# PRIME NUMBER THEOREM

**Overview of Ingredients in Proof of Prime Number Theorem.** Before we prove the Prime Number Theorem, we present an overview of various ingredients which go into its proof, in order to understand the proof when it is presented. The Prime Number Theorem states that $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$, where $\pi(x)$ is the number of primes $< x$.

The most important ingredient is the product formula for the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ \text{prime}} \frac{1}{1 - \frac{1}{p^s}}.$$

All primes $p$ occur inside the infinite product.

Infinite products are handled by transformation to infinite sums by the logarithm function. We get

$$\log \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^s},$$

where $\Lambda_1(n)$ is equal to $\frac{1}{m}$ if $n = p^m$ for some prime $p$ and is otherwise 0.

Because of the factor $\log x$ in the denominator of $\pi(s) \sim \frac{x}{\log x}$, to make $\log p$ appear in $\log n = m \log p$ which comes from differentiating $n^s$ with respect to $s$, we take the derivative of $\log \zeta(s)$ to get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some } m \in \mathbb{N} \text{ and some prime number } p \\ 0 & \text{otherwise,} \end{cases}$$

which is known as the von Mangoldt function. Let $\psi(x) = \sum_{n<x} \Lambda(n)$, known as the second Chebyshev function A simple inequality argument shows that $\pi(x) \sim \frac{x}{\log x}$ is equivalent to $\psi(x) \sim x$.

As the Dirichlet-series analogue of using the Cauchy kernel to single out terms in a power series, Perron's formula

$$\sum_{n<x}\frac{a_n}{n^s}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s+w)\frac{x^w}{w}\,dw,$$

singles out the partial sum of the Dirichlet series $f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ with abscissa of convergence $\sigma_0$, where $x>0$ is a non-integer positive $x$ and $c>0$ and $\sigma=\operatorname{Re}s>\sigma_0-c$. Applied to the Dirichlet series $f(s)=-\frac{\zeta'(s)}{\zeta(s)}$ with $\sigma_0=1$ and $\sigma=0$, Perron's formula yields

$$\psi(x)=\sum_{n<x}\Lambda(n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds \quad \text{for } \ c>1,$$

after changing the notation $w$ to $s$.

The integral over the vertical line from $c-i\infty$ to $c+i\infty$ can be evaluated by Cauchy's residue theory, just like the evaluation of a definite integral over the horizontal real line $\mathbb{R}$ by completing the interval integral $[-A,A]$ (with $A$ eventually approaching $\infty$) to a closed contour by adding the upper semicircle of radius $A$ centered at the origin and then evaluating the sum of residues inside the closed contour and verifying that the contribution from integration over the artificially added upper semicircle goes to 0 as $A\to\infty$.

For the situation at hand, (with fixed $x$) we consider the line segment $[c-i\hat{T},c+i\hat{T}]$ with $\hat{T}$ eventually approaching $\infty$ and complete it to a closed contour by adding a curve $C_{\hat{T}}$ located to the left of the right half-plane $\operatorname{Re}s>c$. Then we evaluate the sum of the residues of

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

inside the closed contour to be $x$ and verify that the contribution from the integration over the artificially added curve $C_{\hat{T}}$ is of the order $\mathrm{o}(x)$ as $x\to\infty$.

For $\operatorname{Re}s>1$, by comparing the discrete version of the Dirichlet series of $\zeta(s)$ with the continuous version of an integral

$$\int_{t=1}^{\infty}\frac{dt}{t^s}=\left.\frac{t^{-s+1}}{-s+1}\right|_{t=1}^{\infty}=\frac{1}{s-1},$$

we know that after extending $\zeta(s)$ to $\mathbb{C}$ by any of the methods available (*e.g.*, repeated integration by parts to applied to a Mellin transform formula, functional equation of reflection with respect $\operatorname{Re} s = \frac{1}{2}$, etc.), the extension of $\zeta(s)$ is holomorphic on $\mathbb{C} - \{1\}$ with a simple pole at $s = 1$ of residue 1.

To apply Cauchy's residue theory with the contour $[c - i\hat{T}, c + i\hat{T}] \cup C_{\hat{T}}$ to verify $\psi(x) \sim x$ via the Perron formula, we have to make sure that $s = 1$ is the only pole of $\frac{\zeta'(s)}{\zeta(s)}$ (with a good choice of $C_{\hat{T}}$) by showing that $\zeta(s)$ has no zero on $\operatorname{Re} s = 1$. This is done by using a simple completion-of-square formula

$$3 + 4\cos t + \cos 2t = 2\left(1 + \cos t\right)^2 \geq 0$$

and the fact that $s = 1$ is a simple pole of $\zeta(s)$. A good choice for $C_{\hat{T}}$ is given in the following figure and is composed of the broken line

$$\begin{aligned}&\text{from } c - i\hat{T} \ \text{ to } \ 1 - i\hat{T}, \ \text{to } \ 1 - iT, \ \text{to } \ 1 - \delta_T - iT, \ \text{to} \\ &1 - \delta_T + iT, \ \text{to } 1 + iT, \ \text{to } 1 + i\hat{T} \ \text{ and finally to } \ c + i\hat{T},\end{aligned}$$

where the positive number $\delta_T$ is chosen to be sufficiently small so that the closed rectangle marked as $R_T$ contains no zero of $\zeta(s)$.

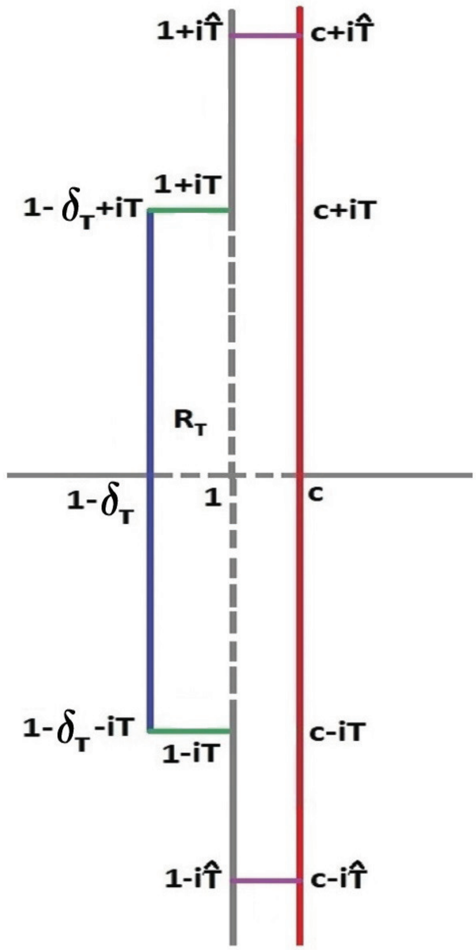


Figure 1: Good Closed Contour with Artificially Added Curve for Application of Residue Theory

Finally to verify that the integral over $C_{\hat{T}}$ contributes only $\mathrm{o}(x)$ as $x \to \infty$, we have to use some subtle estimates of the Dirichlet series $\zeta(s)$ along vertical lines.

