Schwarz-Christoffel Transformations

We start our discussion of Schwarz-Christoffel transformations by putting the Jacobian elliptic sine function $z = \operatorname{sn} w$ (with modulus k and lattice $\mathbb{Z}4K + \mathbb{Z}2iK'$) in the context of a Schwarz-Christoffel transformation from the upper half-plane $\mathbb{H}$ to the rectangle R with vertices at $\pm K$, $\pm K + iK'$. The inverse of such a Schwarz-Christoffel transformation from R to the upper half-plane $\mathbb{H}$ represents the restriction of $z = \operatorname{sn} w$ to one-quarter of the fundamental parallelogram of $\mathbb{Z}4K + \mathbb{Z}2iK'$. To get the full fundamental parallelogram of the lattice $\mathbb{Z}4K + \mathbb{Z}2iK'$, we have to reflect R with respect to its bottom side $[-K, K]$ to get the rectangle $\tilde{R}$ with vertices $\pm K + \pm iK'$ and then reflect $\tilde{R}$ with respect to its right side $[K - iK', K + iK']$ to get a full fundamental parallelogram with vertices at $3K \pm iK'$, $-K \pm iK'$.

In the following figure we use two Riemann spheres with two cross-cuts $[-\frac{1}{k}, -1]$ and $[1, \frac{1}{k}]$ identified in the criss-cross manner to yield a Riemann surface of genus 1 which is a branched cover over the standard Riemann sphere with four branch-points at $\pm\frac{1}{k}$, ± 1. The upper half-plane in the Schwarz-Christoffel transformation corresponds to the front half of the upper Riemann sphere.

A rectangle which corresponds to the front half of the upper Riemann sphere is labelled as " top front". A rectangle which corresponds to the front half of the lower Riemann sphere is labelled as " bottom front". A rectangle which corresponds to the back half of the upper Riemann sphere is labelled as " top back". A rectangle which corresponds to the back half of the lower Riemann sphere is labelled as " bottom back".

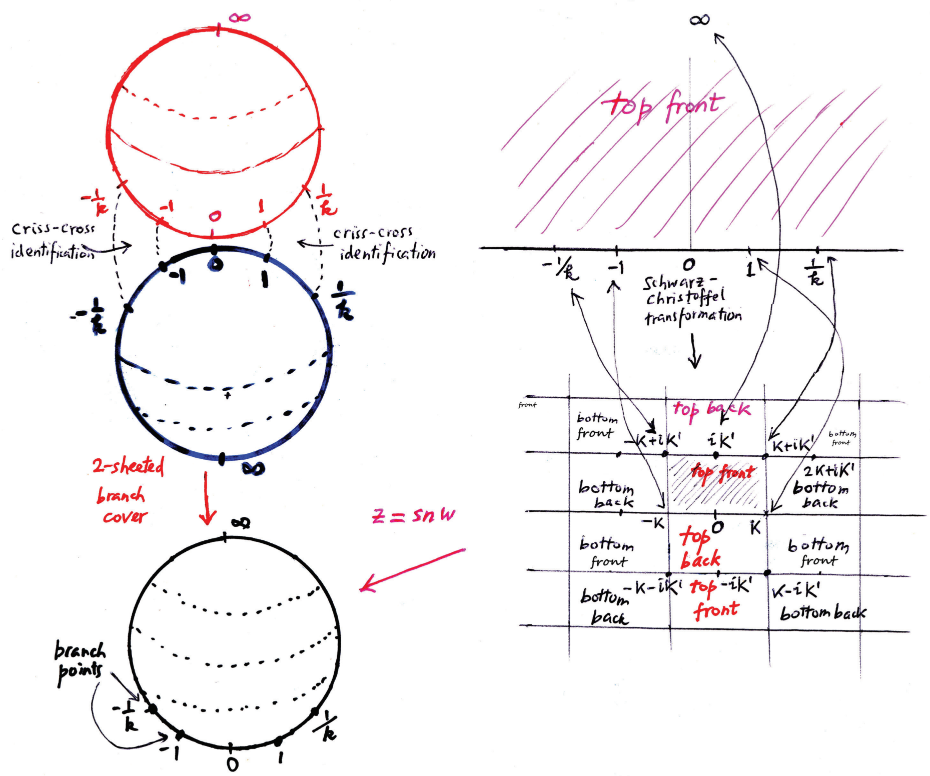

Figure 1: Schwarz-Christoffel Transformation from Upper Half-Plane to Rectangle as Restriction of Inverse of Elliptic Sine Function to One Quarter of Fundamental Parallelogram

We now introduce the notion of a general Schwarz-Christoffel transformation. Schwarz-Christoffel transformations are used to map the upper half-plane $\mathbb{H} = \left\{ x + iy \,\middle|\, y > 0 \right\}$ to a given polygon which may be bounded or unbounded. An unbounded polygon simply means a domain in $\mathbb{C}$ defined by a finite number of line-segments and rays. Let us start out with an n-sided polygon Ω with vertices $w_1, \cdots, w_{n-1}$. We do not put down the n-th vertex w_n to allow the possibility that w_n is ∞, in which case the n-sided polygon Ω is unbounded. We would like to study the problem by constructing a holomorphic map $w = f(z)$ from $\mathbb{H}$ to Ω so that $n-1$ real points $x_1 < x_2 < \cdots < x_{n-1}$ are mapped respectively to the prescribed $n-1$ vertices $w_1, w_2, \cdots, w_{n-1}$ of the n-sided polygon Ω. The important requirement is that the holomorphic map $w = f(z)$ should map the straight line segment $[x_j, x_{j+1}] \subset \mathbb{R}$ to the line-segment in $\mathbb{C}$ joining w_j to w_{j+1}. Let us recall how a holomorphic map transforms the direction of the tangent to a curve in $\mathbb{C}$.

We now recall the mapping behavior of the holomorphic map $w = f(z)$ at a point z_0 where its derivative $f'(z_0)$ is nonzero. Take a smooth curve $t \mapsto z(t)$ passing through the point z_0 so that $z(0) = z_0$. The image of the curve $t \mapsto w(t) := f(z(t))$ passes through the point $w_0 := f(z_0)$. By the chain rule and the Cauchy-Riemann equation we have

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + i\frac{\partial f}{\partial x}\frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) \\
&= \frac{\partial f}{\partial x}\frac{dz}{dt} = f'(z)\frac{dz}{dt}.
\end{aligned}$$

This means that the angle made between the tangent to the curve $t \mapsto z(t)$ at z_0 and the real axis in the z variable is equal to the angle made between the tangent to the curve $t \mapsto w(t)$ at w_0 and the real axis in the w variable plus the angle of the polar representation of $f'(z_0)$. So, if the angle of $f'(z)$ (which we will from now on call the argument of $f'(z)$ and denote by $\arg f'(z)$) stays unchanged as z varies along $[x_j, x_{j+1}]$ from x_j to x_{j+1}, then the line $[x_j, x_{j+1}]$ will be mapped to part of a straight line, because the curve (x_j, x_{j+1}) is mapped to its image curve C in $\mathbb{C}$ so that the tangent direction at every point of C is the same as the tangent direction of (x_j, x_{j+1}) plus the constant value of $\arg f'(z)$ for $z \in (x_j, x_{j+1})$.

