REVIEW OF BASIC COMPLEX ANALYSIS
(Part V throught Part VIII)

PART V. SERIES EXPANSION OF CAUCHY'S KERNEL. The key points are as follows.

(1) Power and Laurent series expansion from geometric series expansion of Cauchy's kernel.

(2) Coefficients of power and Laurent series expansion from integration against monomials, to yield uniqueness of coefficients and the identity theorem for holomorphic functions.

(3) Classification of three kinds of isolated singularities from principal parts of Laurent series expansion.

Power Series and Laurent Series Expansion of Cauchy Kernel. Since holomorphic functions are generated by the Cauchy kernel $\frac{1}{\zeta - z}$ in integrals, some properties of the Cauchy kernel pass over to holomorphic functions. One such property is the power series expansion and Laurent series expansion of the Cauchy kernel by using geometric series. For $a \in \mathbb{C}$, when $|z-a| < |\zeta - a|$ the geometric series expansion

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a} \frac{1}{1 - \frac{z-a}{\zeta-a}} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}}$$

converges and gives a power series expansion of the Cauchy kernel.

For $a \in \mathbb{C}$, when $|z - a| > |\zeta - a|$, with the exchange of the roles of z and ζ, the geometric series expansion

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{-1}{(z-a) - (\zeta - a)} = \frac{-1}{z-a} \frac{1}{1 - \frac{\zeta - a}{z-a}} \\ &= -\sum_{n=0}^{\infty} \frac{(\zeta - a)^n}{(z-a)^{n+1}} = -\sum_{m=-\infty}^{-1} \frac{(z-a)^m}{(\zeta - a)^{m+1}} \end{aligned}$$

(by using $n = -(m+1)$ in the change of dummy index)

converges and gives a Laurent series expansion of the Cauchy kernel

Power Series Expansion of Holomorphic Function on Disk. Suppose f is a holomorphic function on $|z-a|<R$. For z in the disk $|z-a|<R$, choose $0<r<R$ with $|z-a|<r$ to get the power series expansion

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)d\zeta}{\zeta-a} = \frac{1}{2\pi i}\int_{|\zeta|=r} f(\zeta)\left(\sum_{n=0}^{\infty}\frac{(z-a)^n}{(\zeta-a)^{n+1}}\right)d\zeta \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta\right)(z-a)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n, \end{aligned}$$

where for the last step the Cauchy integral formula for derivatives of holomorphic functions is used.

Uniqueness of Power Series Expansion of Holomorphic Function on Disk. The power series expansion of a holomorphic function f on an open disk $|z-a|<R$ is unique, because if $f(z)$ is equal to the convergent power series

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

on $|z-a|<R$, then for $0<r<R$ and any nonnegative integer k,

$$\int_{|z-a|=r}\frac{f(z)dz}{(z-a)^{k+1}} = \sum_{n=0}^{\infty} c_n\int_{|z-a|=r}(z-a)^{n-k-1}dz = 2\pi i c_k$$

so that $c_n = \frac{f^{(n)}(a)}{n!}$ by the Cauchy integral formula for derivatives of holomorphic functions.

Laurent Series Expansion of Holomorphic Function on Annulus. Suppose f is a holomorphic function on $R_1<|z-a|<R_2$. For z in the annulus disk $R_1<|z-a|<R_2$, choose $R_1<r_1<r<r_2<R_2$ with $r_1<|z-a|<r_2$

to get the Laurent series expansion

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=r_2}\frac{f(\zeta)d\zeta}{\zeta-a}-\frac{1}{2\pi i}\int_{|\zeta|=r_1}\frac{f(\zeta)d\zeta}{\zeta-a} \\ &= \frac{1}{2\pi i}\int_{|\zeta|=r_1} f(\zeta)\left(\sum_{n=0}^{\infty}\frac{(z-a)^n}{(\zeta-a)^{n+1}}\right)d\zeta \\ &\quad -\frac{1}{2\pi i}\int_{|\zeta|=r_1} f(\zeta)\left(\sum_{n=-\infty}^{-1}\frac{(z-a)^n}{(\zeta-a)^{n+1}}\right)d\zeta \\ &= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta\right)(z-a)^n, \end{aligned}$$

where for the last step Cauchy's theorem is used.

Uniqueness of Laurent Series Expansion of Holomorphic Function on Annulus. The Laurent series expansion of a holomorphic function f on an open annulus $R_1 < |z-a| < R_2$ is unique, because Suppose $f(z)$ is equal to the Laurent series $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$ on $R_1 < |z-a| < R_2$ in the sense that both series $\sum_{n=0}^{\infty} c_n(z-a)^n$ and $\sum_{n=-\infty}^{-1} c_n(z-a)^n$ converge at every point of $R_1 < |z-a| < R_2$ instead of just the convergence of the principal sum $\sum_{k=-n}^{n} c_k(z-a)^k$ as $n\to\infty$. Then for $R_1 < r < R_2$ and any integer k,

$$\int_{|z-a|=r}\frac{f(z)dz}{(z-a)^{k+1}}=\sum_{n=0}^{\infty}c_n\int_{|z-a|=r}(z-a)^{n-k-1}dz=2\pi i c_k,$$

yielding the uniqueness of c_k.

