OVERVIEW

The syllabus of Math 213a is listed as: Fundamentals of complex analysis, and further topics such as conformal mapping, hyperbolic geometry, canonical products, elliptic functions and modular forms.

The prerequisites are listed as: Basic complex analysis, topology of covering spaces, differential forms.

The course meets from noon to 1:15 p.m. in SC412 Tuesdays and Thursdays. The Exam Group is FAS10_D.

I am going to devote this first lecture to an overview of what is to be covered in Math 213a.

Before I do it, I would like to give you some administrative information about the course. I will post the lecture notes and the homework assignments on the following link

https://people.math.harvard.edu/ siu/math213a/

My office hours are 10:30 a.m. to noon, Tuesdays and Thursdays in SC511. If you would like to see me at other times, please communicate with me by email (siu@math.harvard.edu).

Homework will be assigned every Tuesday and due a week later, to be submitted to the Harvard Canvas website for the course

https://canvas.harvard.edu/courses/137716

The purpose of homework is to give the students an opportunity to think about the material and gain working knowledge of it. The homework does not call for any ingenious special tricks.

The course assistant is Jit Wu YAP (email: jyap@math.harvard.edu), whose office hours are 2 p.m. to 3 p.m. Fridays in SC428g.

Feedback from the students concerning the pace of the course and the level of difficulty of the material is very much welcome, in order to optimize the usefulness of the course to the students.

The purpose of this overview is to point out how the seemingly different topics in the syllabus actually come together to provide a very coherent picture of the theme of this course.

The course will start with a review of the fundamentals of complex analysis. Since one prerequisite for the course is "basic complex analysis", students should have been exposed to some basic theory of complex analysis in the past, possibly in some distant past. To refresh the students' memory of the subject, we do a review of the fundamentals of complex analysis. Our review will be done with a more unified perspective, geared to the natural introduction of the other advanced topics in the course.

The fundamentals of complex analysis are Cauchy's theory consisting of Cauchy's theorem, Cauchy's integral formula, and Cauchy's theory of residues for the explicit computation of certain definite integrals and infinite sums. The main tool is the use of Cauchy's integral formula for derivatives to transform the computation of certain definite integrals to the computation of derivatives.

Besides such computations, complex analysis can be applied to real-world problems involving specical functions, to number theory in the context of the prime number theorem and the Riemann hypothesis, and to value distribution theory (also known as Nevanlinna theory) which studies the distribution of zeroes or inverse images of a varying complex number for general (transcendental) holomorphic and meromorphic functions on $\mathbb{C}$ as a generalization of the fundamental theorem of algebra which gives the number of zeros in terms of the growth order of a polynomial.

To apply complex analysis to solve actual problems, besides general abstract theory, we need the knowledge of enough special functions to express our solutions in. There are two categories of special functions discussed in this course. The first is elliptic and theta functions. The second is the Riemann zeta function and the Gamma function. Elliptic functions can be defined as the inverse of a multi-valued indefinite integral, as a generalization of the logarithmic function and the inverse sine function. Theta functions are defined by infinite series obtained from recurrent relations for the coefficients of Fourier series. The Riemann zeta function can be defined either as a canonical product or a Dirichlet series.

Besides the polynomials, rational functions, and more general algebraic functions $w = w(z)$ which satisfy polynomial equations in z and w, the commonly used functions are the exponential function, the logarithmic function, the trigonometric functions, *etc.*, which are all related. They can be analytically defined as the inverse functions of indefinite integrals. For example,

$$\log x = \int_{t=1}^{x} \frac{dt}{t},$$
$$\sin^{-1} x = \int_{t=0}^{x} \frac{dt}{\sqrt{1-t^2}}.$$

The indefinite integrals are multi-valued and as a result any of their inverse functions has one single primitive period. Historically, the trigonometric functions were first introduced from the geometric properties of triangles, presumably motivated by land survey needs from periodic flooding of the Nile river.