When z goes along $\mathbb{R}$ from the left of x_j to the right of x_j, we want the image to turn an angle of magnitude θ_j when it follows the sides of the n-sided polygon going through the vertex w_j in the counterclockwise sense. To achieve this requirement, we would have to require that the constant value of $\arg f'(z)$ at z on $\mathbb{R}$ to the left of x_j should increase by the amount θ_j to get to the constant value of $\arg f'(z)$ at z on $\mathbb{R}$ to the right of x_j.

We now look for a holomorphic function $g(z)$ to serve as $f'(z)$ which would have this property of being constant along $\mathbb{R}$ to the left of x_j and then increase by the amount θ_j to a constant value along $\mathbb{R}$ to the right of x_j. When $\theta_j = -\pi$, one simple function to fit the requirement is $g(z) = z - x_j$ which satisfies the property that $\arg g(z) = \arg(z - x_j) \equiv \pi$ for $z \in \mathbb{R}$ and $z < x_j$ and $\arg g(z) = \arg(z - x_j) \equiv 0$ for $z \in \mathbb{R}$ and $z > x_j$. To assign a numerical value to the angle $\arg g(z) = \arg(z - x_j)$ would require a choice of the range of the values, which is the same as choosing one branch of the function $\log(z - x_j)$ and then taking its imaginary part $\arg(z - x_j)$. When we say that $\arg g(z) = \arg(z - x_j) \equiv \pi$ for $z \in \mathbb{R}$ and $z < x_j$ and $\arg g(z) = \arg(z - x_j) \equiv 0$ for $z \in \mathbb{R}$ and $z > x_j$, we make the choice of choosing the cut $\{x = x_j,\, y \leq 0\}$ to define a branch of $\log(z - x_j)$. That particular cut is chosen to define $\log(z - x_j)$, because we would like the function $\log(z - x_j)$ to be defined on the upper half-plane $\mathbb{H}$.

We now consider the general value $k_j\pi$ for the amount θ_j of jump instead of the special value $\theta_j = -\pi$. This we can do by multiplying $\arg(z - x_j)$ by the constant $-k_j$, which is the same as using $(z - x_j)^{-k_j}$ instead of $z - x_j$. This takes care of one single jump of $\arg g(z)$ by the amount $k_j\pi$ at the point $z = x_j$. How about all the other jumps at all the other real points among $x_1, x_2, \cdots, x_{n-1}$? An easy solution is to add the individual jumps together. When the jump of $\arg g(z)$ is $k_j\pi$ at $z = x_j$, we should use just take

$$\arg g(z) = \sum_{j=1}^{n-1} (-k_j) \arg(z - x_j)\,.$$

In other words, we should just choose

$$g(z) = \prod_{j=1}^{n-1} (z - x_j)^{-k_j}\,.$$

Recall that $g(z)$ is actually a stand-in for $f'(z)$. So we should use

$$f'(z) = \prod_{j=1}^{n-1} (z - x_j)^{-k_j}$$

in order for the image $w = f(z)$ to achieve the correct turning of corners along the n-sided polygon Ω at the vertices $w_1, \cdots, w_{n-1}$ in the counterclockwise sense.

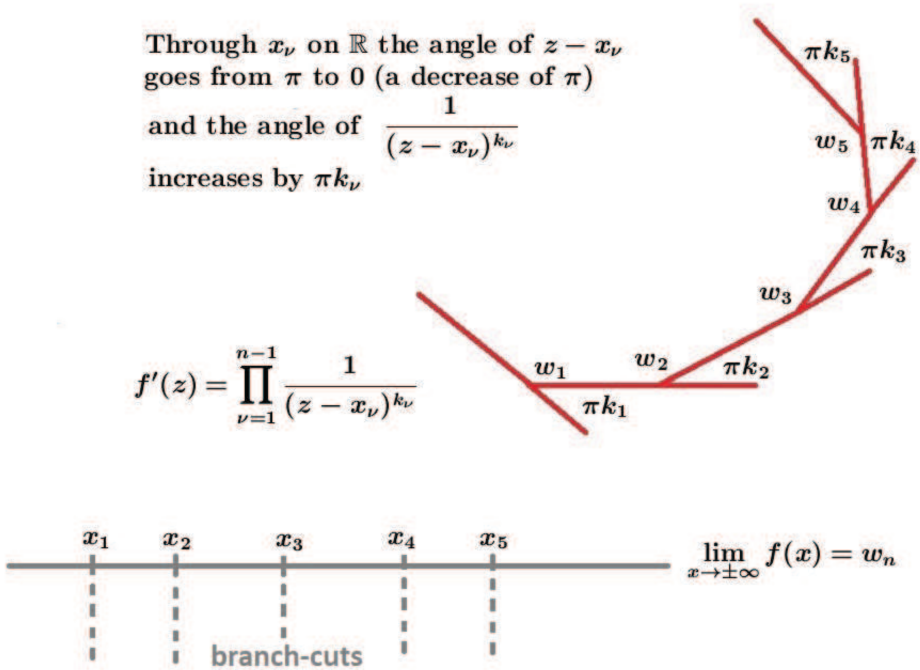

Figure 2: Schwarz-Christoffel Transformation from Upper Half-Plane to Polygon

We should do more than just achieving the correct turning of corners along the n-sided polygon. We should make sure that the ray $(-\infty, x_1]$ is mapped to the side of Ω just before w_1 in the counterclockwise sense. In order to get the initial matching of $(-\infty, x_1]$ with the side of Ω just before w_1, we can multiply $f'(z)$ by a nonzero complex constant A, resulting in the entire image of $f(z)$ being rotated by the argument $\arg A$ of A. So far as matching the orientation is concerned, the absolute value $|A|$ of the complex number A plays no rôle. The absolute value $|A|$ of the complex number A, however, plays the rôle of magnification. Since only the derivative $f'(z)$ is specified, to get back to the holomorphic function $f(z)$ there is the question of the constant of integration. If we use a definite integral with x_1 as the lower limit, we should use w_1 as the constant of integration, because we would like x_1 to be mapped to w_1. Thus we have the following formula for the Schwarz-Christoffel transformation

$$f(z) = A \int_{z_0}^{z} \prod_{j=1}^{n-1} (\zeta - x_j)^{-k_j} \, d\zeta + B,$$

where, for example, z_0 and B can be chosen respectively to be x_1 and w_1.

So far we have been talking about fitting the angles and one initial direction and one initial point. How about fitting all the vertices so that the n vertices $w_1, w_2, \cdots, w_{n-1}, w_n$ (with w_n possibly equal to ∞) are precisely the images of $x_1, x_2, \cdots, x_{n-1}, \infty$ under $w = f(z)$ respectively? Of course, once all the angles fit together with one initial direction and one initial point, the only remaining problem is the lengths of the sides. There are precisely n real numbers (including the possibility of ∞) for the lengths of the sides of the n-sided polygon Ω. On the other hand, we have precisely n real degrees of freedom, namely $|A|$, $x_1, \cdots, x_{n-1}$, to do the job.

