Logarithmic Derivative Lemma

At the end of our discussion on Hadamard’s factorization theorem for entire functions of finite order, we mentioned its application to Picard’s little theorem for entire functions of finite order. The statement is that if $f(z)$ is an entire function of finite order which misses 0 and 1, then $f$ must be constant from a trivial case of Hadamard’s factorization theorem. The reasoning is follows (Exercise #11 on p.155 of Stein-Shakarchi).

Since $f(z)$ misses 0, it can be written as $e^{g(z)}$ for some entire function $g(z)$. By applying Hadamard’s factorization theorem to the entire function $f(z)$ of finite order or simply by using Liouville-theorem-type arguments, we conclude that $g(z)$ must be a polynomial. By the same token, since $1 - f(z)$ misses 0, it can be written as $e^{h(z)}$ for some polynomial. Thus $e^{g(z)} + e^{h(z)} \equiv 1$.

We differentiate it with respect to $z$ to obtain $g'(z)e^{g(z)} + h'(z)e^{h(z)} \equiv 0$. Solving these two linear equations together for the two unknowns $e^{g(z)}$ and $e^{h(z)}$ by Cramer’s rule, we get

$$f(z) = e^{g(z)} = \frac{\begin{vmatrix} 1 & 1 \\ 0 & h'(z) \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ g'(z) & h'(z) \end{vmatrix}} = \frac{h'(z)}{h'(z) - g'(z)}$$

and

$$1 - f(z) = e^{g(z)} = \frac{\begin{vmatrix} 1 & 1 \\ g'(z) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ g'(z) & h'(z) \end{vmatrix}} = \frac{-g'(z)}{h'(z) - g'(z)}.$$

The first equation (and also equivalently the second equation) implies that $f(z)$ is a rational function. Since $f(z)$ is entire, the rational function must be polynomial, but polynomial cannot miss any value unless it is constant. Picard’s little theorem yields the same solution without assuming that $f$ is of finite order and can be proved by using the topological covering map of $\mathbb{C} - \{0, 1\}$ defined by the elliptic modular function $J(\tau)$.

Suppose we would like to get rid of the assumption of the finite-order property of the entire function and keep the same argument of applying Cramer’s rule to a system of linear equations. We would be able to get
a contradiction from the growth order argument if we can prove that the logarithmic derivative of $f$ has a lower order than the growth order of $f$ itself, without assuming that $f$ is an entire function of finite order. Since $f$ is not a function of finite order, its growth order has to be defined in another way. We introduced earlier Nevanlinna’s characteristic function $T(R, f)$ for any meromorphic function on $\mathbb{C}$ which is defined as

$$T(R, f) = m(R, f) + N(R, f),$$

where

$$m(R, f) = m(R, f, \infty) = \frac{1}{2\pi} \int_{\varphi=0}^{2\pi} \log^+ |f(Re^{i\varphi})|d\varphi$$

and

$$N(R, f) = N(R, f, \infty) = \sum_{\nu=1}^{N} \log \frac{R}{|b_{\nu}|}$$

when $b_1, \ldots, b_N$ are the poles of $f$ in $|z| < R$ with multiplicities represented by repeated occurrence. In the case of an entire function $f$, the set $\{b_1, \ldots, b_N\}$ is vacuous and $T(R, f)$ is simply the proximity function $m(R, f) = m(R, f, \infty)$ to infinity given by

$$T(R, f) = m(R, f) = \frac{1}{2\pi} \int_{\varphi=0}^{2\pi} \log^+ |f(Re^{i\varphi})|d\varphi.$$

However, the zeroes of $f$ give rise to poles of its logarithmic derivative $(\log f)' = \frac{f'}{f}$. One key technique in Nevanlinna theory is the logarithmic derivative lemma, which essentially says that

$$m(R, (\log f)') = O(\log T(R, f))$$

if $f$ is a meromorphic function on $\mathbb{C}$. The logarithmic derivative lemma can be used in the arguments of applying Cramer’s rule to a system of linear equations to obtain results concerning missing values of entire functions without the assumption of the finite-order property of the entire functions. We will see this when we discuss Borel’s theorem concerning value distribution after our discussion of the logarithmic derivative lemma.