The elliptic functions arose also from practical needs. One need is the exact solution of the simple pendulum problem. The usual sinusoidal solution

$$\sin^{-1} x = \int_{t=0}^{x} \frac{dt}{\sqrt{1-t^2}}$$

is only an approximated solution and not the exact solution. The exact solution replaces the sine function sin by the *Jacobian elliptic sine function*, denoted by sn, whose inverse is given by

$$\mathrm{sn}^{-1} x = \int_{t=0}^{x} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

where the parameter k depends on the configuration of the simple pendulum like the length of the string. A more general form of the indefinite integral (with respect to a more general coordinate system) is

$$\int \frac{dz}{\sqrt{P_4(z)}},$$

where $P_4(z)$ denotes a polynomial of degree 4.

Motivated by the trigonometric situation, Abel pointed out that the correct approach is to invert the indefinite integral. Since the square root occurs in the denominator integrand, in order to make the integrand *single-valued*, an artificial configuration X known as a *Riemann surface* is constructed, which replaces every point in $\mathbb{P}_1$ by two points except at the four zeroes of $P_4(z)$. The integrand now becomes a well-defined 1-form φ on X, which we can integrate from some chosen initial point of X. The result is not single-valued, but becomes single-valued modulo a lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ of rank 2 in $\mathbb{C}$, as illustrated in the following diagram

$$\begin{array}{ccc} X & \stackrel{\Phi}{\longrightarrow} & \mathbb{C}\,/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \\ \downarrow & & \uparrow \pi \\ \mathbb{P}_1 & \stackrel{\mathrm{sn}}{\longleftarrow} & \mathbb{C}\quad. \end{array}$$

The key point is that the map Φ, which is defined by the integration $\int \varphi$ of the 1-form φ on X, is actually biholomorphic, so that the Jacobian elliptic sine function sn can be defined as the composite of the inverse of Φ and the other two arrows. The map from $\mathbb{C} \to X$, obtained by composing π and Φ^{-1}, is the universal covering of the compact Riemann surface X.

In general, an *elliptic function* is defined as any meromorphic function on $\mathbb{C}$ with period lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ spanned by two primitive periods ω_1 and ω_2. Another way of definition is that an elliptic function is a meromorphic function on a torus $\mathbb{C}\,/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, which is a compact Riemann surface of genus 1 (*i.e.*, with one donut hole).

With a Möbius transformation which sends one of the roots of $P_4(z)$ to the infinity point ∞ of the Riemann sphere $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$, the integral

$$\int \frac{dz}{\sqrt{P_4(z)}}$$

is transformed to

$$\int \frac{dz}{\sqrt{P_3(z)}},$$

where $P_3(z)$ is a polynomial of degree 3. The formulation with $P_4(z)$ is the Jacobian formulation for elliptic functions, whereas the formulation with $P_3(z)$ is the Weierstrass formulation for elliptic functions. The advantage of the Weierstrass formulation of using $P_3(z)$ is that the Riemann surface X

can be elegantly defined as a curve in $\mathbb{P}_2$ of degree 3 by $w^2 = P_3(z)$ in the affine part $\mathbb{C}^2$ of $\mathbb{P}_2$. For that reason, the Weierstrass formulation is usually used in algebraic geometry. On the other hand, when analytic computations involving formulas are needed, the Jacobian formulation is usually used.

A reason for calling these functions *elliptic functions* is that they occur in the computation of the arc-length function of an ellipse measured from some fixed point to a variable point.

An ellipse is given by $x = a\cos\theta$ and $y = b\sin\theta$ so that

$$dx^2 + dy^2 = \left(a^2\sin^2\theta + b^2\cos^2\theta\right)d\theta^2$$

and the arc-length function s is given by

$$s = \int\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta = \frac{1}{b}\int\sqrt{1 - k^2\sin^2\theta}\,d\theta$$

with $k = \sqrt{1 - \frac{a^2}{b^2}}$. Though it is not the same as the integral

$$\int\frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} = \int\frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

(with $x = \sin\theta$) which defines the inverse elliptic sine function, the arc-length function can be evaluated by using functions in the elliptic function theory.