Identity Theorem for Holomorphc Functions as Consequence of Power Series Expansion on Open Disk.

Theorem. The zero-set of a non-identically-zero holomorphic function f on a connected open subset Ω of $\mathbb{C}$ must be isolated.

As a corollary, any two holomorphic functions on a connected open subset which agree on a set with an accumulation point must be identical.

To prove the theorem, let Z be zero-set of f in Ω. For $a \in Z$, we have a power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

on any open disk $|z-a| < r$ whose closure is contained in Ω. Either all the coefficients of the power series vanish, in which case f is identically zero on $|z-a| < r$ and Z contains an open neighborhood of a or $f(z) = (z-a)^k g(z)$ with

$$g(z) = \sum_{n=k}^{\infty} c_n (z-a)^{n-k}$$

nonzero at $z = a$ for some nonnegative integer k, in which case a is an isolated point of Z. This means that the set of all accumulation points of Z, if nonempty, is both open and closed and therefore must be equal to the connected set Ω, which contradicts the assumption that f is not identically zero on Ω.

Isolated Singularities of Holomorphic Functions Classified by Principal Part of Laurent Series. A point a of $\mathbb{C}$ is called an *isolated singularity* of a holomorphic function f is f is holomorphic on some punctured neighborhood $0 < |z-a| < R$ of a. Let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

be the Laurent series expansion of f on the open annulus $0 < |z-a| < R$ (with degenerate inner radius). The part

$$\sum_{n=-\infty}^{-1} c_n (z-a)^n$$

is called the *principal part* of the Laurent series. The isolated singularity of f at a is called

- a *removable singularity* if the Laurent series has no principal part;
- a *pole* of order k if the principal part of the Laurent series consists of only a finite number of nonzero terms $\sum_{n=-k}^{-1} c_n (z-a)^n$ with $c_{-k} \neq 0$;

- an *essential singularity* if the principal part of the Laurent series contains an infinite number of nonzero terms.

A *meromorphic function* on an open subset G of $\mathbb{C}$ means a function f on $G - Z$, where Z is a (possibly empty) discrete subset of G, such that every point of Z is a pole of f.

Residue of Isolated Singularity of Holomorphic Function. The *residue* of a holomorphic function f at an isolated singularity a, denoted by $\operatorname{Res}_a(f)$, is defined as the coefficient c_{-1} of the first negative power of the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

which is also equal to

$$\frac{1}{2\pi i} \int_{|z-a|=\varepsilon} f(z)dz$$

for any sufficiently small positive number ε. If the isolated singularity of f at a is a pole of order $\leq k$ (where k is any positive integer), the residue of f at a can be alternatively computed by the formula

$$\operatorname{Res}_a(f) = \frac{1}{(k-1)!} \left(\frac{d^{k-1}}{dz^{k-1}} \left((z-a)^k f(z) \right) \right)_{z=a},$$

because $(z-a)^k f(z)$ is a power series whose $(k-1)$-th power term has coefficient c_{-1}. Residues are introduced for the computation of definite integrals by techniques of complex analysis.

Characterization of Three Kinds of Isolated Singularities of Holomorphic Function. The isolated singularity of a holomorphic function f at a point a is *removable* if and only if f is square integrable on some deleted neighborhood of a, because, computed in polar coordinates (r, θ) centered at

a (*i.e.*, $z = a + re^{i\theta}$) with area element $rdrd\theta$,

$$\begin{aligned}
&\int_{\varepsilon<|z-a|<\rho} \left| \sum_{n=-\infty}^{\infty} c_n (z-a)^n \right|^2 \\
&= \sum_{m,n=-\infty}^{\infty} \int_{\varepsilon<|z-a|<\rho} c_m (z-a)^m \overline{c_n (z-a)^n} \\
&= \sum_{m,n=-\infty}^{\infty} \int_{r=\varepsilon}^{\rho} \left(\int_{\theta=0}^{2\pi} c_m r^m e^{im\theta} \overline{c_n}\, r^n e^{-in\theta} d\theta \right) rdr \\
&= 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 \int_{r=\varepsilon}^{\rho} r^{2n+1} dr \\
&= 2\pi \left(|c_{-1}|^2 \log \frac{\rho}{\varepsilon} + \sum_{n\in\mathbb{Z}-\{0\}} \frac{\rho^{2n+2} - \varepsilon^{2n+2}}{2n+2} |c_n|^2 \right),
\end{aligned}$$

which can remain bounded as $\varepsilon \to 0+$ if and only if $c_n = 0$ for all $n < 0$.