The technical modification of replacing $\psi(x) \sim x$ by $\psi_1(x) \sim \frac{x^2}{2}$ is also introduced to make some of the estimates easier to handle, where

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^{s+1}}{s(s+1)}\, ds \quad \text{for } c > 1$$

is the integral of $\psi(x)$ with respect to $x$.

**Detailed Proof of Prime Number Theorem.** Now we are going to prove the Prime Number Theorem by following the above overview and provide the details. The Prime Number Theorem gives the first term (*i.e.*, the dominant term) for the asymptotic behavior of the prime-counting function $\pi(x)$, whose value at $x \in \mathbb{R}$ is the number of primes not exceeding $x$. This first term is

$$\frac{x}{\log x},$$

which means that the asymptotic distribution of prime numbers in the set of all natural numbers is off by the factor of the reciprocity of the logarithmic function. The Riemann Hypothesis conjectures what the order of the next term should be. Both the Prime Number Theorem and the Riemann Hypothesis can be translated into formulations concerning the location of the zeroes of the Riemann Zeta function. The proof of the Prime Number Theorem starts with the step of taking the logarithmic derivative of the Euler product formula.

**Relation Between Prime Counting and Partial Sum of Coefficients of Logarithmic Derivative of Euler Product Formula of Riemann Zeta Function.** By taking the logarithm of the Euler product formula

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}},$$

we get

$$\begin{aligned} \log \zeta(s) &= - \sum_{p \text{ prime}} \log \left(1 - \frac{1}{p^s}\right) \\ &= \sum_{p \text{ prime}} \left( \sum_{m=1}^{\infty} \frac{1}{m\left(p^s\right)^m} \right) = \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^s}, \end{aligned}$$

where $\Lambda_1(n)$ is equal to $\frac{1}{m}$ if $n = p^m$ for some prime $p$ and is otherwise 0.

By differentiating with respect to $s$, we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } \ n = p^m \ \ \text{for some } m \in \mathbb{N} \text{ and some prime number } p \\ \\ \ \ 0 & \text{otherwise} \end{cases}$$

is called the *von Mangoldt function* (or sometimes simply called the Lambda function). Note that

$$\Lambda(n) = \frac{\Lambda_1(n)}{\log n},$$

where the factor $\log n$ comes from the derivative of $n^s$ with respect to $s$ when $\log \zeta(s)$ is differentiated with respect to $s$. The partial sum

$$\psi(x) := \sum_{n<x} \Lambda(n)$$

is called the second Chebyshev function (also called the summatory von Mangoldt function).

As we have just seen, $\psi(x)$ can be evaluated by the Perron formula. It is a bridge to the prime-counting function $\pi(x)$ defined as the number of primes $< x$. We now reformulate the Prime Number Theorem in terms of the function $\psi(x)$ so that the Prime Number Theorem

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \ \ x \to \infty$$

can be proved by proving that

$$\psi(x) \sim x \quad \text{as} \ \ x \to \infty.$$

The details just involve some simple arithmetic manipulations and are given as follows.

**Reformulation of Prime Number Theorem in Terms of Second Chebyshev Function.** The statement $\psi(x) \sim x$ implies the statement

$$\pi(x) \sim \frac{x}{\log x}$$

for the following reason. First of all, $\Lambda(n)$ is equal to $\log p$ when $n$

$$
\begin{aligned}
\psi(x) &= \sum_{n<x} \Lambda(n) = \sum_{\substack{p \text{ prime}, m \in \mathbb{N}, \\ p^m < x}} \log p \\
&= \sum_{p \text{ prime}, \ p<x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \\
&\qquad \text{(because the number of times } \log p \text{ occurs is } m \text{ with } p^m < x) \\
&\leqslant \sum_{p \text{ prime}, \ p<x} \frac{\log x}{\log p} \log p = \pi(x) \log x
\end{aligned}
$$

implies

$$\frac{\psi(x)}{x} \leqslant \pi(x) \frac{\log x}{x}$$

and

$$1 = \lim_{x \to \infty} \frac{\psi(x)}{x} \leqslant \liminf_{x \to \infty} \pi(x) \frac{\log x}{x}.$$

On the other hand, for any $0 < \alpha < 1$ from

$$
\begin{aligned}
\psi(x) \geqslant \sum_{p \text{ prime}, \ p<x} \log p &\geqslant \sum_{p \text{ prime}, \ x^\alpha \leqslant p < x} \log p \\
&\geqslant \left(\pi(x) - \pi(x^\alpha)\right) \log x^\alpha \geqslant \left(\pi(x) - x^\alpha\right) \alpha \log x
\end{aligned}
$$

it follows that

$$1 = \lim_{x \to \infty} \frac{\psi(x)}{x} \leqslant \limsup_{x \to \infty} \left(\pi(x) - x^\alpha\right) \alpha \frac{\log x}{x} = \alpha \limsup_{x \to \infty} \pi(x) \frac{\log x}{x}$$

for any $0 < \alpha < 1$ and

$$1 \geqslant \limsup_{x \to \infty} \pi(x) \frac{\log x}{x}.$$

Hence

$$\lim_{x \to \infty} \pi(x) \frac{\log x}{x} = 1.$$

The only important step is the technique of arguing with an arbitrary $0 < \alpha < 1$ which is to approach 1 at the end.

**Contour of Integration to Obtain Asymptotic Behavior of Second Chebyshev Function from Perron Formula.** By applying the special case of the

Perron formula to the Dirichlet series of

$$-\frac{\zeta'(s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

for $\operatorname{Re} s>1$, we get

$$\psi(x)=\sum_{n<x}\Lambda(n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds\quad\text{for }\ c>1.$$

The point $s=1$ is a simple pole of the Riemann Zeta function and therefore a simple pole of

$$\frac{\zeta'(s)}{\zeta(s)}$$

with residue 1. It means that the residue of

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

is $x$ at $s=1$, which implies that the integral

$$\frac{1}{2\pi i}\int_{|s-1|=\varepsilon}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

is equal to $x$ for any sufficiently small $\varepsilon>0$.

The trouble is that the integral here is

$$\int_{|s-1|=\varepsilon}$$

over a small circle of radius $\varepsilon$ centered at $s=1$, whereas we need an integral over a contour, as shown in the figure below, which contains the vertical line-segment $[c-iT,c+iT]$ so that when we let $T\to\infty$, the contribution from the remaining part $C_T$ of the contour can be ignored as $x\to\infty$, which means $o(x)$ as $x\to\infty$.

