integral around the boundary of the domain which is a large disk minus a slightly thickened positive real axis.

Problem 2 *(Integrals Computed in Terms of Gamma and Beta Functions).* Let $b > a > 1$. Prove

$$\int_{\theta=0}^{\frac{\pi}{2}} \cos^a \theta \cos(b\theta)\, d\theta = \frac{\pi\Gamma(a+1)}{2^{a+1}\Gamma\left(\frac{a}{2}+\frac{b}{2}+1\right)\Gamma\left(\frac{a}{2}-\frac{b}{2}+1\right)}.$$

Hint: Consider the integral of $\left(z+\frac{1}{z}\right)^a z^{b-1}dz$ over the right half of the unit circle and use the symmetric version of the integral formula for the beta function.

Problem 3 (*Evaluation of Infinite Sum*). Use the integral of

$$\frac{\sec \pi z}{z^3}\, dz$$

over the rectangle with vertices at $N(\pm 1 \pm i)$, as $N \to \infty$, to compute

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)^3} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

and express the result in the form $a\pi^3$ for some explicit rational number a.

Problem 4 (*Infinite Product and Partial Fraction Expansions*). For complex numbers a, b (with $a - b$ not in $2\pi i\mathbb{Z}$), prove

$$e^{az} - e^{bz} = (a-b)ze^{\frac{1}{2}(a+b)z}\prod_{n=1}^{\infty}\left(1+\frac{(a-b)^2z^2}{4n^2\pi^2}\right)$$

by first deriving the partial fraction expansion of the logarithmic derivative of $e^{az} - e^{bz}$ by using the integral with respect to a modified Cauchy kernel on a sequence of contours expanding to infinity.

Problem 5 *(Periods of Jacobian Elliptic Functions as Solutions of the Hypergeometric Differential Equation of Gauss).* The elliptic integrals K and K' as functions of $0 < k < 1$ are defined by

$$K(k) = \int_{x=0}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad K'(k) = \int_{x=1}^{\frac{1}{k}} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}.$$

Let $t = k^2$. prove that $u = K$ and $u = K'$ as functions of t satisfy the hypergeometric differential equation of Gauss

$$t(1-t)\frac{d^2u}{dt^2} + (c-(a+b+1)t)\frac{du}{dt} - ab\, u = 0$$

in the special case of $a = b = \frac{1}{2}$ and $c = 1$.

Hint: Compare with Problem 2 of Homework Assignment #6 (assigned on October 15, 2024).

Problem 6 (*Weierstrass $\wp$ Function and Weierstrass σ Function*). Let $\wp$ be the Weierstrass $\wp$-function for a lattice L and σ be the Weierstrass σ function for L.

(a) Show that for any positive integer n the function

$$\frac{\sigma(nz)}{\sigma(z)^{n^2}}$$

is elliptic in the sense that it is a meromorphic function on all of $\mathbb{C}$ and L is precisely the set of all its periods.

(b) Show that

$$\frac{\sigma(2z)}{\sigma(z)^4}$$

is equal to $-\wp'(z)$.

Problem 7 (*Formula to Determine the Order of a Finite-Order Entire Function in Terms of Coefficients of its Power Series Expansion*). Let $f(z)$ be an entire function on $\mathbb{C}$ whose power series expansion is

$$\sum_{n=0}^{\infty} a_n z^n.$$

The growth order ρ_0 of f is defined as the infimum of all $\rho > 0$ such that

$$|f(z)| \leq A_\rho e^{|z|^\rho}$$

on $\mathbb{C}$ for some positive number A_ρ. Let

$$\mu = \liminf_{n\to\infty} \frac{\log \frac{1}{|a_n|}}{n \log n}.$$

Assume that μ is a positive number. Prove that $\rho_0 = \frac{1}{\mu}$.