The isolated singularity of a holomorphic function f at a point a is a *pole* of order k (for some positive integer k) if and only if $|f(z)|$ is comparable to $\frac{1}{|z-a|^k}$ as $z \to a$ in the sense that

$$\frac{A}{|z-a|^k} \le |f(z)| \le \frac{B}{|z-a|^k}$$

for some positive constants A, B when z is in a sufficiently small deleted neighborhood of a, because $|f(z)| \sim \frac{1}{|z-a|^k}$ means that the isolated singularity at a of the holomorphic function $(z-a)^k f(z)$, which uniformly bounded on some deleted neighborhood of a, is removable and has a nonzero value at a.

The isolated singularity of a holomorphic function f at a point a is an *essential singularity* if and only if the image of *any* deleted neighborhood $0 < |z-a| < r$ under f is dense in $\mathbb{C}$, because such an image cannot be dense in $\mathbb{C}$ in the case of removable or pole singularity and if one such image is disjoint from some nonempty open disk $|z-b| \ge \delta$, then $\frac{1}{f(z)-b}$ is uniformly bounded on some deleted neighborhood of a and is holomorphic function on some neighborhood of a, making it possible to conclude that the isolated singularity of $f(z)$ is either removable or a pole.

By the big Picard theorem (which follows from Nevanlinna's theory of defect relation to be discussed later), the image of a holomorphic function near an essential singularity cannot miss more than two points.

PART VI. HOLOMORPHIC FUNCTION BEHAVIOR IN THE INTERIOR AND AT BOUNDARY. The key points are as follows.

(1) Number of zeroes in the interior, in the form of the argument principle and Rouché's theorem, obtained from residues of logarithmic derivative.

(2) Open mapping property, maximum modulus principle and Schwarz lemma, concerning boundary behavior of holomorphic functions and in particular, growth property at boundary.

Argument Principle and Rouché's Theorem (Counting of Zeros and Poles from Residues of Logarithmic Derivative). Let G be a bounded domain in $\mathbb{C}$ with piecewise smooth boundary and Ω be an open neighborhood of the closure $\bar{G}$ of G. Let f be a meromorphic function on Ω without any zeroes or poles on the boundary ∂G of G. Let $a_1, \cdots, a_k$ be the set of all zeroes of f in G with multiplicities $p_1, \cdots, p_k$ and let $b_1, \cdots, b_\ell$ be the set of all poles of f in G with multiplicities $q_1, \cdots, q_\ell$. By using the residues of the logarithmic derivative $(\log f)'$ of f defined as $\frac{f'}{f}$, we can compute $\sum_{\mu=1}^k p_\mu - \sum_{\nu=1}^\ell q_\nu$ as follows. Let

$$f(z) = \frac{(z-a_1)^{p_1} \cdots (z-a_k)^{p_k}}{(z-b_1)^{q_1} \cdots (z-b_\ell)^{q_\ell}} g(z)$$

so that $g(z)$ is holomorphic and nonzero at every point of $\bar{G}$ and

$$\frac{f'}{f} = \sum_{\mu=1}^k \frac{p_\mu}{z-a_\mu} - \sum_{\nu=1}^\ell \frac{q_\nu}{z-b_\nu} + \frac{g'}{g}.$$

Then

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial G}\frac{f'}{f} &= \sum_{\mu=1}^k \operatorname{Res}_{a_\mu}\frac{f'}{f} + \sum_{\nu=1}^\ell \operatorname{Res}_{b_\nu}\frac{f'}{f} \\ &= \sum_{\mu=1}^k \operatorname{Res}_{a_\mu}\frac{p_\mu}{z-a_\mu} - \sum_{\nu=1}^\ell \operatorname{Res}_{b_\nu}\frac{q_\nu}{z-b_\nu} \\ &= \sum_{\mu=1}^k p_\mu - \sum_{\nu=1}^\ell q_\nu.\end{aligned}$$

Since $\frac{f'}{f} = (\log f)'$, it follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial G}\frac{f'}{f} &= \frac{1}{2\pi i}\int_{\partial G} d\log f\\ &= \frac{1}{2\pi i}\int_{\partial G}\left(d\log|f| + i\arg f\right)\\ &= \frac{1}{2\pi}\int_{\partial G} d\arg f\\ &= \frac{1}{2\pi}\Delta_{\partial G}\arg f.\end{aligned}$$

Here $\arg f$ is the argument of f which means θ when $f = |f|e^{i\theta}$. Though $\arg f$ can be defined as an angle or direction, yet its numerical value is not well-defined and can only be chosen up to an integral multiple of 2π.

We can choose a point P in a component γ of ∂G and arbitrarily choose the value of the multi-valued function $\arg f$ at P and let it vary continuously along the positive direction of γ until back at P again where the new value may be different. The new value minus the old value of $\arg f$ is the increase in the argument of f along γ, which we denote by $\Delta_\gamma f$. Its sum over all components γ of ∂G is denoted by $\Delta_{\partial G}\arg f$, which we refer to as the total increase of $\arg f$ along ∂G.