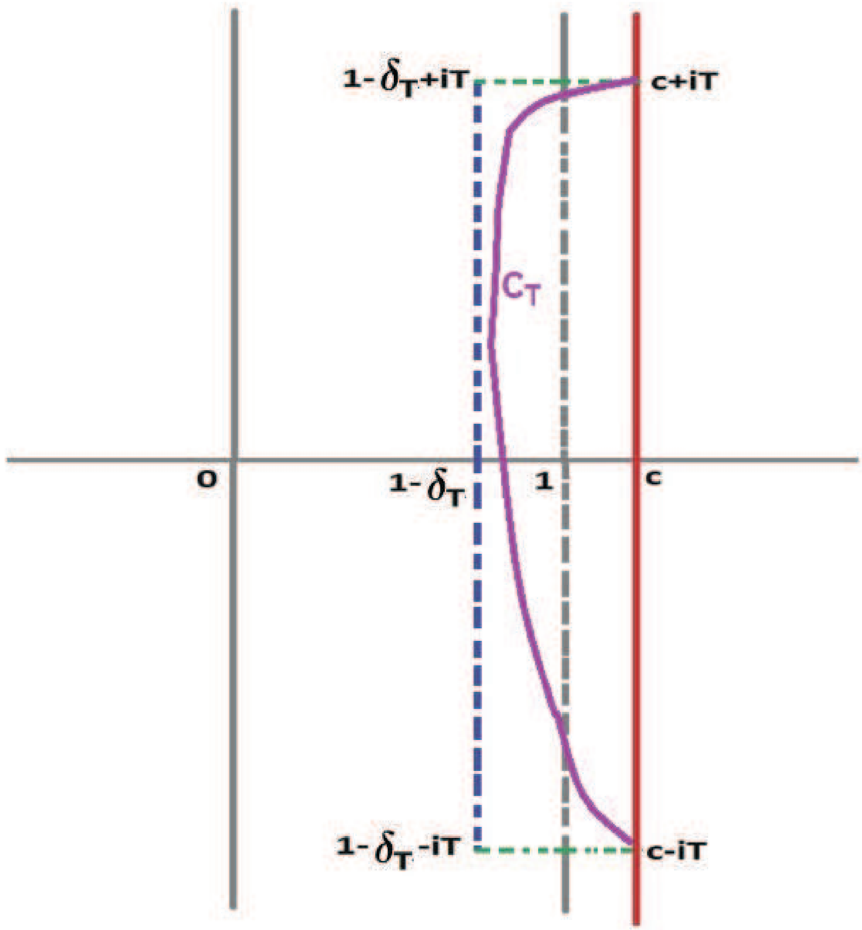


Figure 2: Contour Needed for Application of Residue Theory to Proof of Prime Number Theorem

For this approach to work, besides the residue $x$ at $s=1$ there should not be any additional residue of

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

should be inside the domain bounded by the contour composed of $C_T$ and $[c-iT, c+iT]$. It means that $\zeta(s)$ (which is holomorphic on $\mathbb{C}-\{1\}$) should not have any zeroes inside the domain bounded by the contour of $C_T$ and

$[c-iT, c+iT]$. By choosing $\delta_T > 0$ sufficiently small, this can be achieved if we know that $\zeta(s)$ has no zeroes on the vertical line $\mathrm{Re}\, s = 1$. We now prove that $\zeta(s)$ is nowhere vanishing on $\mathrm{Re}\, s = 1$.

**Nonvanishing of $\zeta(s)$ on $\mathrm{Re}\, s = 1$.** The key step in the argument is the following inequality of trigonometric functions

$$\begin{aligned}&3 + 4\cos t + \cos 2t \\ &= 3 + 4\cos t + \big(2\cos^2 t - 1\big) \\ &= 2\,(1+\cos t)^2 \geq 0.\end{aligned}$$

A consequence of this inequality is that for $\sigma > 1$,

$$\begin{aligned}&\log|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \\ &= 3\mathrm{Re}\,\log\zeta(\sigma) + 4\mathrm{Re}\,\log\zeta(\sigma+it) + \mathrm{Re}\,\log\zeta(\sigma+2it) \\ &= \sum_{n=1}^{\infty}\Lambda_1(n)\left(3\mathrm{Re}\,\frac{1}{n^\sigma} + 4\mathrm{Re}\,\frac{1}{n^{\sigma+it}} + \mathrm{Re}\,\frac{1}{n^{\sigma+2it}}\right) \\ &= \sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^\sigma}\,(3 + 4\cos t + \cos 2t) \geqslant 0.\end{aligned}$$

We now verify the nowhere vanishing of $\zeta(s)$ on $\mathrm{Re}\, s = 1$ by arguing by contradiction. Suppose $\zeta(s)$ vanishes at some point of $s = 1 + it_0$ of $\mathrm{Re}\, s = 1$, where $t_0$ is some nonzero real number. Then

$$\zeta(\sigma + it_0) = O((\sigma - 1)) \quad \text{as} \quad \sigma \to 1^+.$$

Since $\zeta(s)$ is holomorphic at $1 + 2it_0$, it follows that

$$\zeta(\sigma + 2it_0) = O(1) \quad \text{as} \quad \sigma \to 1^+.$$

Because $s = 1$ is a simple pole of $\zeta(s)$, we know that

$$\zeta(\sigma) = O\left(\frac{1}{\sigma - 1}\right) \quad \text{as} \quad \sigma \to 1^+.$$

Putting these three estimates together, we get

$$\zeta(\sigma)^3\zeta(\sigma+it_0)^4\zeta(\sigma+2it_0) = O\left(\frac{(\sigma-1)^4}{(\sigma-1)^3}\right) = O(\sigma - 1)$$

as $\sigma \to 1+$. This contradicts

$$|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \geqslant 1 \quad \text{for} \quad \sigma > 1 \quad \text{and} \quad t \in \mathbb{R}.$$

This finishes the proof of the nowhere vanishing of $\zeta(s)$ on $\mathrm{Re}\, s = 1$.

To make the contribution of the integral over $C_T$ to be of order $o(x)$ after passing to limit $T \to \infty$, we need to choose $C_T$ in a special way and we need some information on the the vertical growth order of

$$-\frac{\zeta'(s)}{\zeta(s)}.$$

It means that we need the vertical growth order estimate for $\zeta'(s)$ as well as the vertical lower bound estimate for $\zeta(s)$. Recall the following three results on the vertical growth order of $\zeta(s)$ and its derivative $\zeta'(s)$, which are proved in our lecture notes on Dirichlet series.

(1) Inside and near the right edge of the critical strip for the Riemann zeta function,

$$\zeta(s) = O\left(|t|^{1-\sigma+\varepsilon}\right) \quad \text{for } 0 < \sigma \leq 1 \text{ and } \varepsilon > 0.$$

(2) To the right of the critical strip for the Riemann zeta function,

$$|\zeta(\sigma + it)| \leqslant C_\varepsilon |t|^\varepsilon \quad \text{for } \sigma \geqslant 1 \text{ and } |t| \geqslant 1$$

for some $C_\varepsilon > 0$ depending on $\varepsilon$ for $\varepsilon > 0$.

(3) To the right of the critical strip for the derivative of the Riemann zeta function,

$$|\zeta'(\sigma + it)| \leqslant C'_\varepsilon |t|^\varepsilon \quad \text{for } \sigma \geqslant 1 \text{ and } |t| \geqslant 1$$

for some $C'_\varepsilon > 0$ depending on $\varepsilon$ for $\varepsilon > 0$.