**Precise Formulation of Logarithmic Derivative Lemma.** Let $f(z)$ be a meromorphic function on $\mathbb{C}$. The logarithmic derivative lemma states that

$$m\left(r, \frac{f'}{f}, \infty\right) \leq O(\log T(r, f) + \log r),$$
where the notation $\|\cdot\|$ at the end of the inequality means that the inequality holds outside a set of finite measure with respect to $\frac{dt}{t}$. We are going to give a proof by using methods of harmonic analysis.

First Step of Proof of Logarithmic Derivative Lemma – Application of (1,0) Differentiation to Poisson-Jensen Formula. Let $f$ be a meromorphic function on $|z| \leq R$ without zeroes and poles at $z = 0$ and on $|z| = R$. Denote by $\{a_\mu\}$ all the zeroes of $f$ (with multiplicities represented by repeated occurrences) in $\Delta_R = \{|z| < R\}$ and denote by $\{b_\nu\}$ all the poles of $f$ (with multiplicities represented by repeated occurrences) in $\Delta_R$. The Poisson-Jensen formula yields, for $z = re^{i\theta}$ with $r < R$,

$$
\log |f(z)| = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \log |f(Re^{i\phi})| \left( \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi) + r^2} \right) d\phi \\
+ \sum_\mu \log \left| \frac{R(z - a_\mu)}{R^2 - \overline{a_\mu}z} \right| - \sum_\nu \log \left| \frac{R(z - b_\nu)}{R^2 - \overline{b_\nu}z} \right|.
$$

Taking $\frac{\partial}{\partial z}$ of both sides and using

$$
\frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi) + r^2} = \text{Re} \left( \frac{Re^{i\phi} + z}{Re^{i\phi} - z} \right) = \frac{1}{2} \frac{Re^{i\phi} + z}{Re^{i\phi} - z} + \frac{1}{2} \frac{Re^{-i\phi} + \overline{z}}{Re^{-i\phi} - \overline{z}}
$$

and

$$
\frac{d}{dz} \left( \frac{az + b}{cz + d} \right) = \frac{ad - bc}{(ca + d)^2},
$$

from $\log |f| = \frac{1}{2} \log f + \frac{1}{2} \log \overline{f}$ we get

$$
\frac{f'(z)}{f(z)} = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \log |f(Re^{i\phi})| \left( \frac{2Re^{i\phi}}{(Re^{i\phi} - z)^2} \right) d\phi \\
- \sum_{|a_\mu| < R} \left( \frac{1}{a_\mu - z} - \frac{\overline{a_\mu}}{R^2 - \overline{a_\mu}z} \right) + \sum_{|b_\nu| < R} \left( \frac{1}{b_\nu - z} - \frac{\overline{b_\nu}}{R^2 - \overline{b_\nu}z} \right).
$$

In order to estimate the right-hand side, we keep $r$ to be strictly less than $R$ so that we can estimate

$$
\frac{Re^{i\phi}}{(Re^{i\phi} - z)^2}
$$
by a positive constant times $\frac{R}{(R-r)^2}$ and estimate

$$\frac{\overline{a}_\mu}{R^2 - \overline{a}_\mu z} \quad \text{and} \quad \frac{\overline{b}_\nu}{R^2 - \overline{b}_\nu z}$$

by a positive constant times $\frac{1}{R-r}$. Of course we have to bound also the number of such terms by $n(R, f, 0)$ and $n(R, f, \infty)$ respectively. The estimates for the terms

$$\frac{1}{a_\mu - z} \quad \text{and} \quad \frac{1}{b_\nu - z}$$

are trickier, because $z$ can be equal to $a_\mu$ or $b_\nu$.