Argument Principle. For a meromorphic function f on an open neighborhood of a bounded domain G with piecewise smooth boundary which does not have any zero or pole on the boundary ∂G of G, the total number of zeroes of f in G minus the total number of poles of f in G is equal to $\frac{1}{2\pi}$ times the total increase of the argument of f along ∂G.

We now look at the effect of a "small" perturbation of f on the total increase of the argument of f along ∂G. Suppose we have another meromorphic function g on Ω without zero or pole on ∂G such that $|g| < |f|$ on ∂G. Rouché's theorem tells us that the total increase of the argument of $f + g$ along ∂G is the same as the total increase of the argument of f along ∂G so that the total number of zeroes of $f + g$ in G minus the total number of poles of $f + g$ in G is the same as the total number of zeroes of f in G minus the total number of poles of f in G. The proof of Rouché's theorem is as follows.

The function

$$t \mapsto \frac{1}{2\pi i}\int_{\partial G}\frac{f'+tg'}{f+tg}=\frac{1}{2\pi}\Delta_{\partial G}\arg(f+tg)$$

for $0 \le t \le 1$ is continuous with integral values so that it is independent of $t \in [0,1]$, because $f(z)+tg(z)$ is nowhere zero for $t \in [0,1]$ and $z \in \partial G$ from the assumption that $|f| > |g|$ on ∂G.

This means that the value

$$\frac{1}{2\pi i}\int_{\partial G}\frac{f'+g'}{f+g}$$

at $t = 1$ is the same as the value

$$\frac{1}{2\pi i}\int_{\partial G}\frac{f'}{f},$$

which yields Rouché' theorem.

Rouché's theorem. For meromorphic functions f and g on an open neighborhood of a bounded domain G with piecewise smooth boundary such that both f and g have no zeroes or poles on the boundary ∂G of G, if $|f| > |g|$ on ∂G, then the total number of zeroes of $f+g$ in G minus the total number of poles of $f+g$ in G is equal to the total number of zeroes of f in G minus the total number of poles of f in G .

Open Mapping Theorem for Holomorphic Function and Maximum Modulus Principle.

Open Mapping Theorem for Holomorphic Function. If f is a nonconstant holomorphic function on connected open subset Ω of $\mathbb{C}$, then the map $\Omega \to \mathbb{C}$ defined by f is open (*i.e.,* the image of an open set is open).

Proof. Since f' is not identically zero, the zeroes of f' are isolated. Let G be an open subset of Ω and z_0 be any point of G. If f' is nonzero at z_0, the inverse mapping theorem guarantees that f can be locally inverted at z_0 and some open neighborhood U_{z_0} of z_0 in G is mapped by f to some open subset of $\mathbb{C}$. If f' is zero at z_0, the vanishing order of f' at P is some positive integer k. Write $f(z) = f(z_0) + (z - z_0)^\ell g(z)$ with $g(z)$ holomorphic and nonzero at z_0. Then $\ell = k + 1$.

Since $g(z_0) \neq 0$, we can choose a holomorphic function $h(z)$ on some open neighborhood at z_0 such that $h(z)^{k+1} = g(z)$ (by using a branch of $\log g(z)$ and setting $h(z) = e^{\frac{1}{k+1}\log g(z)}$). With respect to the local coordinate $\zeta = (z - z_0)h(z)$ of Ω at z_0, the map $f = f(z_0) + \zeta^{k+1}$, as a power map followed by a translation, maps some open neighborhood U_{z_0} of z_0 in $G$to an open subset of $\mathbb{C}$. $f(G) = \cup_{P\in G} f(U_P)$ is open in $\mathbb{C}$. Q.E.D.

Maximum Modulus Principle. If f is a nonconstant holomorphic function on a connected open subset Ω of $\mathbb{C}$ such that the supremum M of $|f|$ on Ω is finite, then there does not exist any point P of Ω such that $|f(P)| = M$.

Proof. If such a point P exists, since f as a map is open, $f(U_P)$ is an open subset of $\mathbb{C}$ for some open neighborhood U_P of P in Ω. Let D be the closed disk in $\mathbb{C}$ with radius M and center 0. From $|f(P)| = M = \sup_\Omega |f|$, we get the contradiction that $f(U_P)$ is an open neighborhood of the boundary point $f(P)$ of D and yet is contained in the subset $f(\Omega)$ of D.

Schwarz Lemma (Application of Maximum Modulus Principle).

Schwarz Lemma. Let f be a holomorphic function on the open unit disk $\mathbb{D}$ such that $|f| \leq 1$ on $\mathbb{D}$ and $f(0) = 0$. Then $|f(z)| \leq |z|$ on $\mathbb{D}$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \in \mathbb{D} - \{0\}$ or $|f'(0)| = 1$, then $f(z) = e^{i\alpha} z$ for some $\alpha \in \mathbb{R}$.

The quotient function $\frac{f(z)}{z}$ is holomorphic on $\mathbb{D}$ when its value at $z = 0$ is defined to be $f'(0)$ at $z = 0$. On $|z| \leq r$ for any $0 < r < 1$, it follows from the maximum modulus principle that for $|z| < r$,

$$\left|\frac{f(z)}{z}\right| \leq \sup_{|\zeta|\leq r}\left|\frac{f(\zeta)}{\zeta}\right| \leq \frac{1}{r}.$$

The conclusion follows from letting $r \to 1$.