One simple way of getting the value for B is to choose $z_0 = x_1$. Then $B = w_1$. Since the line segment $[x_1, x_2]$ is mapped to the side $[w_1, w_2]$, of the n-sided polygon Ω, for $1 \leq \nu \leq n-2$ we have the $n-2$ equations

$$(*)_\nu \qquad A \int_{t=x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} (t - x_j)^{-k_j} \, d\zeta = w_{\nu+1} - w_\nu.$$

For $t \in \mathbb{R}$ the complex number $(t - x_j)^{-k_j}$ is given by

$$(\ddagger) \qquad (t - x_j)^{-k_j} = \begin{cases} |t - x_j|^{-k_j} & \text{for } \ t > x_j \\ \\ |t - x_j|^{-k_j} \, e^{-ik_j\pi} & \text{for } \ t < x_j \end{cases}$$

according to the choice of the branch for the function $(\zeta - x_j)^{-k_j}$ defined for $\operatorname{Im} \zeta \geq 0$. The equation $(*)_\nu$ can now be rewritten as

$$A \left(\prod_{j=\nu+1}^{n-1} e^{-ik_j\pi} \right) \int_{x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} |t - x_j|^{-k_j} \, dt = w_{\nu+1} - w_\nu,$$

because $t < x_j$ for $\nu + 1 \leq j \leq n-1$ when $x_\nu < t < x_{\nu+1}$. This gives right away the value for $\arg A$, for example, from the equation $(*)_1$, namely

$$\arg A = \arg (w_2 - w_1) + \sum_{j=2}^{n-1} k_j \pi.$$

At this point we are left with the n unknowns $|A|, x_1, \cdots, x_{n-1}$. By taking the absolute values of both sides of $(*)_\nu$, we get

$$(\dagger)_\nu \qquad \int_{x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} \left|t - x_j\right|^{-k_j} dt = \frac{|w_{\nu+1} - w_\nu|}{|A|}$$

for $1 \leq \nu \leq n-1$. Since we have n unknowns $|A|, x_1, \cdots, x_{n-1}$ to determine, the $n-2$ equations $(*)_1, \cdots, (*)_{n-2}$ are not enough. We still need two more equations. One piece of information we have not yet used, namely the vertex w_n to worry about. This vertex w_n should be reached by the limit $f(t)$ both

(i) by letting $x_{n-1} < t < \infty$ go to $+\infty$ and

(i) by letting $-\infty < t < x_1$ go to $-\infty$.

So we have the two equations

$$A \int_{t=x_{n-1}}^{+\infty} \prod_{j=1}^{n-1} \left(t - x_j\right)^{-k_j} dt = w_n - w_{n-1},$$

$$A \int_{t=-\infty}^{x_1} \prod_{j=1}^{n-1} \left(t - x_j\right)^{-k_j} dt = w_1 - w_n.$$

Using $(\ddagger)$, we can rewrite these two equations as

$$A \int_{t=x_{n-1}}^{+\infty} \prod_{j=1}^{n-1} \left|t - x_j\right|^{-k_j} dt = w_n - w_{n-1},$$

$$A \left(\prod_{j=1}^{n-1} e^{-ik_j\pi} \right) \int_{t=-\infty}^{x_1} \prod_{j=1}^{n-1} \left(t - x_j\right)^{-k_j} dt = w_1 - w_n.$$

Again we can take the absolute value of both sides of the two equations and get

$$(\dagger)_{n-1} \qquad \int_{t=x_{n-1}}^{+\infty} \prod_{j=1}^{n-1} \left|t - x_j\right|^{-k_j} dt = \frac{|w_n - w_{n-1}|}{|A|},$$

$$(\dagger)_n \qquad \int_{t=-\infty}^{x_1} \prod_{j=1}^{n-1} \left|t - x_j\right|^{-k_j} dt = \frac{|w_1 - w_n|}{|A|}.$$

We now use the n equations $(\dagger)_\nu$ for $1 \leq \nu \leq n$ to solve for the n unknowns $|A|, x_1, \cdots, x_{n-1}$. These n equations are integral equations with the $n-1$ unknowns $x_1, \cdots, x_{n-1}$ both in the integrands as well as in the upper limits of the integrals. In general, it is very difficult to solve these n equations simultaneously. However, the solvability of the equations can be proved by using the Riemann mapping theorem to map the polygon with vertices $w_1, \cdots, w_n$ biholomorphically onto the upper half-plane $\mathbb{H}$ and then using the continuous extension of a biholomorphic map between two domains with piecewise smooth boundaries. The discussion on the Riemann mapping theorem and the continuous extension of a biholomorphic map between two domains with piecewise smooth boundaries, as well as Schwarz reflection principle, will be given at the end of this set of lecture notes on the Schwarz-Christoffel transformation.

Justification of Schwarz-Christoffel Transformation from Riemann Mapping Theorem and Continuous Extension of Biholomorphic Maps to Boundary. First, use the Riemann mapping theorem f to map the open upper half-plane $\mathbb{H}$ biholomorphically onto the n-gon with vertices $w_1, \cdots, w_n$ and then use the continuity of f to the boundary so that we have n points $x_1, \cdots, x_n$ on the real line corresponding to $w_1, \cdots, w_n$. The upper half-plan is considered as a domain in the Riemann sphere so that after applying a biholomorpism of $\mathbb{H}$ we can assume without loss of generality that none of the n vertices $w_1, \cdots, w_n$ of the n-gon corresponds to the boundary point ∞ of $\mathbb{H}$. We claim that the logarithmic derivative $\frac{f''}{f'}$ of the derivative f' of f is a meromorphic function on the Riemann sphere (*i.e.,* a rational function on $\mathbb{C}$) such that its only poles are simple at $x_1, \cdots, x_n$ with residues $-k_1, \cdots, -k_n$ respectively and it vanishes at ∞. The technique to prove this is by applying Schwarz reflection with respect to $\mathbb{R}$ to the composite of f with a straightening fractional power map in the w-plane centered at w_ν (for $1 \leq \nu \leq n$) and applying Schwarz reflection to the restriction of f to the outside of a large circle with respect to $\mathbb{R}$.

Let $q_\nu = 1 - k_\nu$ so that πq_ν is the interior angle at the vertex w_ν. Use the straightening fractional power map

$$g_\nu(w) = (w - w_\nu)^{\frac{1}{q_\nu}}$$

which straightens the two sides $[w_{k-1}, w_\nu]$ and $[w_\nu, w_{k+1}]$ making an angle πq_ν into a straight line L_k segment passing through 0. The branch of $g_\nu(w)$

is chosen to be defined on the polygon. We now compose the straightening fractional power map g_ν with f to define

$$h_\nu(z) = (g_\nu \circ f)(z) = (f(z) - w_\nu)^{\frac{1}{q_\nu}}$$

so that the line segment $[x_{\nu-1}, x_{\nu+1}]$ in the z-plane is mapped to the line segment L_ν in the w-plane. Express $f(z)$ in terms of h_ν to get

$$f(z) = h_\nu(z)^{q_\nu} + w_\nu$$

so that

$$f'(z) = q_\nu h_\nu(z)^{-k_\nu} h'_\nu(z)$$

and its logarithmic derivative is

$$\frac{f''(z)}{f'(z)} = -k_\nu \frac{h'_\nu(z)}{h_\nu(z)} + \frac{h''_\nu(z)}{h'(z)}.$$

The function h_ν obtained by Schwarz reflection is holomorphic on $\mathbb{C}-((-\infty, x_{\nu-1}] \cup [x_{\nu+1}, \infty))$ with nonzero derivative on $[x_{\nu-1}, x_{\nu+1}]$ and with only one zero at x_ν. Hence

$$\frac{f''(z)}{f'(z)} = \frac{-k_\nu}{z - a_\nu} + E_\nu(z)$$

with $E_\nu(z)$ holomorphic on $\mathbb{C} - ((-\infty, x_{\nu-1}] \cup [x_{\nu+1}, \infty))$. This means that

$$\frac{f''(z)}{f'(z)} - \frac{-k_\nu}{z - a_\nu}$$

is holomorphic on $\mathbb{C} - ((-\infty, x_{\nu-1}] \cup [x_{\nu+1}, \infty))$ and

$$\frac{f'(z)}{f''(z)} - \sum_{\nu=1}^{n} \frac{-k_\nu}{z - a_\nu}$$

is holomorphic on $\mathbb{C}$.