Second Step of Proof of Logarithmic Derivative Lemma – Estimation in Terms of the Minimum Distance from the Variable Point to the Zeroes and Poles. Let $\delta(z)$ be the minimum of $|z - a_\mu|$ and $|z - b_\nu|$ for all $\mu$ and $\nu$. We can bound

$$\sum_{|a_\mu| < R} \left| \frac{1}{a_\mu - z} \right| + \sum_{|b_\nu| < R} \left| \frac{1}{b_\nu - z} \right|$$

by

$$(n(R, f, 0) + n(R, f, \infty)) \frac{1}{\delta(z)}$$

which we can rewrite as

$$\left( \frac{n(R, f, 0) + n(R, f, \infty)}{r} \right) \left( \frac{r}{\delta(z)} \right)$$.

The reason for the rewriting is that we need to compare $\delta(z)$ with $r$ in a multiplicative way instead of an additive way. In other words, we are interested in an inequality of the form $\delta(z) < \eta r$ for some $0 < \eta < 1$ small rather than in an inequality of the form $|\delta(z) - r| < \eta$.

In the literature there are no names for the functions $n(R, f, 0)$ and $n(R, f, \infty)$. To facilitate the explanation of the steps of the proof, we are going to refer to them respectively as the number-of-zeros function and the number-of-poles function.

Since $\delta(z) = \delta(re^{i\theta})$ can be zero, we can do the estimates only when we take $\log^+$ and average over the variable $z = re^{i\theta}$ in the circle $\partial \Delta_r$. For the integration over $\partial \Delta_r$, for the sake of notational simplicity we put together
\{a_\mu\} \text{ and } \{b_\nu\} \text{ into } \{c_\lambda\}_{\lambda=1}^m = \{a_\mu\} \cup \{b_\nu\}, \text{ where } m = n(R, f, 0) + n(R, f, \infty).

We divide up the circle \(\partial \Delta_r\) as the union of the good part

\[ E_0 = \left\{ re^{i\theta} \mid \delta(re^{i\theta}) = \min_{1 \leq \lambda \leq m} |re^{i\theta} - c_\lambda| \geq \frac{r}{m} \right\} \]

and the bad part

\[ E_\lambda = \left\{ re^{i\theta} \mid |re^{i\theta} - c_\lambda| \leq \frac{r}{m} \right\} \]

for \(1 \leq \lambda \leq m\). By the triangle inequality, for \(m \geq m_0\) (for some universal \(m_0 \geq 2\)) the set \(E_\lambda\) is contained in an arc of angle \(\leq \frac{3}{m} < \frac{\pi}{2}\) (that is, of length \(\leq \frac{3\pi}{2m}\)). The use of \(m_0\) is for the statement that when the ratio of the length of a chord to the radius is sufficiently small \((\leq \frac{2}{m_0})\) the arc-length is no more than \(\frac{3}{2}\) times the chord length. The length of the arc spanned by points of \(E_\lambda\) is computed from the chord length from the point \(c_\lambda\), with the arc-length no more than \(\frac{3}{2}\) times the chord length. We know that the chord length measured from the point \(c_\lambda\) is no more than twice of \(\frac{r}{m}\). So we have the bound of \(\frac{3}{2}\) times \(2\) times \(\frac{r}{m}\), which is \(\frac{3\pi}{m}\). When we measure in terms of the angular variable instead of the arc-length, we have to divide by the radius \(r\) of the circle so that we get \(\frac{3}{m}\). This explains the upper limit \(\frac{3}{m}\) of the integral below, with the integral multiplied by 2.