An important application of the Schwarz lemma is the following characterization of biholomorphic maps of the open unit disk preserving the origin.

Any biholomorphic map f of $\mathbb{D}$ preserving the origin is of the form $f(z) = e^{i\alpha} z$ for some $\alpha \in \mathbb{R}$.

Let g be the inverse map of f. By the chain rule $g'(0)f'(0) = 1$. The Schwarz lemma applied separately to f and g implies that $|f'(0)| \leq 1$ and $|g'(0)| \leq 1$. Hence $|f'(0)| = 1$ and $f(z) = e^{i\alpha}z$ for some $\alpha \in \mathbb{R}$ by the Schwarz lemma.

PART VII. COMPUTATION OF DEFINITE INTEGRALS BY CAUCHY RESIDUE THEORY. The key points are as follows.

1. Integral of rational functions of sine and cosine by using residue theory for the unit disk.
2. Boundary of upper half disk, with radius going to infinity, for integral of rational functions on real line and their products with sine and cosine.
3. Techniques involving branches of multi-valued functions.

Application of Cauchy's Residue Theory to Evaluation of Definite Integrals and Representation of Functions as Infinite Sums of Partial Fractions and as Infinite Products. Cauchy's Residue theory can be applied to evaluate certain definite integrals, especially the following types.

- Integral of a rational function of the sine and cosine functions over an interval of length 2π.
- Integral of a rational function over the real line with the degree of the denominator at least 2 more than that of the numerator.
- Integral of a product of either sine or cosine and a rational function with real coefficients over the real line with the degree of the denominator at least 1 more than that of the numerator.
- Integral involving branches of certain holomorphic functions.
- Expansion of meromorphic functions as infinite sums of partial fractions.
- Expansion of entire functions as infinite products.

Integral of Rational Function of Sine and Cosine Functions over Interval of Length 2π. For the evaluation of definite integrals of the form

$$\int_{\theta=0}^{2\pi} R\left(\cos\theta, \sin\theta\right) d\theta,$$

where $R\left(x,y\right)$ is a rational function (with the value of $R\left(\cos\theta,\sin\theta\right)$ assumed finite at every point $\theta\in[0,2\pi]$), The substitution

$$z=e^{i\theta},\quad \cos\theta=\frac{z+\frac{1}{z}}{2},\quad \sin\theta=\frac{z-\frac{1}{z}}{2i},\quad d\theta=\frac{dz}{iz}$$

is used to convert the integral into a contour integral

$$\int_{|z|=1}f(z)dz$$

of a meromorphic function $f(z)$ of z over the contour $|z|=1$, which can then be evaluated by using Cauchy's residue theory.

Integral of Rational Function over Real Line with Degree of Denominator At Least 2 More Than That of Numerator. For the evaluation of

$$\int_{x=-\infty}^{\infty}\frac{P(x)dx}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials with $\deg Q(x)\geq\deg P(x)+2$ and no zero of $Q(x)$ for real x, the meromorphic function used is

$$\frac{P(z)}{Q(z)}$$

over the contour of the boundary of the upper half disk of radius R centered at the origin as $R\to\infty$.

The condition of $\deg Q\geq 2+\deg Q$ is needed to conclude that

$$\lim_{R\to\infty}\int_{C_R}^{\infty}\frac{P(z)dz}{Q(z)}=0$$

with $C_R=\left\{Re^{i\theta}\,\middle|\,0\leq\theta\leq\pi\right\}$.

The formula is

$$\int_{x=-\infty}^{\infty}\frac{P(x)dx}{Q(x)}=2\pi i\sum_{\operatorname{Im}a>0}\operatorname{Res}_a\frac{P(z)}{Q(z)}.$$

Integral of Product of Either Sine or Cosine Function and Rational Function with Real Coefficients over Real Line with Degree of Denominator At Least 1 More Than That of Numerator. For the evaluation of

$$\int_{x=-\infty}^{\infty}\frac{P(x)\cos x dx}{Q(x)}\quad\text{or}\quad\int_{x=-\infty}^{\infty}\frac{P(x)\sin x dx}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials of real coefficients with $\deg Q(x)\geq\deg P(x)+1$ and no zero of $Q(x)$ for real x except simple ones at the zeros of respectively $\cos x$ and $\sin x$, the meromorphic function used is

$$\frac{P(z)e^{iz}}{Q(z)}$$

over the contour of the boundary of the upper half disk of radius R centered at the origin as $R\to\infty$.