We now discuss the vertical lower bound estimate for $\zeta(s)$ which we need in order to get a vertical growth estimate for

$$-\frac{\zeta'(s)}{\zeta(s)}.$$

**Lower Bound of Riemann Zeta Function on Vertical Line.** We are going to show that for any $\varepsilon > 0$ there exists some $c^\flat_\varepsilon > 0$ (depending on $\varepsilon$) such that

$$|\zeta(\sigma + it)| \geqslant c^\flat_\varepsilon |t|^{-\varepsilon}$$

for $\sigma \geqslant 1$ and $|t| \geqslant 1$.

*Use of Simple Trigonometric Inequality to Obtain Good and Bad Contributions.* An important step is to use the inequality

$$|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)|\geqslant 1 \quad \text{for} \quad \sigma>1,$$

obtained from the simple trigonometric inequality

$$\begin{aligned}&3+4\cos t+\cos 2t\\&=3+4\cos t+\left(2\cos^2 t-1\right)\\&=2\,(1+\cos t)^2\geq 0.\end{aligned}$$

This will be coupled with

$$|\zeta(\sigma)|\leqslant\frac{C}{\sigma-1} \quad \text{for} \quad \sigma>1$$

(from the simple pole of $\zeta(s)$ at $s=1$) and the upper bound estimate

$$|\zeta(\sigma+2it)|\leqslant C_\varepsilon|t|^{4\varepsilon}$$

for $|t|\geqslant 1$ and $\sigma\geqslant 1$ for some positive constant $C_\varepsilon$ for $\varepsilon>0$, so as to get a lower bound estimate for $|\zeta(\sigma+it)|$. It is not completely straightforward. A rather subtle step is involved.

A straightforward estimate is

$$|\zeta(\sigma+it)|\geqslant\frac{1}{(C^3C_\varepsilon)^{\frac{1}{4}}}\frac{(\sigma-1)^{\frac{3}{4}}}{|t|^\varepsilon}$$

for $|t|\geqslant 1$ and $\sigma\geqslant 1$, in which there is a good factor $|t|^\varepsilon$ and a bad factor $(\sigma-1)^{\frac{3}{4}}$. The strategy is to absorb the bad factor $(\sigma-1)^{\frac{3}{4}}$ at the expense of replacing the good factor $|t|^\varepsilon$ by another factor $|t|^{(1+\gamma)\varepsilon}$ less good but still good for some $\gamma>0$. That is, we would like to get the result of

$$|\zeta(\sigma+it)|\geqslant c_1\frac{1}{|t|^{(1+\gamma)\varepsilon}},$$

with $\gamma>0$ to be determined later.

*Dividing $(\sigma,\tau)$-Space into Good and Bad Sets.* In the space of the two variables $\sigma$, $\tau$, we consider the division into two sets and the division depends on $\gamma$. The first set (which is the good set) is defined by the inequality $(\sigma-1)^{\frac{3}{4}} \geqslant \frac{1}{|t|^{\gamma\varepsilon}}$, while the second set (which is the bad set) is defined by $(\sigma-1)^{\frac{3}{4}} < \frac{1}{|t|^{\gamma\varepsilon}}$. When $(\sigma-1)^{\frac{3}{4}} \geqslant \frac{1}{|t|^{\gamma\varepsilon}}$ (*i.e.*, in the good set), the inequality

$$|\zeta(\sigma+it)| \geqslant c_1 \frac{1}{|t|^{(1+\gamma)\varepsilon}},$$

clearly holds with

$$c_1 = \frac{1}{(C^3 C_\varepsilon)^{\frac{1}{4}}}.$$

*Moving a Point in Bad Set to Nearest Point in Good Set and Estimating the Effect by Mean Value Theorem and Vertical Growth Estimate of Derivative of Riemann Zeta Function.* When $(\sigma-1)^{\frac{3}{4}} < \frac{1}{|t|^{\gamma\varepsilon}}$ (which is the bad set), we move along the fixed value of $t$ to get to the good set by letting $\sigma' > \sigma$ be the smallest number $> 1$ which satisfies $(\sigma'-1)^{\frac{3}{4}} = \frac{1}{|t|^{\gamma\varepsilon}}$ so that

$$|\zeta(\sigma'+it)| \geqslant c_1 \frac{1}{|t|^{(1+\gamma)\varepsilon}}.$$

Then $\sigma'-\sigma \leqslant \sigma'-1 = \frac{1}{|t|^{\frac{4\gamma}{3}\varepsilon}}$. We are going to use this estimate of $\sigma'-\sigma$ to apply the mean value theorem and the vertical growth order for $\zeta'(\sigma+it)$ to get the required lower bound for $|\zeta(\sigma+it)|$ from the lower bound of $|\zeta(\sigma'+it)|$.

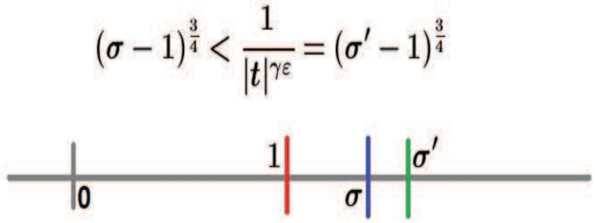


Figure 3: Choice of Point on Real Line for Proof of Vertical Lower Estimate of $\zeta(s)$

We now choose $\gamma > 0$ such that $\frac{4\gamma}{3} > 1 + \gamma$ (which means $\gamma > 3$). Choose $0 < \delta < \left(\frac{4\gamma}{3} - (1+\gamma)\right)\varepsilon$. By the upper bound estimate

$$|\zeta'(\sigma + it)| \leqslant c'|t|^{\delta}$$

for $\sigma \geqslant 1$ and $|t| \geqslant 1$ (where $c' > 0$ depends on $\delta$), we have from the mean value theorem

$$|\zeta(\sigma + it) - \zeta(\sigma' + it)| \leqslant c'' \frac{1}{|t|^{\frac{4\gamma}{3}\varepsilon}} |t|^{\delta} = c'' \frac{1}{|t|^{\frac{4\gamma}{3}\varepsilon - \delta}}$$

so that

$$\begin{aligned} |\zeta(\sigma + it)| &\geqslant |\zeta(\sigma' + it)| - |\zeta(\sigma + it) - \zeta(\sigma' + it)| \\ &\geqslant c_1 \frac{1}{|t|^{(1+\gamma)\varepsilon}} - c'' \frac{1}{|t|^{\frac{4\gamma}{3}\varepsilon - \delta}} \\ &\geqslant c^{\#} \frac{1}{|t|^{(1+\gamma)\varepsilon}}, \end{aligned}$$

because $\frac{4\gamma}{3}\varepsilon - \delta > (1+\gamma)\varepsilon$.

Note that the exponent $\delta$ in the vertical growth estimate of the derivative of the Riemann zeta function with $\frac{4\gamma}{3}\varepsilon - \delta > (1+\gamma)\varepsilon$ is chosen to make sure that the effect of the use of the mean value theorem would not affect the final conclusion.