Any meromorphic function on $\mathbb{P}_1$ (*i.e.,* any rational function) can be described additively by specifying all its principal parts, up to an additive constant. Multiplicatively, any mermorphic function on $\mathbb{P}_n$ can be described by its pole-set and its zero-set, up to a multiplicative constant. Analogously, a general elliptic function (*i.e.,* a meromorphic function on a torus) can be described by its principal parts, up to an additive constant, but subject to the additional condition that the sum of the residues is 0. This statement evolves into the theorem of Riemann-Roch for a general compact Riemann surface. Multiplicatively, a general elliptic function can be described by its pole-set and its zero-set, up to a multiplicative constant, but subject to the additional condition that the sum of the coordinates of its poles is equal to the sum of the coordinates of its zeroes modulo the period matrix. This statement evolves into the theorem of Abel for a general compact Riemann surface. The theorems of Riemann-Roch and Abel are two fundamental theorems in the theory of compact Riemann surfaces. In this course we will not go into the general theory of compact Riemann surfaces. Note that for any meromorphic function on any compact Riemann surface the number of zeroes (with multiplicities counted) is always equal to the number of zeroes (with multiplicities counted), as a consequence of the vanishing of the sum of the residues of the differential of its logarithm.

The fundamental theorem of algebra tells us that a polynomial of degree n (over $\mathbb{C}$) can be factored into n linear factors. Each linear factor has only one single zero. Any meromorphic function on $\mathbb{P}_1$ (which is a compact Riemann surface of genus 0) is the quotient whose numerator and denominator are products of linear factors. One asks whether there is a similar result for a general elliptic function (which is a meromorphic function on a torus) so that each factor $\vartheta(z)$ has one single zero. Of course, we cannot expect each factor to be doubly periodic, otherwise Liouville's theorem is violated when it is applied to the entire function on $\mathbb{C}$ defined by $\vartheta(z)$.

Jacobi came up with the idea of weakening the property of double periodicity to keep the property of a single zero for $\vartheta(z)$. With the period lattice normalized to be $\omega_1 = \pi$ and $\omega_2 = \pi\tau$ with $\operatorname{Im}\tau > 0$, Jacobi kept the period π for $\vartheta(z)$ but introduced a *periodicity factor* for the period $\pi\tau$, which means that $\vartheta(z + \pi\tau) = g(z)\vartheta(z)$ for some factor $g(z)$. Since the purpose of decomposing of an elliptic function into the quotients of products of factors with single zeros is to focus on the zero-set and the pole-set of the elliptic function, we need to avoid any additional zeroes introduced by the periodicity factor $g(z)$. So we want $g(z)$ to be nowhere zero and therefore can be written as $e^{h(z)}$. The simplest choice for $h(z)$ is that it is a polynomial of degree 1 in z. This then becomes the definition of a *Jacobian theta function* which allows periodicity factors equal to the exponential of polynomials of degree 1. One now expresses a general elliptic function f as a constant factors times

$$\frac{\prod_{j=1}^{k} \vartheta(z - a_j)}{\prod_{j=1}^{k} \vartheta(z - b_j)}.$$

Jacobi explicitly wrote down a theta function with a single zero at the origin. Then he translated it by half periods to form three other Jacobian theta functions so that the four Jacobian theta functions $\vartheta_1(z), \vartheta_2(z), \vartheta_3(z), \vartheta_4(z)$

have single zeroes at $0, \frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi\tau}{2}, \frac{\pi\pi}{2}$ respectively. The reason for Jacobi to introduce half-period translations of a theta function is the following.

When $\operatorname{Im} \tau$ goes to infinity, the rank-2 period lattice $\mathbb{Z}\pi + \mathbb{Z}\pi\tau$ becomes a rank-1 period lattice $\mathbb{Z}\pi$. The trigonometric sine function $\sin z$ can be regarded as having a period factor -1 for the period π. It half-period translate is the trigonometric cosine function $\cos z$. An important algebraic relation between the two is $\sin^2 z + \cos^2 z = 1$. Analogously there are similar (but more complicated) algebraic relation among the four Jacobian theta functions $\vartheta_1(z), \vartheta_2(z), \vartheta_3(z), \vartheta_4(z)$ (which are related to one another by half-period translations).

There are well-known addition formulas and differential relations for the trigonometric functions $\sin z$ and $\cos z$. Similarly, there are corresponding, more complicated, addition formulas and differential relations for the Jacobian theta functions $\vartheta_1(z), \vartheta_2(z), \vartheta_3(z), \vartheta_4(z)$ (and for Jacobian elliptic functions). The addition formulas are related to the operation of addition for the torus $\mathbb{C}/\mathbb{Z}\pi + \mathbb{Z}\pi\tau$.