Since f maps the line-segments (x_n, ∞) and (∞, x_1) respectively to the line-segments $(w_n, f(\infty))$ and $(f(\infty), w_1)$, for $R > \max(|x_1|, \cdots, |x_n|)$, we can reflect the bounded holomorphic function f on $\{|z| > R, \operatorname{Im} z > 0\}$ with respect to the part $(x_n, \infty) \cup (\infty, x_1)$ of $\mathbb{R}$ so that f can be continued to a deleted neighborhood $\{|z| > R\}$ of ∞ in the Riemann sphere and then by its boundedness on $\{|z| > R\}$ can be extended to be holomorphic in a neighborhood of ∞ in the Riemann sphere.

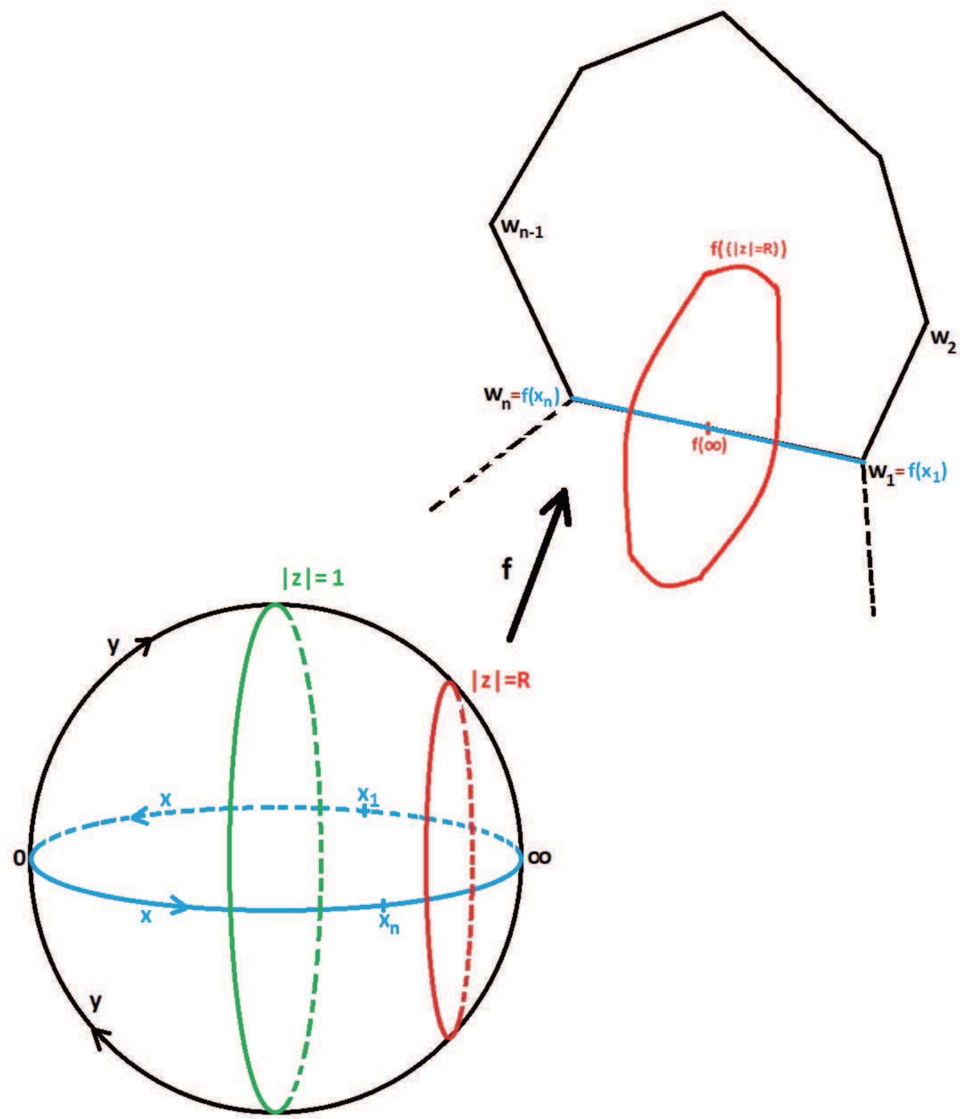

Figure 3: Schwarz Reflection with Respect to Real Line and Sides of Polygon

By using

$$f(z)=c_0+\sum_{\nu=p}^{\infty}\frac{c_\nu}{z^\nu}$$

with $c_p\neq 0$ (actually $p=1$ and $c_0=f(\infty)$) and computing $f'(z)$ and $f''(z)$, we conclude that

$$\frac{f''(z)}{f'(z)}=-(p+1)\left(1+o(1)\right)\frac{1}{z}$$

which approaches 0 as $z\to\infty$. This means that

$$\frac{f'(z)}{f''(z)}=\sum_{\nu=1}^{n}\frac{-k_\nu}{z-a_\nu}$$

on all of $\mathbb{C}\cup\{\infty\}$ and

$$(\log f'(z))'=\left(\log\frac{1}{\prod_{\nu=1}^{n}(z-a_\nu)^{k_\nu}}\right)'.$$

Integrating once and exponentiating, we obtain

$$f'(z)=\frac{A}{\prod_{\nu=1}^{n}(z-a_\nu)^{k_\nu}}$$

for some nonzero constant A. Another integration yields

$$f(z)=A\int\frac{dz}{\prod_{\nu=1}^{n}(z-a_\nu)^{k_\nu}}+B.$$

We now look at the case $a_n=\infty$. When $a_n=\infty$, we choose a biholomorphism $\zeta=b_n-\frac{1}{z}$ of $\mathbb{H}$ to map $a_1,\cdots,a_{n-1},\infty$ in the z-plane to the n points $b_1,\cdots,b_n$ on $\mathbb{R}$ in the ζ-plane. Let $g(\zeta)=f(z)$. Then

$$\frac{dg}{d\zeta}=\frac{\hat{A}}{(\zeta-b_1)^{\alpha_1}\cdots(\zeta-b_n)^{\alpha_n}}$$

so that by using $\alpha_1 + \cdots + \alpha_n = 2$ and $b_\nu = b_n - \frac{1}{a_\nu}$ for $1 \leq \nu \leq n-1$, we get

$$\begin{aligned}\frac{df}{dz} &= \frac{dg}{d\zeta}\frac{d\zeta}{dz} \\ &= \frac{\hat{A}}{\left(b_n - b_1 - \frac{1}{z}\right)^{\alpha_1} \cdots \left(b_n - b_{n-1} - \frac{1}{z}\right)^{\alpha_{n-1}} \left(-\frac{1}{z}\right)^{\alpha_n}} \frac{1}{z^2} \\ &= \frac{\hat{A}}{\left((b_n - b_1)z - 1\right)^{\alpha_1} \cdots \left((b_n - b_{n-1})z - 1\right)^{\alpha_{n-1}} \left(-1\right)^{\alpha_n}} \\ &= \frac{A}{(z-a_1)^{\alpha_1} \cdots (z - a_{n-1})^{\alpha_{n-1}}},\end{aligned}$$

where

$$A = \frac{\hat{A}}{(b_n - b_1)^{\alpha_1} \cdots (b_n - b_{n-1})^{\alpha_{n-1}} (-1)^{\alpha_n}}.$$