We have the estimate that

\[
\int_{E_\lambda} \log^+ \left( \frac{r}{|re^{i\theta} - c_\lambda|} \right) d\theta = \int_{E_\lambda} \log^+ \left( \frac{1}{|e^{i\theta} - \frac{c_\lambda}{r}|} \right) d\theta
\]

\[
\leq 2 \int_{\theta=0}^{\frac{3\pi}{2m}} \log \frac{1}{\sin \theta} d\theta \quad \text{(use only Im}(e^{i\theta} - \frac{c_\lambda}{r}))
\]

\[
\leq 2 \int_{\theta=0}^{\frac{3\pi}{2m}} \log \frac{\pi}{2\theta} d\theta
\]

\[
= \frac{3}{m} \log \frac{\pi}{2} - \frac{2}{m} \int_{\theta=0}^{\frac{3\pi}{2m}} \log \theta d\theta
\]

\[
= \frac{3}{m} \log \frac{\pi}{2} - \left[ \theta \log \theta - \theta \right]_{\theta=0}^{\frac{3\pi}{2m}}
\]

\[
= \frac{3}{m} \log m + \frac{3}{m} \left( \log \frac{\pi}{3} + 1 \right),
\]
where the inequality \( \sin \theta \geq \frac{\theta}{\pi/2} \) for \( 0 \leq \theta \leq \frac{\pi}{2} \) (from the fact that on the interval \([0, \pi/2]\) the graph of the function \( \sin \theta \) is above the line segment joining the origin to the point \((\pi/2, 1)\)) is used.

We now handle the overlap of the sets \( E_\lambda \) by removing the overlapping parts so that we end up with a disjoint union for the estimation of the integrals. For \( 1 \leq \lambda \leq m \) let \( E'_\lambda \) be the set of all \( re^{i\theta} \) such that
\[
\delta (re^{i\theta}) = \left| re^{i\theta} - c_\lambda \right| \leq \frac{r}{m}
\]
and
\[
\delta (re^{i\theta}) < \left| re^{i\theta} - c_\lambda \right|
\]
for \( 1 \leq \lambda < \lambda \). Then \( E'_\lambda \subset E_\lambda \) and \( \bigcup_{\lambda=1}^m E'_\lambda \) equal to the disjoint union of \( E'_\lambda \) for \( 1 \leq \lambda \leq m \). We have
\[
\int_{\bigcup_{\lambda=1}^m E'_\lambda} \log^+ \frac{r}{\delta (re^{i\theta})} d\theta = \int_{\bigcup_{\lambda=1}^m E_\lambda} \log^+ \frac{r}{\delta (re^{i\theta})} d\theta = \sum_{\lambda=1}^m \int_{E'_\lambda} \log^+ \frac{r}{|re^{i\theta} - c_\lambda|} d\theta \leq \sum_{\lambda=1}^m \int_{E_\lambda} \log^+ \frac{r}{|re^{i\theta} - c_\lambda|} d\theta \leq 3 \log m + 3 \left( \log \frac{\pi}{3} + 1 \right).
\]
We have thus taken care of all the bad parts. The estimation of the good part from \( E_0 \) is now easy. On \( E_0 \) from
\[
\delta (re^{i\theta}) = \min_{1 \leq \lambda \leq m} \left| re^{i\theta} - c_\lambda \right| \geq \frac{r}{m}
\]
we get
\[
\int_{E_0} \log^+ \frac{r}{\delta (re^{i\theta})} d\theta \leq 2 \pi \log m.
\]
Putting everything together we have the estimate
\[
\left| \frac{f''(z)}{f(z)} - \infty \right| \leq C_1 \left( \log^+ T(R, f) + \log \frac{1}{R-r} + \log (n(R, f, 0) + n(R, f, \infty)) + \log R \right)
\]
for some positive constant \( C_1 \).
Third Step of Proof of Logarithmic Derivative Lemma – Replacing Number-of-Poles Function and Number-of-Zeros Function by Respective Counting Functions. There is one problem which we have to handle, namely, in order to get the logarithmic derivative lemma we need to replace
\[ n(R, f, 0) + n(R, f, \infty) \]
by
\[ N(R, f, 0) + N(R, f, \infty) \leq 2T(R, f) \]
on the right-hand side of the estimate. For this we insert \( \rho \) as the midpoint between \( r \) and \( R \), that is, \( \rho = \frac{R + r}{2} \) and rewrite the above estimate with \( R \) replaced by \( \rho \). The rewritten estimate is
\[ m \left( r, \frac{f'(z)}{f(z)}, \infty \right) \leq C_2 \left( \log^+ T(\rho, f) + \log \frac{1}{\rho - r} + \log (n(\rho, f, 0) + n(\rho, f, \infty)) + \log \rho \right). \]
The reason for inserting the mid-point \( \rho \) of \( r \) and \( R \) is to use the following inequality
\[
N(R, f, \infty) = \int_{t=0}^{R} \frac{n(t, f, \infty)}{t} \, dt \geq \int_{t=\rho}^{R} \frac{n(t, f, \infty)}{t} \, dt \\
\geq n(\rho, f, \infty) \int_{t=\rho}^{R} \frac{dt}{t} = n(\rho, f, \infty) \log \frac{R}{\rho} \\
= n(\rho, f, \infty) \log \frac{1}{1 - \frac{R-\rho}{R}} \geq n(\rho, f, \infty) \frac{R - \rho}{R}. 
\]
Similarly we have
\[ N(R, f, 0) \geq n(\rho, f, 0) \frac{R - \rho}{R}. \]
The estimate now can be rewritten as
\[ m \left( r, \frac{f'(z)}{f(z)}, \infty \right) \leq C_2 \left( \log^+ T(R, f) + \log \frac{1}{R - r} + \log R \right) \]
for some positive constant \( C_2 \). Note that we have seen the use of this technique of dominating the counting function \( N(R, f, 0) \) by the number-of-zeros-function \( n(\rho, f, 0) \) in the proof of Hadamard’s factorization theorem.