The theory of theta function admits natural generalizations from the complex torus of one complex dimension to a complex torus of higher complex dimension (and in particular to abelian varieties, which are complex tori admitting sufficiently many meromorphic functions).

The third Jacobian theta function, which we denote by $\vartheta_3(z, \tau)$ to emphasize its dependence on τ, satisfies the equation

$$\sqrt{\frac{\tau}{i}}\vartheta_3(z,\tau) = \exp\left(\frac{-iz^2}{\pi\tau}\right)\vartheta_3\left(\frac{z}{\tau}, -\frac{1}{\tau}\right),$$

because the transformation

$$(z,\tau) \mapsto \left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$$

leaves the single zero

$$z = \pm\frac{\pi}{2} + \pm\frac{\pi\tau}{2} \quad \left(\text{equivalently } \ \frac{z}{\tau} = \pm\frac{\pi}{2\tau} + \pm\frac{\pi}{2}\ \right)$$

of ϑ_3 unchanged. This equation has important number-theoretical consequences. For example, it yields results on the problem of sums of squares, which asks for the number of ways an integer can be written as the sum of squares of two integers or four integers. More importantly, it provides a way of proving *quadratic reciprocity* which Gauss considered as one of his own important mathematical achievements.

An important feature of theta functions and elliptic functions is the transformation law when the variable is subject to the translation by an element of the period lattice transform. The domain is $\mathbb{C}$ which is the universal cover of the Riemann surface. We can look at the case of rank-1 lattice and the case of rank-2 lattice. Clearly it is not possible to have anything more complicated like a rank-3 lattice. The reason why we end up with the domain $\mathbb{C}$ as the universal cover of the compact Riemann surface is that the Riemann surface is the artificially constructed configuration which makes the integrand

$$\frac{dz}{\sqrt{P_4(z)}}$$

single-valued. We can use more complicated integrands, for example,

$$\frac{dz}{\sqrt[p]{P_q(z)}}$$

for larger positive numbers p and q, or even

$$R(z,w)dz$$

for some rational function $R(z,w)$ subject to the polynomial relation $P(z,w) = 0$. The universal cover very likely is not $\mathbb{C}$. One asks what universal cover can occur. This is answered by the *Uniformization Theorem* which says that any simply connected Riemann surface X can only be one of the three: the Riemann sphere $\mathbb{P}_1$, the Gauss plane $\mathbb{C}$, and the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$.

Since both $\mathbb{C}$ and $\mathbb{D}$ are open subsets of $\mathbb{P}_1$, one way to prove the uniformization theorem to seek a good univalent (*i.e.,* one-to-one holomorphic) map f into $\mathbb{P}_1$. If the image of f contains the infinite point ∞ of $\mathbb{P}_1$, its one-to-one property means that it admits precisely one pole P_0 which is simple. One way to get such an f is to use electrostatic potential by putting a point charge at P_0 and considers the electrostatic potential u due to it by minimizing the Dirichlet integral which is the integral of its gradient square and then consider du with a single pole at P_0. There is the great technical difficulty of using some offset function to modify the definition of the Dirichlet integral to make it finite.

Another way due to Koebe is to construct f locally first and then use Riemann mapping theorem, Schwarz reflection and the Koebe distortion theorem enlarge the domain of f so that one can finally go to the limit as the solution.

The theory of conformal mapping fits as part of the theory of the uniformization theorem.

When one uses $\mathbb{D}$ as the universal of the compact Riemann surface (instead of $\mathbb{C}$R), the role of the action on $\mathbb{C}$ by translation by an element of the period lattice is to be replaced by the action of an element of a discrete subgroup of the biholomorphism group of $\mathbb{D}$. The transformation law involving periodicity factors is replaced by that for the factors of automorphy in the context of automorphic functions and forms. So instead of the impossible rank-3 period lattice, discrete group actions on $\mathbb{D}$ are considered.