Since the constant $c^{\#}$ is independent of $\sigma > 1$, we can go to the limit $\sigma \to 1$ to get

$$|\zeta(\sigma + it)| \geqslant c^{\#} \frac{1}{|t|^{(1+\gamma)\varepsilon}}$$

for $\sigma \geqslant 1$ and $|t| \geqslant 1$. This finishes the derivation of the lower bound estimate of the Riemann zeta function on a vertical line.

**Contour of Integration to Obtain Asymptotic Behavior of Second Chebyshev Function from Perron Formula.** The application of the Perron formula to the Dirichlet series $-\frac{\zeta'(s)}{\zeta(s)}$ yields

$$\psi(x) = \sum_{n<x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}\, ds \quad \text{for } c > 1.$$

To prove $\psi(x) \sim x$ as $x \to \infty$, now that we know the nowhere vanishing of $\zeta(s)$ on $\operatorname{Re} s = 1$, one hopes to choose some contour $C_T \cup [c - iT, c + iT]$, inside the rectangle

$$1 - \delta_T \leq \operatorname{Re} s \leq c, \quad -T \leq \operatorname{Im} s \leq T$$

(with $\delta_T > 0$ sufficiently small) where $\zeta(s)$ is nowhere zero, so that from the simple pole of $\zeta(s)$ at $s = 1$ the application of residue calculus together with vertical growth estimates of

$$\frac{\zeta'(s)}{\zeta(s)}$$

would yield $\psi(x) \sim x$ as $x \to \infty$.

When it comes to the estimate of an integral along a vertical line-segment of length $L$ which goes to infinity, the quotient

$$\frac{ds}{s}$$

contributes order $\log L$. Even along the vertical line with $\sigma \geq 1$ the growth rate of

$$\frac{\zeta'(s)}{\zeta(s)}$$

is known to be $C_\varepsilon |t|^\varepsilon$ for any $\varepsilon > 0$, with contribution of order $L^\varepsilon$ in an integral along a vertical line-segment of length $L$, we would have trouble to get a useful estimate for the integral

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}$$

over a vertical length $L$ which goes to infinity.

To facilitate such a kind of estimate, it would be better if we can replace the integral of

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}\, ds$$

by the integral of

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^{s+1}}{s(s+1)}\, ds$$

which is obtained by integration with respect to $x$, because in the latter integral along a vertical line-segment of length $L$ which goes to infinity, the quotient

$$\frac{ds}{s(s+1)}$$

contributes order $\frac{1}{L}$.

This motivates us to define

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}\,ds \quad \text{for } c>1$$

and to expect that the proof of the statement $\psi(x) \sim x$ is replaced by the proof of the statement

$$\psi_1(x) \sim \frac{x^2}{2} \quad \text{as } x \to \infty.$$

Note that, instead of integrating

$$\psi(x) = \sum_{n<x}\Lambda(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds \quad \text{for } c>1$$

once with respect to $x$, we can also repeat the proof of the Perron formula with

$$\frac{x^s}{s}$$

replaced by

$$\frac{x^{s+1}}{s(s+1)}$$

to directly prove that

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}\,ds \quad \text{for } c>1,$$

with

$$\frac{d}{dx}\psi_1(x) = \psi(x).$$

The proof of the Prime Number Theorem is now reduced to evaluating the definite integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}\,ds \quad \text{for } c>1$$

to show that its asymptotic value is $\frac{x^2}{2}$ as $x\to\infty$.

The modified Perron lemma is the following with $\frac{ds}{s}$ changed to $\frac{ds}{s(s+1)}$.

*Lemma.* Let $c>0$.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{n}\right)^s\frac{ds}{s(s+1)}$$

is equal to $1-\frac{n}{x}$ if $n<x$ and is equal to $0$ if $n>x$.

*Proof of Lemma.* Let $a=\frac{x}{n}$. The identity can be rewritten as follows.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^s}{s(s+1)}\,ds$$

is equal to $1-\frac{1}{a}$ if $a>1$ and is equal to $0$ if $0<a<1$. As in the original Perron lemma, the verification is done in the following two cases separately.

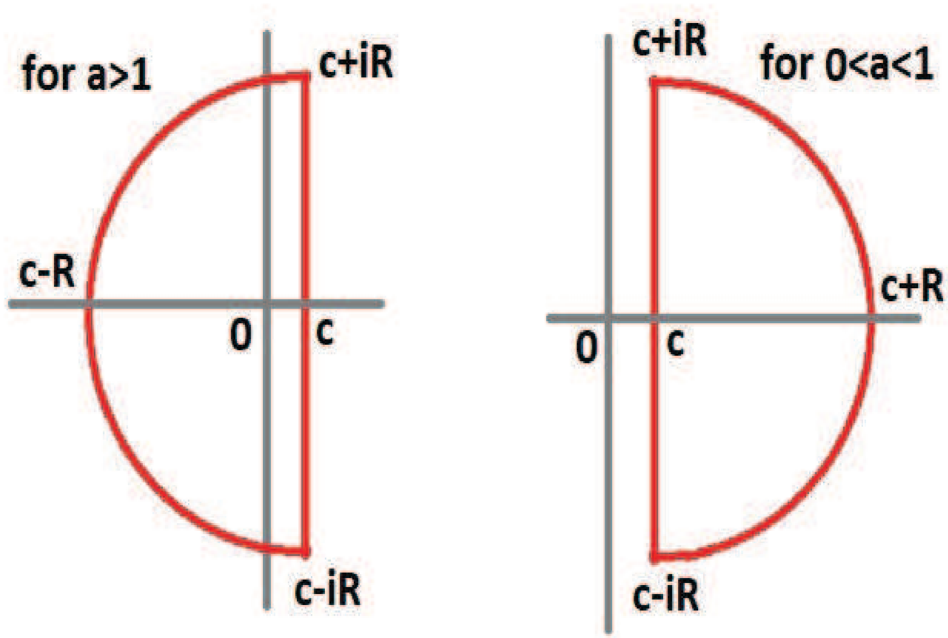


Figure 4: Contours for Proof of Perron's Lemma After Changing $\frac{ds}{s}$ to $\frac{ds}{s(s+1)}$

Assume first $a > 1$. Then $\log a > 0$ and we do the integration from $c - iR$ to $c + iR$ and then the left circle $C_R$ whose diameter is on $[c - iR, c + iR]$. For $x < n$ the integral along $C_R$ goes to zero as $R \to \infty$ for the following reason. Given any $\varepsilon > 0$ there exists some $A = A_\varepsilon$ such that the integral over $C_R \cap \{\mathrm{Re}\, s < -A\}$ is less than $\varepsilon$, because each

$$|a^s| < e^{-A \log a}$$

on $\{\mathrm{Re}\, s < -A\}$ and, for fixed $A$, the expression

$$\left| \frac{1}{s(s+1)} \right| = O\left(\frac{1}{R^2}\right)$$

on $C_R \cap \{\mathrm{Re}\, s \geqslant -A\}$ as $R \to \infty$ while $|a^s|$ on $C_R \cap \{\mathrm{Re}\, s \geqslant -A\}$ remains bounded as $R \to \infty$. Finally the pole of $\frac{a^s}{s(s+1)}$ at $s = 0$ contributes the residue 1 and the pole of $\frac{a^s}{s(s+1)}$ at $s = -1$ contributes the residue $\frac{-1}{a}$. Thus the integral we seek is $1 - \frac{1}{a}$.