When $\deg P=1+\deg Q$, to conclude

$$\lim_{R\to\infty}\int_{C_R}\frac{P(z)e^{iz}dz}{Q(z)}=0,$$

one needs to use integration by parts

$$\int_{C_R}\frac{P(z)e^{iz}dz}{Q(z)}=\left[\frac{P(z)e^{iz}}{iQ(z)}\right]_{z=-R}^{z=R}-\int_{C_R}\left(\frac{d}{dz}\frac{P(z)}{Q(z)}\right)\frac{e^{iz}}{i}dz$$

and the fact that on the right-hand side of

$$\frac{d}{dz}\frac{P(z)}{Q(z)}=\frac{P'(z)Q(z)-P(z)Q'(z)}{Q(z)^2}$$

the degree of the denominator is at least 2 more than that of the numerator.

When there is a simple zero of $Q(x)$ which is cancelled by the zeroes of sine or cosine in the numerator, a detour of a small half-circle around the simple zero has to be use to get the contribution of one half of the residue from the simple zero of $Q(x)$. At the end the real part and the imaginary part of the result from residue theory. The final formula is the following.

$$\begin{aligned}&\int_{x=-\infty}^{\infty}\frac{P(x)\cos x\,dx}{Q(x)}\\&=\operatorname{Re}\left(2\pi i\sum_{\operatorname{Im}a>0}\operatorname{Res}_a\frac{P(z)e^{iz}}{Q(z)}+\pi i\sum_{\operatorname{Im}a=0}\operatorname{Res}_a\frac{P(z)e^{iz}}{Q(z)}\right),\\&\int_{x=-\infty}^{\infty}\frac{P(x)\sin x\,dx}{Q(x)}\\&=\operatorname{Im}\left(2\pi i\sum_{\operatorname{Im}a>0}\operatorname{Res}_a\frac{P(z)e^{iz}}{Q(z)}+\pi i\sum_{\operatorname{Im}a=0}\operatorname{Res}_a\frac{P(z)e^{iz}}{Q(z)}\right).\end{aligned}$$

Definite Integrals Evaluated by Using Branches of Holomorphic Functions. Some definite integrals which can be evaluated by the theory of residues involve branches of holomorphic functions. For this type of definite integrals there are no general formulas for evaluation. However, there are some natural ways of choosing contours and branches for the meromorphic functions used. Some examples of such definite integrals are

$$\begin{gathered}\int_{x=0}^{\infty}\frac{\log x\,dx}{x^2+1},\qquad\int_{x=0}^{\infty}\frac{(\log x)^2\,dx}{x^4+1},\\\int_{x=0}^{\infty}\frac{x^\alpha\,dx}{1+x}\quad\text{with}\quad-1<\alpha<0,\\\int_{x=0}^{1}\frac{dx}{x^\alpha(1-x)^{1-\alpha}}\quad\text{with}\ \ 0<\alpha<1.\end{gathered}$$

For the evaluation of

$$\int_{x=0}^{\infty}\frac{\log x\,dx}{x^2+1}$$

the contour used is the boundary of the indented upper half disk

$$\left\{z\in\mathbb{C}\,\middle|\,\varepsilon<|z|<R,\ \operatorname{Im}z>0\right\}$$

with radius R and an indentation of radius $\varepsilon>0$. Then pass to limit as $R\to\infty$ and $\varepsilon\to0$. For the evaluation of

$$\int_{x=0}^{\infty}\frac{x^\alpha\,dx}{1+x}\quad\text{with}\quad-1<\alpha<0$$

the contour used is the boundary of the domain which is the annulus

$$\left\{z \in \mathbb{C} \,\middle|\, \varepsilon < |z| < R\right\}$$

with inner radius ε and outer radius R minus the right half-strip

$$\left\{z \in \mathbb{C} \,\middle|\, \operatorname{Re} z \geq 0, -\varepsilon \leq \operatorname{Im} z \leq \varepsilon\right\}$$

with width 2ε and axis $\operatorname{Im} z = 0$. Then pass to limit as $R \to \infty$ and $\varepsilon \to 0$.

The meromorphic function used involves choosing branches for the logarithmic function and the fraction power function which can also be expressed in terms of the logarithmic function.

Branches of Multivalued Holomorphic Functions. Some holomorphic functions such as the logarithmic function and the fractional power function are multi-valued. A *continuous* function obtained by choosing one value of a multi-valued function f for each point of a domain D is called a branch of the multi-valued function of f on D. The domain D usually is constructed by removing certain line-segments called "branch-cuts" from the natural maximum domain of definition for the multi-valued function. The value of the branch function at a point approaching one side of a branch-cut differs from the value when the point approaches from the other side. That is the reason why the branch-cut is needed to guarantee that the branch function is a continuous function. For example, for $\log z$ a branch can be defined on $\mathbb{C} - [0, \infty)$ by choosing $\arg z$ to be in $(2k\pi, (2k+1)\pi)$ for any $k \in \mathbb{Z}$.