Example of a Conformal Mapping from the Upper Half Plane to an Equilateral Triangle. Given an equilateral triangle Ω with $w_1 = 0$ and $w_2 > 0$ and $\operatorname{Im} w_3 > 0$. We seek a conformal map from the upper half-plane $\mathbb{H} = \{\operatorname{Im} z > 0\}$ to Ω with $x_1 = -1$ and $x_2 = 1$. We do not specify $|w_2 - w_1| = |w_2|$ but instead normalize $A = 1$. The conformal map $w = f(z)$ will be given by the Schwarz-Christoffel transformation

$$f(z) = \int_{-1}^{z} \frac{d\zeta}{(\zeta+1)^{\frac{2}{3}} (\zeta - 1)^{\frac{2}{3}}}.$$

We would like to determine the length of a side $|w_2|$ of the equilateral triangle Ω so that we know precisely what the equilateral triangle Ω with the normalization $|A| = 1$. In order for the Schwarz-Christoffel transformation to map 1 to $w_2 > 0$, according to the preceding discussion we must have $\arg A = \frac{2\pi}{3}$. We have the equation

$$\int_{-1}^{1} \frac{dt}{(t+1)^{\frac{2}{3}} (t-1)^{\frac{2}{3}}} = w_2.$$

From the above discussion on the values of the chosen branches of the factors in the integrand, we have

$$e^{-i\frac{2\pi}{3}} \int_{-1}^{1} \frac{dt}{|t+1|^{\frac{2}{3}} |t-1|^{\frac{2}{3}}} = w_2.$$

By taking the absolute values of both sides of the equation, we get

$$\int_{-1}^{1} \frac{dt}{|1+t|^{\frac{2}{3}}\,|1-t|^{\frac{2}{3}}} = |w_2|\,.$$

We rewrite the equation as

$$\int_{-1}^{1} \frac{dt}{(1-t^2)^{\frac{2}{3}}} = w_2,$$

because $w_2 > 0$. To evaluate the definite integral

$$\int_{-1}^{1} \frac{dt}{(1-t^2)^{\frac{2}{3}}} = 2\int_{0}^{1} \frac{dt}{(1-t^2)^{\frac{2}{3}}},$$

we use the transformation $\tau = t^2$ and get $d\tau = 2tdt = 2\tau^{\frac{1}{2}}dt$ and

$$2\int_{0}^{1} \frac{dt}{(1-t^2)^{\frac{2}{3}}} = \int_{0}^{1} \frac{d\tau}{\tau^{\frac{1}{2}}(1-\tau)^{\frac{2}{3}}}$$

which is equal to the value $B\left(\frac{1}{2}, \frac{1}{3}\right)$ of the beta function

$$B(x,y) = \int_{0}^{1} \tau^{x-1}(1-\tau)^{y-1}\,d\tau.$$

Our final conclusion of this example is that the holomorphic map

$$z \mapsto e^{i\frac{2\pi}{3}} \int_{-1}^{z} \frac{d\zeta}{(\zeta+1)^{\frac{2}{3}}(\zeta-1)^{\frac{2}{3}}}$$

maps the upper half-plane to the equilateral triangle in the upper half-plane whose base is $\left[0, B\left(\frac{1}{2}, \frac{1}{3}\right)\right]$ and is the image of $[-1, 1]$ with the sense preserved.

Example of Using a Scharz-Christoffel Transformation to Show that the Sine Function Maps a Vertical Upper Half-Strip to the Upper Half Plane. Let our 3-sided polygon Ω to be the vertical upper half-strip $\left\{-\frac{\pi}{2} < x < \frac{\pi}{2},\, y > 0\right\}$ with the two vertices $w_1 = -\frac{\pi}{2}$ and $w_2 = \frac{\pi}{2}$. We would like to write down the Schwarz-Christoffel transformation to map the upper half-plane $\mathbb{H} =$

$\{\operatorname{Im} z > 0\}$ to Ω with $x_1 = -1$ and $x_2 = 1$. The two angles at w_1 and w_2 are both $\frac{\pi}{2}$. The Schwarz-Christoffel transformation $w = f(z)$ is

$$f(z) = -\frac{\pi}{2} + A \int_{-1}^{z} \frac{d\zeta}{(\zeta+1)^{\frac{1}{2}}(\zeta-1)^{\frac{1}{2}}}.$$

for some $A > 0$, because $w_1 = -\frac{\pi}{2}$. To determine A, we put in the value $z = 1$ to get

$$\frac{\pi}{2} = -\frac{\pi}{2} + A \int_{-1}^{1} \frac{d\zeta}{(\zeta+1)^{\frac{1}{2}}(\zeta-1)^{\frac{1}{2}}}.$$

because $w_2 = \frac{\pi}{2}$. From the above discussion on the values of the chosen branches of the factors in the integrand, after we move $-\frac{\pi}{2}$ from the left-hand side of the equation to the right-hand, we get

$$\pi = A \int_{-1}^{1} \frac{i\, dt}{|1+t|^{\frac{1}{2}}\, |1-t|^{\frac{1}{2}}}.$$

We know that

$$\int_{-1}^{1} \frac{dt}{|1+t|^{\frac{1}{2}}\, |1-t|^{\frac{1}{2}}} = \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t \,\Big|_{t=-1}^{t=1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Hence $A = -i$. Now our Schwarz-Christoffel transformation is

$$f(z) = -\frac{\pi}{2} + A \int_{-1}^{z} \frac{d\zeta}{(\zeta+1)^{\frac{1}{2}}(\zeta-1)^{\frac{1}{2}}}.$$

For $z = t$ in the interval $(-1, 1)$, according to the values of the chosen branches of the factors in the integrand we have

$$f(t) = -\frac{\pi}{2} + A \int_{-1}^{t} \frac{i\, ds}{\sqrt{1-s^2}} = -\frac{\pi}{2} + \int_{-1}^{t} \frac{ds}{\sqrt{1-s^2}} = \int_{0}^{t} \frac{ds}{\sqrt{1-s^2}} = \sin^{-1} t,$$

because

$$\int_{-1}^{0} \frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{2}.$$

The range of $\sin^{-1} t$ for $-1 < t <$ would have to be the range of $f(t)$ for $-1 < t <$ and hence must be $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Taking the inverse of $\sin^{-1} z$, we conclude that the holomorphic map $z = \sin w$ maps the vertical upper half-strip

$$\left\{ w = u + iv \,\middle|\, -\frac{\pi}{2} < u < \frac{\pi}{2},\, v > 0 \right\}$$

onto the upper half-plane $\{y > 0\}$ with $w = -\frac{\pi}{2}$ corresponding to $z = -1$ and $w = \frac{\pi}{2}$ corresponding to $z = 1$.