Assume now $a > 1$. Then $\log a < 0$ and we use the right half-circle whose diameter is on $[c - iR, c + iR]$. In this case there is no pole inside the contour and the integral we seek is 0. This ends the proof of the Lemma.

*Argument of Integration by Parts Unnecessary in Proof of New Version of Perron's Lemma.* The proof of the modified version is actually a tiny little bit easier, because for the original version, in the proving the vanishing of the integral over $C_R$ as $R \to \infty$ when $\frac{ds}{s}$ is used instead of $\frac{ds}{s(s+1)}$, one integration by parts with respect to $s$ is needed to increase the order of the denominator by 1, just like in the application of residue theory to compute

$$\int_{x=-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix} dx$$

when the degree of the denominator polynomial $Q(x)$ is precisely 1 more than that of the numerator polynomial $P(x)$.

**Choice of Abscissa of the Vertical Line for Integration.** For the equation

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^{s+1}}{s(s+1)}\, ds \quad \text{for } \ c > 1$$

(where $\psi_1(x)$ is a primitive of the second Chebyshev function $\psi$) the reason for choosing $c > 1$ is to use the Dirichlet series

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

with uniform absolute convergence on any half-plane $\operatorname{Re} s \geq c$ for any $c > 1$.

We need this convergence to go from

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^{s+1}}{s(s+1)}\, ds$$

to

$$\sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Lambda(n)}{n^s} \frac{x^{s+1}}{s(s+1)}\, ds$$

and then the modified Perron lemma is applied to each term

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Lambda(n)}{n^s} \frac{x^{s+1}}{s(s+1)}\, ds$$

with the weakened condition $c > 0$ to get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Lambda(n)}{n^s}\frac{x^{s+1}}{s(s+1)}\,ds\\&=\Lambda(n)x\left(1-\frac{n}{x}\right)=\Lambda(n)(x-n)\quad\text{for}\;\; n<x\end{aligned}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Lambda(n)}{n^s}\frac{x^{s+1}}{s(s+1)}\,ds=0\quad\text{for}\;\; n>x.$$

Thus, for $x$ not an integer, we finally end up with

$$\sum_{n<x}\Lambda(n)(x-n)=\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Lambda(n)}{n^s}\frac{x^{s+1}}{s(s+1)}\,ds$$

which means that

$$\psi_1(x)=\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Lambda(n)}{n^s}\frac{x^{s+1}}{s(s+1)}\,ds$$

when

$$\psi_1(x)=\int_{t=0}^{x}\psi(t)dt,$$

because

$$\int_{t=0}^{x}\psi(t)dt=\int_{t=0}^{x}\left(\sum_{n<x}\Lambda(n)\right)dt$$

which is derived as follows, by exchanging the procedure of summation over $n$ for $n < x$ and the procedure of integration over $t$ for $0 \leq t \leq x$. We now fix $n$ and integrate with respect to $t$ first and then sum over $n$. When $n$ is fixed, the interval for the integration over $t$ for $n \leq t \leq x$ so that the integration over $t$ yields $x - n$. Now we sum $\Lambda(n)(x - n)$ over $n$ for the range $n < x$ to get

$$\int_{t=0}^{x}\psi(t)dt=\sum_{n<x}\Lambda(n)(x-n).$$

This finishes the verification of the equation

$$\psi_1(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}\,ds\quad\text{for}\;\; c>1$$

**Choice of Contour for Proof of Prime Number Theorem.** We now discuss the choice of the contour $C_T \cup [c - iT,\, c + iT]$ in Figure 1 for the application of residue theory to the proof of the Prime Number Theorem. Since $\zeta(s)$ is nowhere zero on $\operatorname{Re} s = 1$, for any given $T > 0$ we can choose $\delta_T > 0$ sufficiently small so that inside the rectangle

$$1 - \delta_T \leq \operatorname{Re} s \leq c, \quad -T \leq \operatorname{Im} s \leq T$$

the Riemann zeta function $\zeta(s)$ is nowhere zero.

As shown in Figure 4, let $R_T$ be the closed rectangle

$$1 - \delta_T \leq \operatorname{Re} s \leq 1, \quad -T \leq \operatorname{Im} s \leq T.$$

For $\hat{T} > T$ let $E_{\hat{T}}$ be the closed rectangle with vertices $1 \pm i\hat{T}$ and $c \pm i\hat{T}$. For application of residue theory we use the contour $C_T \cup [c - iT,\, c + iT]$ which is the boundary of $R_T \cup E_{\hat{T}}$.

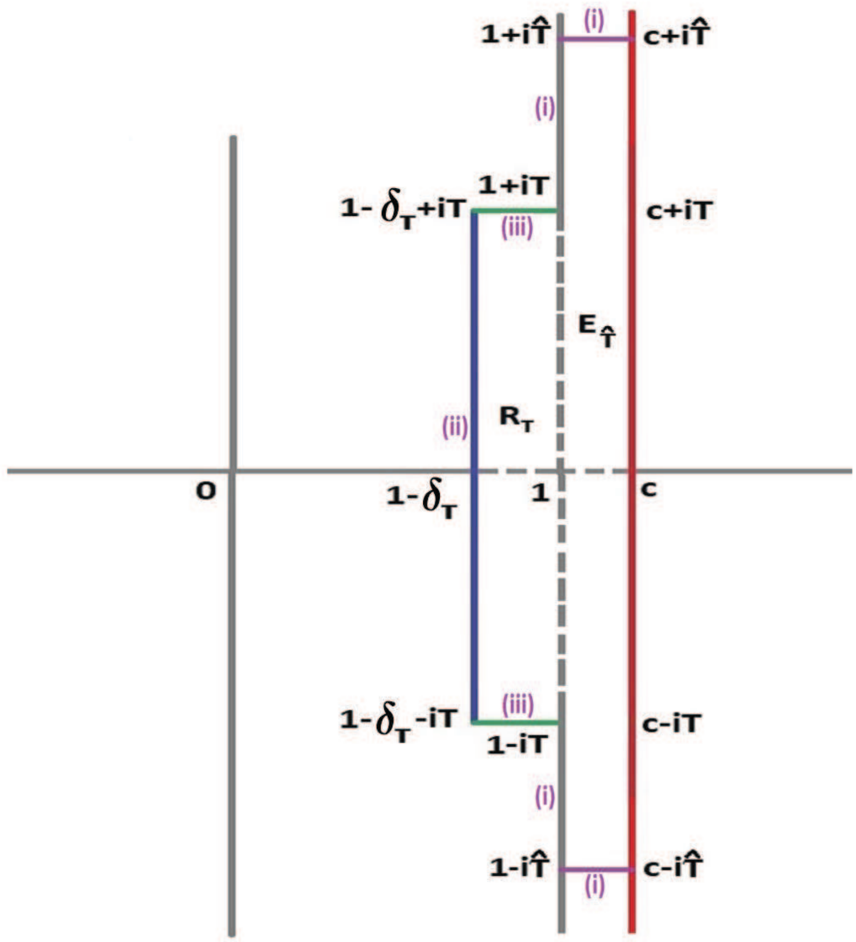


Figure 5: Contour for Application of Residue Theory to Proof of Prime Number Theorem

We claim the following statements. In Figure 4 the segments over which the integrals in Statements (i), (ii), (iii) are taken are respectively labelled by (i), (ii), (iii).