Before we discuss the Riemann zeta function as the second special function, we would like to say something about Nevanlinna theory. The fundamental theorem of algebra tells us that a function f of polynomial order growth n has precisely n roots with multiplicities counted. Moreover, for any complex number c, $f - c$ also has precisely n roots. It means that the function f assumes the value c precisely n times. Theory applies also to rational functions with appropriate formulation. Nevanlinna theory answers the same questions for (meromorphic) transcendental functions (*i.e.,* non-algebraic functions such as $\sin z$, e^z, $\log z$, $\operatorname{sn} z$, $\vartheta_1(z)$, *etc.*). We relate the order of growth of the function f to the order of growth of the zero-set of $f - c$ and also studies when some c gives exceptional results and how to count the number of such exceptional constants c. Because of the role of the constant c, Nevanlinna theory is also known as *value distribution theory.* One of the main tools used is the Poisson-Jensen formula to which Nevanlinna introduced one more radial integration and an ingenious technique of absorbing the boundary term.

The fundamental theorem of algebra factors a polynomial into a product of linear factors. One asks for analogy for transcendental functions. The question is about factoring into infinite produces. Canonical products and the factorization theorems of Weierstrass and Hadamard enter the picture.

The techniques of Nevanlinna theory is very much related to the Gelfond-Schneider method of solving the seventh problem of Hilbert.

The second kind of special function considered in this course is the Riemann zeta function (and its accompanying Gamma function).

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}},$$

whose definition is motivated by the above product formula which comes from the decomposition of an integer into a product of primes. Riemann introduced it to compute the growth order of the prime-counting function $\pi(x)$ which is defined the number of primes not exceeding x. The answer is the *Prime Number Theorem* which states that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty$$

Since infinite products are more difficult to handle than infinite sums, usually the former is transformed to the latter by using the logarithmic function. In this particular case, differentiation is done once to yield

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } \ n = p^m \ \text{ for some } m \in \mathbb{N} \text{ and some prime number } p \\ \\ 0 & \text{otherwise.} \end{cases}$$

This is again a Dirichlet series, with more complicated coefficients, which depend on the distribution of prime numbers. Let $\pi(x)$ be the prime-counting function which is the number of primes not exceeding x. The Prime Number Theorem states that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty$$

which is easily seen to be equivalent to the Chebyshev function

$$\psi(x) = \sum_{p^k \leq x} \log p = \sum_{n \leq x} \Lambda(n)$$

of the same asymptotic order as x, *i.e.,*

$$\psi(x) \sim x \quad \text{as} \quad x \to \infty.$$

Besides the dominant term x for $\psi(x)$, Riemann asked about the next lower order term, which is conjectured to be

$$\psi(x) = x + O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

as $x \to \infty$ for any $\varepsilon > 0$. Just like Nevanlinna theory, it turns out that the zero-set of $\zeta(s)$ is related to its growth order along certain vertical lines and the growth order of $\psi(x) - x$ as $x \to \infty$. Vertical lines are considered instead of large circles because $\zeta(s)$ is a Dirichlet series. One way to express the relation between the zero-set of $\zeta(s)$ and the rowth order of the difference $\psi(x) - x$ as $x \to \infty$ is in the following two statements (a) and (b).

(a) Suppose there exists $0 < \theta < 1$ such that the zero-set Z of $\zeta(s)$ in $\{0 \leq \operatorname{Re} s \leq 1\}$ is contained in $\{1 - \theta \leq \operatorname{Re} s \leq \theta\}$. Then

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\theta} (\log x)^2\right)$$

as $x \to \infty$.

(b) Suppose that for some $0 < \alpha < 1$,

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\alpha}\right)$$

as $x \to \infty$. Then Z is contained in $\{1 - \alpha \leq \operatorname{Re} s \leq \alpha\}$.

From (a) and (b) it follows that the statement that the zero-set Z of $\zeta(s)$ in $\{0 \leq \operatorname{Re} s \leq 1\}$ is lies on the vertical line $\operatorname{Re} s = \frac{1}{2}$ if and only if

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

as $x \to \infty$ for any $\varepsilon > 0$, which automatically implies the stronger statement that

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\frac{1}{2}} (\log x)^2\right)$$

as $x \to \infty$, because $Z \subset \{1 - \theta \leq \operatorname{Re} s \leq \theta\}$ for any $\theta > \frac{1}{2}$ is equivalent to $Z \subset \left\{\frac{1}{2} \leq \operatorname{Re} s \leq \frac{1}{2}\right\}$. The Riemann hypothesis states that $Z \subset \left\{\frac{1}{2} \leq \operatorname{Re} s \leq \frac{1}{2}\right\}$.