Riemann Mapping Theorem

The statement of the Riemann mapping theorem is that if D is a simply connected domain in $\mathbb{C}$ not equal to $\mathbb{C}$, then there exists a biholomorphic map from D to the open unit disk Δ. Since D is simply connected and is not equal to $\mathbb{C}$, there are two distinct points in $\mathbb{C} - D$.

First of all we find one bounded injective holomorphic function on D. We do this by using one branch $f(z)$ of the square root of $\frac{z-a}{z-b}$. Let G be the image of this branch. The negative $-G$ of G must be disjoint from G for the following reason

The two branches of the square root of $\frac{z-a}{z-b}$ are defined from integrating $\frac{1}{2}\frac{df'}{f}$ and exponentiating. If G intersects $-G$, then the value of one branch at some point z_1 of D agrees with the branch at another point z_2 of D. By squaring both and using the univalent property of a Möbius transformation we conclude that $z_1 = z_2$ and the two branches have the same value at the same point, which contradicts the uniqueness of the result of integrating the same integrand from the same initial point with the same initial value.

Let w_0 be a point of $-G$. Then $\frac{1}{f(z)-w_0}$ is a bounded injective holomorphic function on D.

Among all injective holomorphic maps g from D to Δ normalized with $g(0) = 0$ let f be a function with $|f'(0)|$ maximum. Such a function exists because the set of all such functions g is a normal family. We claim that f maps D onto Δ. The idea of the proof is to use the fact that when one takes a square root, the absolute value of the derivative is increased after renormalization. Assume that some point a in Δ is not in the image of f. Let $w_1 = \frac{w-a}{\bar{a}w-1}$ and $w_2 = w_1^{1/2}$ and $b = \sqrt{a}$ and $w_3 = \frac{w_2-b}{\bar{b}w_2-1}$ and $g(z) = w_3$. The function g is obtained by moving a to the origin by a Möbius transformation and then taking a branch of the square root and then moving the final image of the origin back to the origin by a Möbius transformation. By the chain

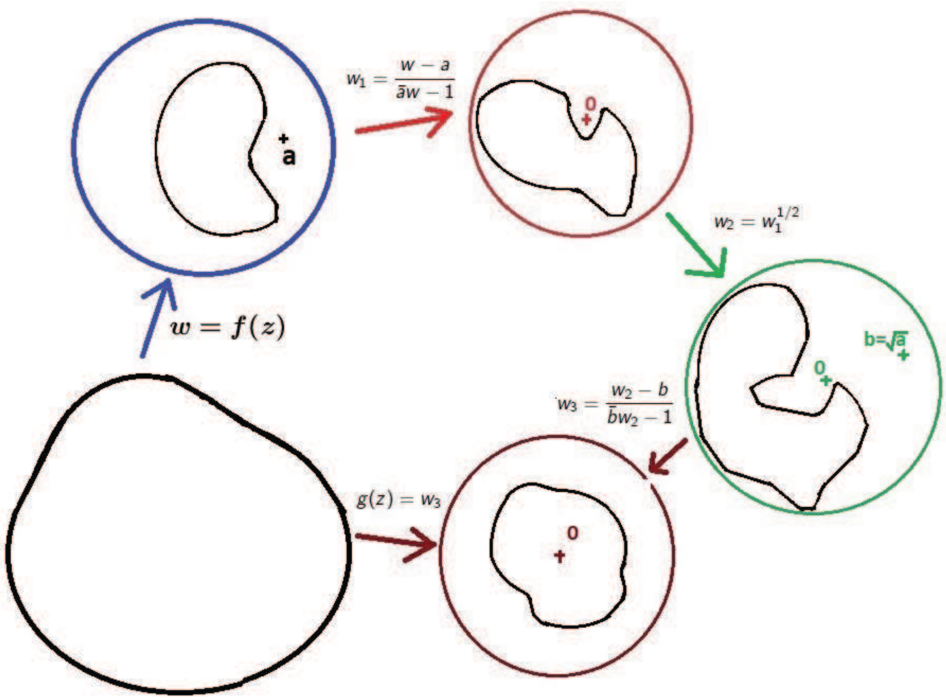

Figure 4: Use of Square Root to Increase Derivative of Non-Surjective Univalent Function at Origin

rule

$$\begin{aligned} g'(0) &= \left.\frac{dw_3}{dw_2}\right|_{w_2=b} \left.\frac{dw_2}{dw_1}\right|_{w_1=a} \left.\frac{dw_1}{dw}\right|_{w=0} f'(0) \\ &= \frac{1}{|b|^2-1}\frac{1}{2b}(|a|^2-1)f'(0) \\ &= \frac{1}{|b|^2-1}\frac{1}{2b}(|b|^4-1)f'(0) \\ &= \frac{1}{2b}(|b|^2+1)f'(0) \end{aligned}$$

whose absolute value is greater than that of $f'(0)$ because $|b| < 1$. Thus we get a contradiction and can conclude that f maps D onto Δ.

Boundary Behavior of Biholomorphism of Piecewise Smooth Domains

The Schwarz-Christoffel transformations were introduced to map conformally the open upper half-plane $\mathbb{H}$ to a given n-gon with vertices $w_1, \cdots, w_{n-1}$ in the w-plane (arranged in the counter-clockwise direction). Denote by $k_j\pi$ the exterior angle at w_j which measures the change of direction as a point going along the boundary of the n-gon passes the point w_j. The method is to determine real points $x_1, \cdots, x_{n-1}$ and the complex number A so that the following n equations are satisfied.

$$\int_{t=-\infty}^{x_1} \prod_{j=1}^{n-1} |t - x_j|^{-k_j} \, dt = \frac{|w_1 - w_n|}{|A|},$$

$$\int_{x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} |t - x_j|^{-k_j} \, dt = \frac{|w_{\nu+1} - w_\nu|}{|A|} \quad \text{for} \quad 1 \leq \nu \leq n-2,$$

$$w_n - w_{n-1} = A \int_{x=x_{n-1}}^{\infty} \frac{dt}{(t - x_1)^{k_1} \cdots (t - x_{n-1})^{k_{n-1}}}.$$

After the $n-1$ real unknowns $x_1, \cdots, x_{n-1}$ are solved from the first $n-1$ real-number equations and the complex unknown A is obtained from the last complex-number equation, the Schwarz-Christoffel transformation is given by

$$z \mapsto w = w_1 + A \int_{\zeta=x_1}^{z} \frac{d\zeta}{(\zeta - x_1)^{k_1} \cdots (\zeta - x_{n-1})^{k_{n-1}}}.$$

It is not clear that it is always possible to solve the first $n-1$ real-number equations for the $n-1$ unknowns $x_1, \cdots, x_{n-1}$, though it is clear that A can be readily obtained from the last complex-number equation of the above set of n equations. The Riemann mapping theorem guarantees that there is a biholomorphic map f from $\mathbb{H}$ to the interior of the n-gon. In order to use the Riemann mapping theorem to how that the above n equations can be solved for $x_1, \cdots, x_{n-1}$ and A, first we need to make sure that f extends continuously from the closure $\bar{\mathbb{H}}$ to the enclosure of the n-gon. This is the step we would like to do. After we finish this step, by composing with an biholomorphic self-map of $\mathbb{H}$, we can assume that f maps ∞ (in the Riemann sphere which contains the real line) to w_n. We can also use the Schwarz reflection principle to each side and also to the straightened corner after the application of a fractional power map to get the differentiable boundary behavior of the biholomorphic map (to be interpreted at a corner as after the application of a fractional power map). We can now identity the points $x_1, \cdots, x_{n-1}$ from the biholomorphic map f from the Riemann mapping theorem and its boundary behavior. The collection of $x_1, \cdots, x_{n-1}, A$ is not unique. There are three real parameters in the biholomorphism group of the open upper half plane $\mathbb{H}$ (or equivalently the open unit disk $\mathbb{D}$). Specifying that ∞ (in the Riemann sphere which contains the real line) is mapped to w_n means specifying only one of the three real parameters. There still two real parameters which we can specify. These two can be specified by the image of i in $\mathbb{H}$ (or equivalently the image of 0 in $\mathbb{D}$).