*Statement (i)* For any $\varepsilon > 0$ there exists $T_\varepsilon$ such that the limit of

$$\left(\int_{c+i\hat{T}}^{1+i\hat{T}} + \int_{1+i\hat{T}}^{1+iT} + \int_{1-iT}^{1-i\hat{T}} + \int_{1-i\hat{T}}^{c-i\hat{T}}\right)\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}ds$$

as $\hat{T} \to \infty$ is dominated by $\varepsilon x^2$ for $T \geq T_\varepsilon$.

*Statement (ii)* The integral

$$\int_{1-\delta_T+iT}^{1-\delta_T-iT}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}ds$$

is dominated by a constant times $x^{2-\delta_T}$ (with the constant depending on $T$).

*Statement (iii)* The integral

$$\left(\int_{1-\delta_T-iT}^{1-iT}+\int_{1+iT}^{1-\delta_T+iT}\right)\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}ds$$

is dominated by a constant times $\frac{x^2}{\log x}$.

The logic is as follows. Take any $\varepsilon > 0$. It is possible to choose $T_\varepsilon > 0$ so large that the integrals in Statement (i) are $O(\varepsilon x^2)$ (as $x \to \infty$) when we choose $T = T_\varepsilon$ as $\hat{T} \to \infty$ and the integrals in Statement (iii) are $O\left(\frac{x}{\log x}\right)$ when $T = T_\varepsilon$. The integral in Statement (ii) is $O(x^{2-\delta_T})$ when $T = T_\varepsilon$. Putting these statements together, for any $\varepsilon > 0$ we can choose $T_\varepsilon$ so large that when $T = T_\varepsilon$, the sum of the all the integrals in Statements (i), (ii), (iii) is $O\left(\varepsilon x + \frac{x}{\log x}\right)$ as $x \to \infty$. Since the positive number $\varepsilon$ is arbitrary, it follows residue theory applied to the residue at $s = 1$ that for any $c > 1$ the limit of

$$\frac{x^2}{2} - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}\,ds = o(x^2)$$

as $x \to \infty$, which means asymptotically

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}\,ds$$

is $\frac{x^2}{2}$ as $x \to \infty$.

Let us analyze the estimates for the four integrals in Statement (i). For the two vertical integrals over $[1+i\hat{T}, 1+iT]$ and $[1-iT, 1-i\hat{T}]$ along $\operatorname{Re} s = 1$ the absolute value of the factor $x^{s+1}$ is $x^2$. For any $\eta > 0$ the contribution of the factor

$$\frac{\zeta'(1+it)}{\zeta(1+it)}$$

is $\leq C_\eta |T|^\eta$ with the constant $C_\eta$ depending on $\eta$. The factor $\frac{ds}{s(s+1)}$ contributes $O\left(T^{-1}\right)$. Hence, for the two vertical integrals over $[1+i\hat{T}, 1+iT]$ and $[1-iT, 1-i\hat{T}]$ along $\operatorname{Re} s = 1$, we get the order $\varepsilon x^2$ when $T \geq T_\varepsilon$ for any $\varepsilon > 0$. Let us look at the other two integrals. For the two horizontal integrals over $[c+i\hat{T}, 1+i\hat{T}]$ and $[1-i\hat{T}, c-i\hat{T}]$ the absolute value of the factor $x^{s+1}$ is $\leq x^{c+1}$. For any $\eta > 0$ the contribution of the factor

$$\frac{\zeta'(1+it)}{\zeta(1+it)}$$

is $\leq C_\eta |\hat{T}|^\eta$ with the constant $C_\eta$ depending on $\eta$. The factor

$$\frac{ds}{s(s+1)}$$

contributes $O\left(\hat{T}^{-2}\right)$. Hence, for the two integrals over $[c+i\hat{T}, 1+i\hat{T}]$ and $[1-i\hat{T}, c-i\hat{T}]$, we end up with 0 as $\hat{T} \to \infty$.

Statement (ii) follows from the fact that the integration is along $\sigma = 1-\delta_T$ so that $x^s$ is dominated by $x^{1-\delta_T}$ while the other factor contributes only a constant bound.

For Statement (iii), we use

$$\int_{\sigma=1-\delta_T}^{1} x^{1+\sigma} d\sigma = \left[\frac{x^{1+\sigma}}{\log x}\right]_{\sigma=1-\delta_T}^{\sigma=1} = \frac{x^2 - x^{2-\delta_T}}{\log x}$$

which is of the order $o(x^2)$, while the other factors only contributes a constant. By residue theory, the integral

$$\frac{1}{2\pi i}\int_{\partial(G_T \cup E_{\hat{T}})} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}\, ds$$

equals the residue of

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}$$

at $s = 1$, which is $\frac{x^2}{2}$.

By Statements (i), (ii) and (iii), we conclude that

$$\psi_1(x) \sim \frac{x^2}{2} \quad \text{as} \;\; x \to \infty,$$

because

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^{s+1}}{s(s+1)} \, ds \quad \text{for} \;\; c > 1.$$

This finishes the proof of the Prime Number Theorem.

*Remarks on Choice of Contour.* On the interval $(1+i\hat{T},\, c+i\hat{T}]$ (and similarly on the interval $(1 - i\hat{T},\, c - i\hat{T}])$, where $\operatorname{Re} s > 1$, the factor $x^{s+1}$ in the integrand

$$\left( -\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^{s+1}}{s(s+1)} \, ds$$

may contribute $x^\gamma$ with $\gamma > 2$. For that reason we need to make the integrals vanish by $\hat{T} \to \infty$, which necessitates the use of $\hat{T} > T$ with $\hat{T} \to \infty$ at the end.

For the integral over $[1 + iT,\, 1 + i\hat{T}]$ (and similarly over $[1 - iT,\, 1 - i\hat{T}]$) the value of $\operatorname{Re} s$ is 1 so that the factor $x^{s+1}$ in the integrand

$$\left( -\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^{s+1}}{s(s+1)} \, ds$$

may contribute $x^2$. We need to make the coefficient of $x^2$ in the contribution less than any prescribed $\varepsilon > 0$. For that reason, we choose $T$ sufficiently large to make it happen. After we make the choice for $T$, we fix $T$ and therefore also fix $\delta_T$.