Hint: To show that $\rho_0 \leq \frac{1}{\mu}$, for any $\varepsilon > 0$ choose n_ε such that

$$\log \frac{1}{|a_n|} > (\mu - \varepsilon) n \log n$$

for $n \geq n_\varepsilon$. For z with $|z| = r$, decompose

$$f(z) = \sum_{n \leq n_\varepsilon} a_n z^n + \sum_{n_\varepsilon < n \leq (2r)^{\frac{1}{\mu-\varepsilon}}} a_n z^n + \sum_{(2r)^{\frac{1}{\mu-\varepsilon}} < n} a_n z^n$$

into three sums. The second sum is no more than a constant times

$$\exp\left((2r)^{\frac{1}{\mu-\varepsilon}} \log r\right)$$

with the constant independent of r. The third sum is bounded by a constant independent of r.

To show that $\frac{1}{\mu} \leq \rho_0$, apply Cauchy's integral formula for the n-th derivative of a holomorphic function to get

$$|a_n| r^n \leq \sup_{|z|=r} |f(z)|$$

for any positive integer n. For any $\varepsilon > 0$ there exists a subsequence n_ν such that

$$\log \frac{1}{|a_{n_\nu}|} < (\mu + \varepsilon) n_\nu \log n_\nu.$$

Let $r_\nu = (2n_\nu)^{\mu+\varepsilon}$. Use the inequality

$$|a_{n_\nu}| r_\nu^{n_\nu} > 2^{(\mu+\varepsilon)n_\nu} = \exp\left(\frac{1}{2}(\mu+\varepsilon)(\log 2) r_\nu^{\frac{1}{\mu+\varepsilon}}\right).$$

Problem 8 (*Zero-Set of a Polynomial Perturbation of the Exponential Function*). Let P be a polynomial which is not identically zero. Prove that there is an infinite number of zeroes for the equation $e^z = P(z)$.

Hint: Use Hadamard's factorization theorem.

Problem 9 (*Three-Lines Lemma Derived from Maximum Modulus Principle – from Problem #3 on p.133 of Stein's Complex Analysis Book*). In this problem, we investigate the behavior of certain bounded holomorphic functions in an infinite strip. The particular result described here is sometimes called the "three-lines lemma".

(a) Suppose $F(z)$ is holomorphic and bounded in the strip $0 < \operatorname{Im}(z) < 1$ and continuous on its closure. If $|F(z)| \leq 1$ on the boundary lines, then $|F(z)| \leq 1$ through out the strip.

Hint: Apply the maximum modulus principle to $F_\varepsilon(z) = F(z)e^{-\varepsilon z^2}$ for $\varepsilon > 0$ and then let $\varepsilon \to 0$.

(b) For the more general F, let $\sup_{x\in\mathbb{R}}|F(x)| = M_0$ and $\sup_{x\in\mathbb{R}}|F(x+i)| = M_1$. Then

$$\sup_{x\in\mathbb{R}}|F(x+iy)| \leq M_0^{1-y}M_1^y \quad \text{if} \quad 0 \leq y \leq 1.$$

Hint: Apply **(a)** to $M_0^{-iz-1}M_1^{iz}F(z)$.

(c) As a consequence, prove that $\log\sup_{x\in\mathbb{R}}|F(z+iy)|$ is a convex function of y when $0 \leq y \leq 1$.

Problem 10 (*Bieberbach Conjecture for Special Case of Real Coefficients – from Problem #8 on p.259 of Stein's Complex Analysis Book*). Let f be an injective holomorphic function in the open unit disk $\mathbb{D}$, with $f(0) = 0$ and $f'(0) = 1$. If we write

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

Bieberbach made the conjecture that $|a_n| \leq n$ for all $n \geq 2$, which was proved by Louis de Branges in 1984. This problem outlines an argument to prove Bieberbach's conjecture under the additional assumption that the coefficients a_n are real.