For the product of the branches of two different multi-valued functions, a line-segment common to both branch-cuts may become unnecessary for the product if the two discrepancies of values across it for the two branch functions cancel out. An example is the function $z^\alpha (1-z)^{1-\alpha}$ (with $0 < \alpha < 1$) for which the branch-cut of $[0, \infty)$ is used to construct a branch-function for the factor z^α while the branch-cut of $[1, \infty)$ is used to construct a branch-function for the factor $(1-z)^{1-\alpha}$. Across the common part $(1, \infty)$ of two branch-cuts, the two discrepancies of values for the two branch functions cancel out so that a branch of the multi-valued function $z^\alpha (1-z)^{1-\alpha}$ can be defined on $\mathbb{C} - [0, 1]$.

PART VIII. PARTIAL FRACTION EXPANSION, INFINITE PRODUCT EXPANSION, AND SUM AT INTEGER POINTS.

(1) Adding power of variable to Cauchy kernel to take care of boundary term, to obtain partial fraction expansion of meromorphic functions.

(2) Infinite produce expansion from partial fraction of logarithmic derivative.

(3) Sum (or alternate sum) of function at integer points by applying residue theory to its product with cotangent or cosecant function on rectangles centered at the origin with sides going to infinity.

Partial Fraction Expansion of Meromorphic Functions (Application of Cauchy's Integral Formula to Meromorphic Function after Modification to Ensure Vanishing of Limit Contour Integral). For a holomorphic function f on an open neighborhood of a bounded domain G with piecewise smooth boundary, Cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\int_{\partial G}\frac{f(\zeta)d\zeta}{\zeta-z}$$

reproduces $f(z)$ from an integral. Suppose $f(z)$ is now assumed to be meromorphic instead of holomorphic. Cauchy's integral formula will not hold in general. There will be some additional terms from the contribution of the pole singularities of f. The additional terms are some partial fractions. To represent f as an infinite sum of partial fractions, one would like to get rid of the integral over ∂G by letting ∂G recede to infinity and at the same time keep these additional partial fraction terms from the pole singularities of f.

The integral over ∂G would go to 0 as ∂G recedes to infinity if certain growth order conditions are met. To get to such a situation, we replace the integrand

$$\frac{f(\zeta)d\zeta}{\zeta-z}$$

by

$$\frac{f(\zeta)d\zeta}{\zeta^{p+1}(\zeta-z)}$$

and impose the growth condition $f(z)=o\left(R_n^{p+1}\right)$ on a contour C_n, where R_n is the distance from the origin, when ∂G is replaced by C_n whose length is of order $O(R_n)$ with $R_n \to \infty$ as $n \to \infty$. We formulate in the following statement the result obtained for the partial fraction expansion of meromorphic functions.

Theorem. Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$ whose poles are simple $\{a_n\}_{1\le n<\infty}$ with $0 < |a_1| \le |a_2| \le \cdots$ so that the residue of $f(z)$ at a_n is b_n. Suppose that there is a sequence of closed contours C_n such that C_n includes $a_1, \cdots, a_n$ but no other poles. Assume that the distance R_n from C_n to the origin goes to infinity as $n \to \infty$ and the length L_n of C_n is of the order $O(R_n)$. Assume that on C_n we have $f(z) = o(R_n^{p+1})$. Then

$$f(z) = \sum_{\nu=0}^{p} \frac{z^\nu}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z-a_n} + \frac{1}{a_n} + \frac{z^2}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}} \right).$$

To prove it, we apply the theorem of residue to the integral

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(\zeta)}{\zeta^{p+1}(\zeta - z)} d\zeta.$$

The residue at $\zeta = 0$ is obtained by expanding $\frac{f(\zeta)}{\zeta - z}$ in Laurent series in ζ around $\zeta = 0$. We have

$$\frac{f(\zeta)}{\zeta^{p+1}(\zeta - z)} = \frac{-1}{\zeta^{p+1}} \left(\frac{1}{z} + \frac{\zeta}{z^2} + \frac{\zeta^2}{z^3} + \cdots \right) \left(f(0) + f'(0)\zeta + \frac{1}{2} f''(0)\zeta^2 + \cdots \right)$$

and the coefficient of $\frac{1}{\zeta}$ is

$$-\frac{1}{z} \left(\frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!} \right).$$

The residue at $\zeta = z$ is given by $\frac{f(z)}{z^{p+1}}$. The residue at a_n is $\frac{b_n}{a_n^{p+1}(a_n - z)}$. Since as $n \to \infty$ the integral becomes zero, we get

$$-\frac{1}{z} \left(\frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!} \right) + \frac{f(z)}{z^{p+1}} + \sum_{n=1}^{\infty} \frac{b_n}{a_n^{p+1}(a_n - z)} = 0,$$

which means that

$$\begin{aligned} f(z) &= f(0) + z\, f'(0) + \cdots + \frac{z^p}{p!} f^{(p)}(0) + \sum_{n=1}^{\infty} \frac{b_n z^{p+1}}{a_n^{p+1}(a_n - z)} \\ &= \sum_{\nu=0}^{p} \frac{z^\nu}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z-a_n} + \frac{1}{a_n} + \frac{z^2}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}} \right). \end{aligned}$$

The last expression comes from writing

$$z^{p+1} = (z^{p+1} - a_n^{p+1}) + a_n^{p+1}$$

and then factoring

$$z^{p+1} - a_n^{p+1} = (z - a_n)\sum_{\nu=0}^{p} z^\nu a_n^{p-\nu}.$$

Note that

$$\frac{1}{a_n} + \frac{z^2}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}}$$

is the p-th Taylor polynomial for the function

$$\frac{1}{z - a_n}$$

at $z = 0$ from geometric series expansion.