For the integral over $[1 - \delta_T - iT,\, 1 - \delta_T + iT]$ the value of $\operatorname{Re} s$ is $1 - \delta_T$ so that the factor $x^s$ in the integrand

$$\left( -\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^{s+1}}{s(s+1)} \, ds$$

may only contribute $x^{2-\delta_T}$, which is $o(x)$ and is fine.

For the integral over the interval $[1-\delta_T-iT,\,1-iT]$ (and similarly over the interval $[1-\delta_T+iT,\,1+iT]$) the value of $\operatorname{Re} s$ varies from $1-\delta_T$ to 1. The contribution of $x^{s+1}$ in the integrand

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}\,ds$$

is explicitly estimated to be of order $\frac{x^2}{\log x}$, which is $o(x^2)$ and is fine.

**Growth Order of Error Term for Second Chebyshev Function and Width of Strip Containing All Essential Zeroes.** The Prime Number Theory gives the first-term $x$ of the asymptotic growth order of the second Chebyshev function $\psi(x)=\sum_{n<x}\Lambda(n)$. Its proof consists of using

(i) the simple pole of the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ at $s=1$ (from the integral test by comparing with $\int_{x=1}^{\infty}\frac{dx}{x^s}$) and

(ii) the application of the completion of squares $3+4\cos t+\cos 2t)=2(1+\cos t)\geq 1$ to $|\zeta(\sigma)^2\zeta(\sigma+it)^4\zeta(\sigma+2it)|\geq 1$

to show that

(a) there are no zeroes of $\zeta(s)$ on the right edge $\{\operatorname{Re} s=1\}$ of the critical strip $\{0\leq \operatorname{Re} s\leq 1\}$ and, more precisely, quantitatively

(b) there is a lower bound estimate $|\zeta(\sigma+it)|\geq C_\varepsilon|t|^{-\varepsilon}$ for any $\varepsilon>0$ and any $\sigma>1$.

The Riemann hypothesis is a statement concerning the order of the next term in the asymptotic growth order of the second Chebyshev function

$$\psi(x)=\sum_{n<x}\Lambda(n).$$

It is equivalent to the statement that all the nontrivial zeroes of the Riemann zeta function $\zeta(s)$ are contained in the vertical line $\{\operatorname{Re} s=\frac{1}{2}\}$, where the set of all nontrivial zeroes of $\zeta(s)$ means all its zeroes on the critical strip $\{0\leq \operatorname{Re} s\leq 1\}$. There is a general theorem (given below) which relates the growth order of the error term for the second Chebyshev function to the minimum width of the strip which contains all the nontrivial zeroes of the Riemann zeta function. The Riemann hypothesis states that the assumption in Part(a) of the general theorem below is fulfilled with $\theta = \frac{1}{2}$. However, the general theorem below has nothing to do with giving or attempting to give a proof of the Riemann hypothesis.

Theorem *(on Relating Growth Order of Error Term for Second Chebyshev Function to Width of Strip Containing All Essential Zeroes).* (a) Suppose there exists $0 < \theta < 1$ such that the zero-set $Z$ of $\zeta(s)$ in $\{0 \leq \operatorname{Re} s \leq 1\}$ is contained in $\{1 - \theta \leq \operatorname{Re} s \leq \theta\}$. Then

$$\sum_{n \leq x} \Lambda(n) = x + \mathrm{O}\left(x^{\theta} (\log x)^2\right).$$

(b) Suppose that for some $0 < \alpha < 1$,

$$\sum_{n \leq x} \Lambda(n) = x + \mathrm{O}\left(x^{\theta}\right)$$

for every $\theta > \alpha$. Then $Z$ is contained in $\{1 - \alpha \leq \operatorname{Re} s \leq \alpha\}$.

Graphically the equivalent relations in the theorem above are represented in the following figure

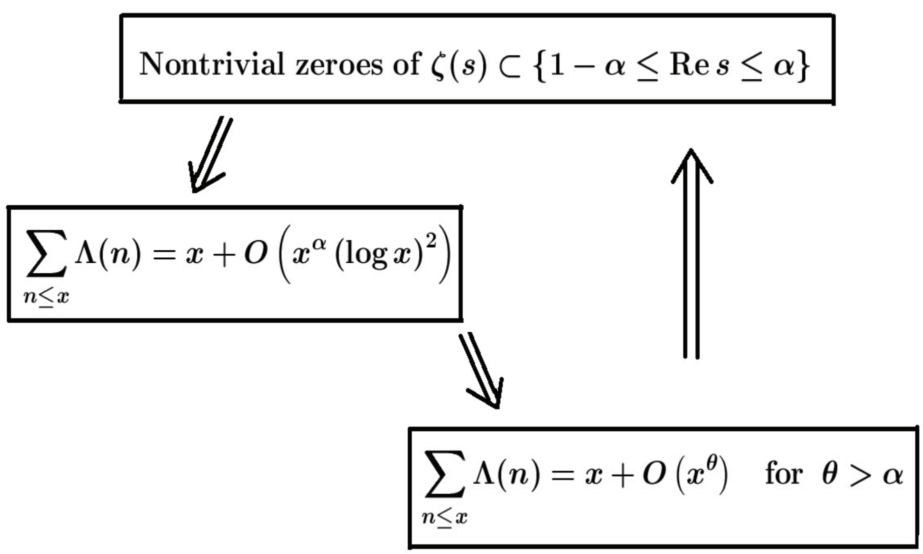


Figure 6: *Relation Between Width of Strip Containing All Nontrivial Zeroes of Riemann Zeta Function and Order of Next Term in Growth Estimate of Second Chebyshev Function*

The idea of the proof of the theorem above is as follows. For the proof of the Prime Number Theorem, we apply Perron's formula to the Dirichlet series

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

to get

$$\psi(x) = \sum_{n<x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}\, ds \quad \text{for } c > 1.$$

Now, to prove the theorem above, we apply Perron's lemma and Perron's formula *with error estimates*. We will not go any further in this direction, but will only give examples of statements of Perron's lemma with error estimates and the application of Perron's formula to $-\frac{\zeta'(s)}{\zeta(s)}$ with error estimates.

*Perron Lemma with Error Estimate.* For $c > 0$ and $T > 0$ and $y \neq 1$,

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^s \frac{ds}{s} = \chi_{(1,\infty)} + \mathrm{O}\left(\frac{y^c}{T\,|\log y|}\right).$$

*Application of Perron Formula to $-\frac{\zeta'(s)}{\zeta(s)}$ with Error Estimate.*

$$\begin{aligned} \int_{c-iT}^{c+iT} \frac{\zeta'(s)}{\zeta(s)}\, x^s \frac{ds}{s} &= \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} \\ &= \sum_{n<x} \Lambda(n) + \mathrm{O}\left(\frac{x\,(\log x)^2}{T}\right) \end{aligned}$$

for $x \in \mathbb{N} + \frac{1}{2}$.