We now present the technique concerning the boundary behavior of biholomorphic mappings between domains of piecewise smooth boundaries. Suppose we have a univalent holomoprhic map f between two domains D and Ω with piecewise smooth boundary. We want to show that f extends continuously to the boundary.

What precisely is needed for the continuous extension of f to $\bar{D} \to \bar{\Omega}$ is the following boundary condition of Ω. The boundary $\partial\Omega$ is parametrized continuously by the circle. For two distinct points w^* and w^{**} on $\partial\Omega$ such that to go from w^* to w^{**} along $\partial\Omega$ is in the counterclockwise sense (according to the continuous parametrization by the circle), there exists a closed disk $\bar{\Delta}$ of positive radius such that the center of $\bar{\Delta}$ is in Ω and the intersection of $\bar{\Delta}$ and $\partial\Omega$ is contained in the interior of the part of $\partial\Omega$ from w^* to w^{**} and there is a closed arc of $\bar{\Delta}$ of positive length (for example, subtended by an angle $\frac{2\pi}{k}$ at the center for some integer $k \geq 3$) which is outside of $\bar{\Omega}$. An example of a domain not satisfying this condition is the right-half of the open unit disk minus the line segment $(0, \frac{1}{2}]$ with the two points w^* and w^{**} on $(0, \frac{1}{2})$.

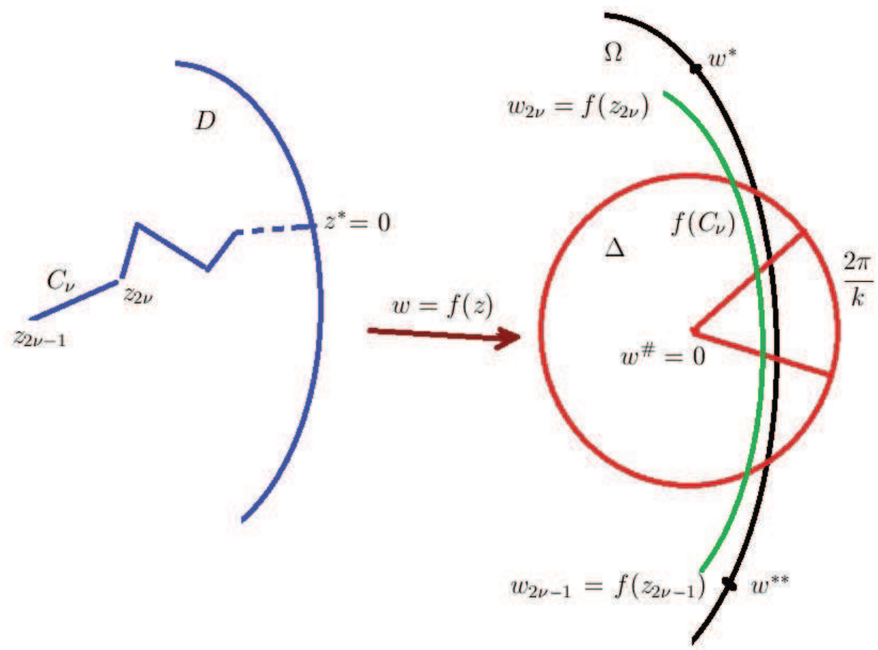

Figure 5: Different Limits of Images of Two Subsequences of Sequence Convergent to Boundary of Domain

We now prove that the biholomorphic map $f : D \to \Omega$ can be continuously extended to $\bar{D} \to \bar{\Omega}$. Suppose the contrary. Then we can find a sequence z_ν in D converging to some point z^* on the boundary of D such that $w_{2\nu} = f(z_{2\nu})$ converges to a boundary point w^* of Ω and $w_{2\nu+1} = f(z_{2\nu+1})$ converges to another boundary point w^{**} of Ω. Since the boundary of D is piecewise smooth, we can join $z_{2\nu-1}$ to $z_{2\nu}$ by a curve C_ν with the property that for any open disk U_ϵ of positive radius ϵ centered at z^* all the curves C_ν are inside U_ϵ when ν is greater than some $\nu_0(\epsilon)$. Since w^* and w^{**} are distinct, we can find a disk Δ centered at some point $w^\#$ of Ω so that an arc ω of Δ with angle $\frac{2\pi}{k}$ for some k is outside Ω and for all ν sufficiently large the image of C_ν under f separates Δ into two parts with one part containing the $w^\#$ and one part containing the arc ω. Without loss of generality we can assume that z^* is the origin and $w^\#$ is also the origin.

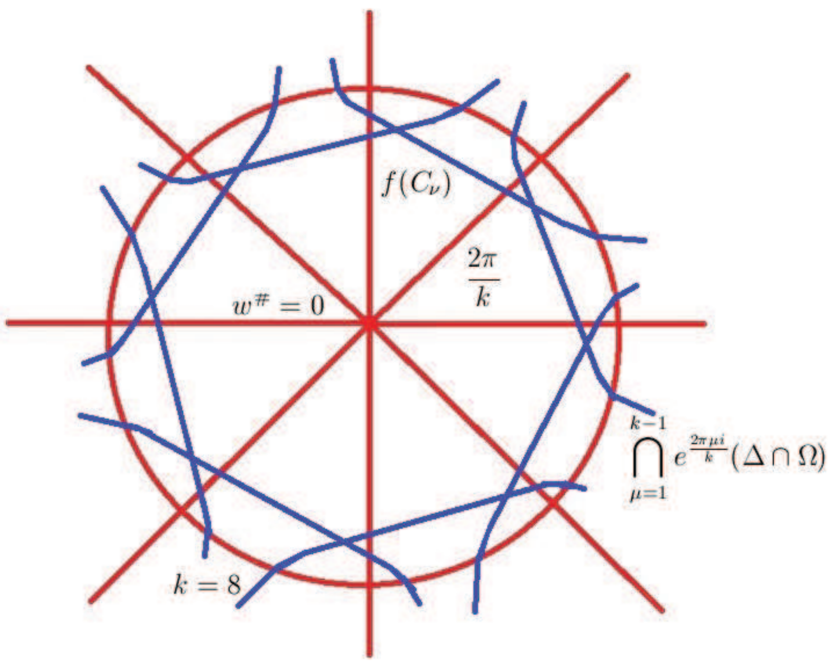

Figure 6: Application of Maximum Modulus Principle to Intersection of Rotated Domains