An Example of Partial Fraction Expansion. The meromorphic function is $f(z) = \cot z - \frac{1}{z}$ and C_n is the square with corners at $(n + \frac{1}{2})(\pm 1 \pm i)\pi$. When $|y| > \frac{\pi}{2}$ we have

$$|\cot z| \leq \left|\frac{e^{2iz}+1}{e^{2iz}-1}\right| \leq \frac{e^{2y}+1}{e^{2y}-1} = 1 + \frac{2}{e^{2y}-1} \leq 1 + \frac{2}{e^{\pi}-1}$$

and hence uniformly bounded. The function $\cot z$ is bounded on the line joining $\frac{1}{2}(1-i)\pi$ to $\frac{1}{2}(1+i)\pi$ and we can use periodicity and conclude that $\operatorname{cosec} z$ is uniformly bounded on C_n. Hence $\cot z - \frac{1}{z} = \sum_{n\in\mathbb{Z}-\{0\}}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)$ because the residue of $\cot z$ at $n\pi$ is 1.

Infinite Product Expansion. From an infinite product expansion, one can get a partial fraction expansion by taking the derivative of the logarithm of the infinite product. This process can be reversed to yield an infinite product expansion of an entire function from a partial fraction expansion of the logarithmic derivative of the entire function. As an example, we derive the infinite product expansion of $\sin \pi z$ by using the partial fraction expansion of $\pi \cot \pi z = \frac{d}{dz} \log \sin \pi z$. By replacing z by πz in the above partial fraction expansion, we get

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\in\mathbb{Z}-\{0\}}\left(\frac{1}{z-n} + \frac{1}{n}\right),$$

from which the infinite product expansion of $\sin \pi z$ will derived.

We are going to exponentiate the indefinite integral of

$$\frac{d}{dz}\left(\log\frac{\sin\pi z}{\pi z}\right)=\pi\cot\pi z=\sum_{n\in\mathbb{Z}-\{0\}}\left(\frac{1}{z-n}+\frac{1}{n}\right).$$

Since

$$\frac{d}{dz}\log\left(\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}\right)=\frac{1}{z-n}+\frac{1}{n},$$

when we integrate from $z=0$ to z, we get

$$\int_{\zeta=0}^{\zeta=z}\left(\frac{1}{\zeta-n}+\frac{1}{n}\right)d\zeta=\log\left(\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}\right).$$

After exponentiating, we obtain

$$\frac{\sin\pi z}{\pi z}=C\prod_{n\in\mathbb{Z}-\{0\}}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}.$$

Passing to limit as $z\to 0$, we determine C to be 1 and get

$$\sin\pi z=\pi z\prod_{n\in\mathbb{Z}-\{0\}}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}.$$

Evaluation of Sum of Values of Rational Functions at Integral Points by Residues and Cotangent Function. Suppose $f(z)$ is a rational function whose poles are simple nonintegers $a_1,\cdots,a_k$ with residues $b_1,\cdots,b_k$ such that the degree of the denominator of f is at least two more than that of its numerator. Let C_n be the square with corners at $(n+\frac{1}{2})(\pm 1\pm i)$. The integral

$$\int_{C_n}\pi\ \cot\pi z\,f(z)dz$$

goes to zero as $n\to\infty$. The use residues in evaluating

$$\int_{C_n}\pi\ \cot\pi z\,f(z)dz$$

and passing to limit as $n\to\infty$ yields

Theorem. For a rational function with simple poles at nonintegers $a_1, \cdots, a_k$ with residues $b_1, \cdots, b_k$,

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{\nu=1}^{k} b_\nu \cot\, \pi\, a_\nu.$$

If we use $\operatorname{cosec} \pi z$ instead of $\cot\, \pi z$ we can obtain the sum of the series $\sum_{n=-\infty}^{\infty} (-1)^n f(n)$.

Theorem. For a rational function with simple poles at nonintegers $a_1, \cdots, a_k$ with residues $b_1, \cdots, b_k$,

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{\nu=1}^{k} b_\nu \operatorname{cosec} \pi\, a_\nu.$$

As an example, we sum the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}$$

for $a > 0$ by using the rational function

$$f(z) = \frac{1}{z^2 + a^2}$$

which has simple poles at $z = ai$ and at $z = -ai$ with residues respectively $\frac{-i}{2a}$ and $\frac{i}{2a}$ to get

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -\pi \left(\frac{-i \cot \pi ai}{2a} + \frac{i \cot \pi\, (-ai)}{2a} \right) = \frac{\pi}{a} \coth \pi a.$$