(a) Let $z = re^{i\theta}$ with $0 < r < 1$, and show that if $v(r,\theta)$ denotes the imaginary part of $f(re^{i\theta})$, then

$$a_n r^n = \frac{2}{\pi}\int_{\theta=0}^{\pi} v(r,\theta)\sin n\theta\, d\theta.$$

Hint: Consider the Fourier coefficients of the function $v(r,\theta) = \operatorname{Im} f(re^{i\theta})$ which is periodic in θ of period 2π and is an odd function of θ.

(b) Show that for $0 \leq \theta \leq \pi$ and $n \in \mathbb{N}$, we have $|\sin n\theta| \leq n\sin\theta$.

(c) Use the fact that $a_n \in \mathbb{R}$ to show that the image $f(\mathbb{D})$ of the open unit disk $\mathbb{D}$ under f is symmetric with respect to the real axis, and use this fact to show that f maps the upper half-disk into either the upper or lower part of $f(\mathbb{D})$.

(d) Show that for r small,

$$v(r,\theta) = r\sin\theta\,(1 + O(r))$$

and use the previous part to conclude that $v(r,\theta)\sin\theta \geq 0$ for all $0 < r < 1$ and $0 \leq \theta \leq \pi$.

(e) Prove that $|a_n r^n| \leq nr$, and let $r \to 1$ to conclude that $|a_n| \leq n$.

(f) Check that the function $f(z) = \frac{z}{(1-z)^2}$ satisfies all the hypothesis and that $|a_n| = n$ for all n.

Problem 11 (*Riemann Mapping Theorem and Bergman Kernel*). Let Ω be a simply connected bounded domain in $\mathbb{C}$. Let $f_\nu(z)$ $(1 \leq \nu < \infty)$ be an orthonormal basis of the Hilbert space of all square integrable holomorphic functions on Ω. Let

$$K(z, \bar{w}) = \sum_{\nu=1}^{\infty} f_\nu(z) \overline{f_\nu(w)}.$$

According to Problem 4 of Homework Assignment #2 (assigned on September 17, 2024), the above series which defines $K(z, \bar{w})$ converges uniformly on compact subsets of $\Omega \times \Omega$ so that $K(z, \bar{w})$ is holomorphic in z and anti-holomorphic in w. The function $K(z, \bar{w})$ is known as the *Bergman kernel function.*

Show that for any point w_0 in Ω the map

$$z \mapsto \left(\frac{\partial}{\partial \bar{w}} K(z, \bar{w}) \right)_{w=w_0} K(z, \bar{w}_0)^{-1}$$

maps Ω biholomorphically onto an open disk in $\mathbb{C} \cup \{\infty\}$.

Hint: First, verify from the definition of the Bergman kernel function $K_{\mathbb{D}}(z, \bar{w})$ for the open unit disk $\mathbb{D}$ that

$$K_{\mathbb{D}}(z, \bar{w}) = \frac{1}{\pi} \frac{1}{(1 - z\bar{w})^2}.$$

Then use the Riemann mapping theorem to get a biholomorphic map φ from $\mathbb{D}$ to Ω and observe how the Bergman kernel function transforms under the biholomorphic map φ.

Terminology and Formulas for Problem 12 Below. For a meromorphic function f on $\mathbb{C}$, the *proximity function* $m(r, f, \infty)$ of f to ∞ is defined by

$$m(r, f, \infty) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

(where $\log^+$ means the maximum of 0 and log) and for a point a of $\mathbb{C}$ the *proximity function* $m(r, f, a)$ of f to a is defined as $m\left(r, \frac{1}{f-a}, \infty\right)$, that is,

$$m(r, f, a) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| d\theta.$$

The *counting function* $N(r, f, \infty)$ of f for ∞ is defined as

$$N(r, f, \infty) = \sum_{|b_\nu| < r} \log \frac{r}{|b_\nu|},$$

where b_ν is the sequence of all the poles of f, with each pole b_ν in the sequence repeated as many times as its multiplicity. The *truncated counting function* $\bar{N}(r, f, \infty)$ of f for ∞ is defined as