Consider the function g on $\Delta \cap \Omega$ which is the inverse of the map f. Then $|g| \leq \epsilon$ on $f(C_\nu)$ for $\nu \geq \nu_0(\epsilon)$. Consider the function

$$G(w) = \prod_{\mu=0}^{k-1} g(e^{2\pi\mu\sqrt{-1}/k} w).$$

Then G is holomorphic on

$$\bigcap_{\mu=0}^{k-1} e^{2\pi\mu\sqrt{-1}/k} \cdot (\Delta \cap \Omega),$$

where

$$e^{2\pi\mu\sqrt{-1}/k} \cdot (\Delta \cap \Omega)$$

is the domain obtained by rotating $\Delta \cap \Omega$ by an angle of $\frac{2\pi\mu}{k}$. On the k curves

$$e^{2\pi\mu\sqrt{-1}/k} \cdot f(C_\nu)$$

obtained by rotating the curve $f(C_\nu)$ by an angle of $\frac{2\pi\mu}{k}$, the function G is bounded by $A^{k-1}\epsilon$, where A is the radius of some disk centered at the origin which contains D. Since $w^\#$ is contained in the compact subset of

$$\bigcap_{\mu=0}^{k-1} e^{2\pi\mu\sqrt{-1}/k} \cdot (\Delta \cap \Omega)$$

bounded by

$$\bigcup_{\mu=0}^{k-1} e^{2\pi\mu\sqrt{-1}/k} \cdot f(C_\nu),$$

it follows from the maximum modulus principle that $G(w^\#) = g(w^\#)^k$ is bounded by $A^{k-1}\epsilon$. Thus $|g(w^\#)| \leq (A^{k-1}\epsilon)^{1/k}$. Since ϵ is just any positive number, we conclude that $g(w^\#) = 0$. The only condition on $w^\#$ is the existence of the disk Δ. Points sufficiently close to $w^\#$ would also satisfy this condition. So we get the vanishing of g in a neighborhood of $w^\#$ contradicting the univalency of f.

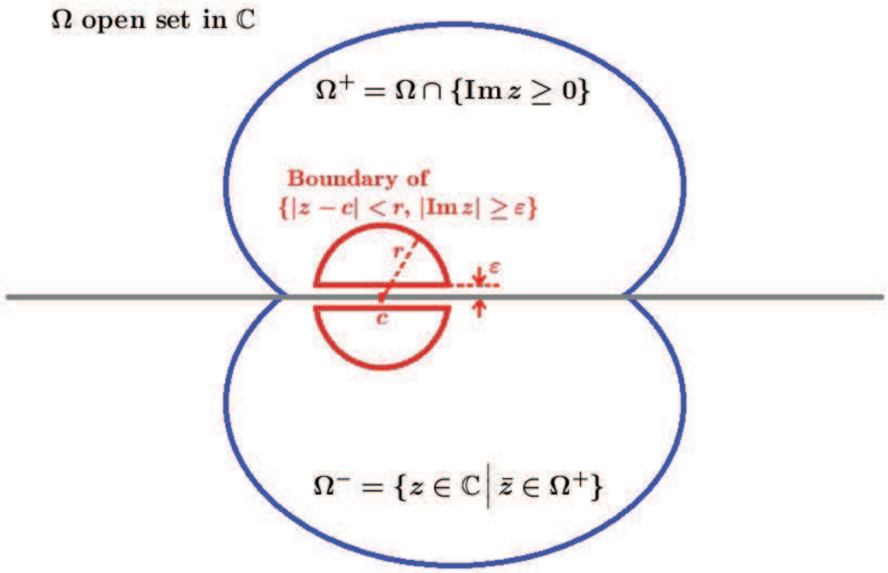

Figure 7: Schwarz Reflection with Respect to Real Line

Schwarz Reflection Principle

Suppose Ω is an open subset of $\mathbb{C}$. Let

$$\begin{aligned} \Omega^+ &= \Omega \cap \{\operatorname{Im} z \geq 0\}, \\ \Omega^- &= \left\{z \in \mathbb{C} \,\middle|\, \bar{z} \in \Omega^+\right\}, \\ \widetilde{\Omega} &= \Omega^+ \cup \Omega^-. \end{aligned}$$

If f is a continuous function on Ω^+ which is holomorphic on its interior $\Omega \cap \{\operatorname{Im} z > 0\}$ and which is identically zero on $\widetilde{\Omega} \cap \mathbb{R}$, then its extension $\widetilde{f}$ on $\widetilde{\Omega}$ defined by reflection with respect to the real axis

$$\widetilde{f}(z) = \begin{cases} f(z) & \text{if } \operatorname{Im} z \geq 0 \\ \\ \overline{f(\bar{z})} & \text{if } \operatorname{Im} z < 0 \end{cases}$$

is holomorphic on $\tilde{\Omega}$.

The proof goes as follows. First of all the vanishing of $\widetilde{f}$ on $\widetilde{\Omega}\cap\mathbb{R}$ implies that $\widetilde{f}$ is continuous on $\widetilde{\Omega}$. Since any power series expansion

$$f(z)=\sum_{n=0}^{\infty}\alpha_n(z-a)^n$$

on any open disk in $\widetilde{\Omega}\cap\{\operatorname{Im} z>0\}$ of radius r centered at a yields the power series expansion

$$\overline{f(\bar{z})}=\sum_{n=0}^{\infty}\bar{\alpha}_n(z-\bar{a})^n$$

on the open disk in $\widetilde{\Omega}\cap\{\operatorname{Im} z<0\}$ of radius r centered at $\bar{a}$, it follows that $\widetilde{f}$ is holomorphic on $\widetilde{\Omega}\cap\{\operatorname{Im} z\neq 0\}$.

To verify that $\widetilde{f}$ is holomorphic in an open neighborhood of any point c in $\widetilde{\Omega}$, we consider an open disk $D_c(r)$ of positive radius r whose closure is contained in $\widetilde{\Omega}$ and define the function

$$g(z)=\frac{1}{2\pi i}\int_{|\zeta-c|=r}\frac{\widetilde{f}(\zeta)d\zeta}{\zeta-z}$$

for $z\in D_c(r)$. Then $g(z)$ is holomorphic on $D_c(r)$. For $\varepsilon>0$ let

$$E_\varepsilon=\left\{\, z\in\mathbb{C}\;\middle|\;|z|<r,\ -\varepsilon<\operatorname{Im} z>\varepsilon\,\right\}.$$

Since $\widetilde{f}$ is continuous on $\widetilde{\Omega}$, it follows that

$$g(z)=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\zeta\in\partial E_\varepsilon}\frac{\widetilde{f}(\zeta)d\zeta}{\zeta-z}.$$

On the other hand, by the holomorphicity of $\widetilde{f}$ on $\widetilde{\Omega}\cap\{\operatorname{Im} z\neq 0\}$, we know by Cauchy's integral formula that

$$\widetilde{f}(z)=\frac{1}{2\pi i}\int_{\zeta\in\partial E_\varepsilon}\frac{\widetilde{f}(\zeta)d\zeta}{\zeta-z}$$

for any z in E_ε. Thus $g(z)$ agrees with $\widetilde{f}(z)$ on E_ε for any $\varepsilon>0$ which from the continuity of $\widetilde{f}$ and g implies that $\widetilde{f}$ agrees with g on $D_c(r)$ and is holomorphic on $D_c(r)$. This finishes the proof of the Schwarz reflection principle.