$$\bar{N}(r, f, \infty) = \sum_{|b'_\nu| < r} \log \frac{r}{|b'_\nu|},$$

where $\{b'_\nu\}$ is the set of all the poles of f, each pole occurring only once regardless of its multiplicity. For $a \in \mathbb{C}$ the *counting function* $N(r, f, a)$ of f for a is defined as $N\left(r, \frac{1}{f-a}, \infty\right)$ and the *truncated counting function* $\bar{N}(r, f, a)$ of f for a is defined as $\bar{N}\left(r, \frac{1}{f-a}, \infty\right)$.

The Nevanlinna characteristic function $T(r, f)$ of f is defined by

$$T(r, f) = m(r, f, \infty) + N(r, f, \infty).$$

For $a \in \mathbb{C} \cup \{\infty\}$ and a meromorphic function f on $\mathbb{C}$, the *defect* of f at a, denoted by $\delta(f, a)$, is defined as

$$\delta(f, a) = \liminf_{r \to \infty} \frac{m(r, f, a)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, f, a)}{T(r, f)}.$$

The *defect of f at a for the truncated counting function*, denoted by $\bar{\delta}(f, a)$, is defined as

$$\bar{\delta}(f, a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}(r, f, a)}{T(r, f)}.$$

The defect relation for the truncated function is

$$\sum_{a \in \mathbb{P}_1} \bar{\delta}(f, a) \leqslant 2.$$

Problem 12 below provides an example of equality in the above defect relation with $\bar{\delta}(f, z)$ nonzero precisely at three values of z.

Problem 12 (*Nevanlinna Defects for Elliptic Functions Constructed from Schwarz-Christoffel Transformations and Schwarz Reflection Principle*). Let a, b, c be distinct complex numbers not on a straight line. Let Ω be the disk whose boundary contains the three points a, b, and c. Let m, n, p be positive integers such that

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = 1.$$

Consider the holomorphic function

$$z = g(w) = \int_{\zeta=0}^{w} (\zeta - a)^{\frac{1}{m}-1}(\zeta - b)^{\frac{1}{n}-1}(\zeta - c)^{\frac{1}{p}-1} d\zeta$$

defined on Ω, where the integrand is any chosen branch defined on Ω. This is a Schwarz-Christoffel transformation whose domain is a disk instead of the upper half-plane.

(a) Let $A = g(a)$, $B = g(b)$, and $C = g(c)$. Denote by ΔABC the triangle with vertices A, B, C. Verify that the interior angles of the triangle ΔABC at the vertices A, B, C are respectively $\frac{\pi}{m}$, $\frac{\pi}{n}$, and $\frac{\pi}{p}$.

(b) Verify that $z = g(w)$ maps the disk Ω biholomorphically onto the triangle ΔABC.

(c) Prove that the inverse function $w = F(z)$ of $z = g(w)$ can be holomorphically continued to a meromorphic function f on all of $\mathbb{C}$ by repeatedly applying the Schwarz reflection principle. Moreover, show that the meromorphic function f on $\mathbb{C}$ is an elliptic function with two primitive periods.

(d) Verify the following relation between the counting function and the truncated counting function for the three points a, b, and c.

$$\bar{N}(r, f, a) = \frac{1}{m} N(r, f, a), \quad \bar{N}(r, f, b) = \frac{1}{n} N(r, f, b), \quad \text{and} \quad \bar{N}(r, f, c) = \frac{1}{p} N(r, f, c).$$

Hint: Consider $\frac{d}{dw} g(w)$.

(e) Verify that $\delta(f, z) = 0$ for $z \neq a, b, c$ and that

$$\bar{\delta}(f, a) = 1 - \frac{1}{m}, \quad \bar{\delta}(f, b) = 1 - \frac{1}{n}, \quad \text{and} \quad \bar{\delta}(f, c) = 1 - \frac{1}{p}.$$