Final Step of Proof of Logarithmic Derivative Lemma – Use of Borel’s Lemma in Choosing Radius of Circle in Poisson-Jensen Formula in Terms of Radial
Coordinate of Variable Point. The radius $R$ of the circle used in the Poisson-Jensen formula appears on the right-side of the estimate for the proximity function of the logarithmic derivative to infinity. To finish the proof of the logarithmic derivative lemma, we need to replace expressions involving $R$ by expressions in $r$. For that, we have to find a good way to choose the radius of the circle $R$ used in the Poisson-Jensen formula. The tool used in this choice of $R$ in terms of the radial coordinate $r$ of the variable point $z$ is Borel’s lemma which we now state and prove.

Borel’s Lemma. Let $T(r)$ be a continuous, increasing function with $T(r) \geq 1$ for $r_0 \leq r < \infty$. Then the set $Z$ of points $r_0 \leq r < \infty$ such that the inequality

$$(*)_r \quad T \left( r + \frac{1}{T(r)} \right) < 2T(r)$$

fails to hold is of total linear measure $\leq 2$.

Proof of Borel’s Lemma. Inductively define $r_{\nu}$ and $r'_{\nu}$ for positive integers $\nu$ as follows. Set $r_1$ to be the infimum of points $r \geq r_0$ such that the inequality $(*)_r$ fails. Inductively, set

$$r'_{\nu} = r_{\nu} + \frac{1}{T(r_{\nu})}$$

and set $r_{\nu+1}$ to be the infimum of points $r \geq r'_{\nu}$ such that the inequality $(*)_r$ fails. By the choice of $r_{\nu+1}$ we know that the open interval $(r'_{\nu}, r_{\nu+1})$ must be disjoint from $Z$.

We claim that $Z$ is contained in the union of $[r_{\nu}, r'_{\nu}]$ for $1 \leq \nu < \infty$. For this claim it suffices for us to show that $r_{\nu} \to \infty$ as $\nu \to \infty$, because $r_{\nu} < r'_{\nu} \leq r_{\nu+1}$ and each $(r'_{\nu}, r_{\nu+1})$ is disjoint from $Z$. Assume the contrary so that the supremum $r_{\infty}$ of $r_{\nu}$ for $1 \leq \nu < \infty$ is finite. From the inequality $r_{\nu} < r'_{\nu} \leq r_{\nu+1}$ it follows that $r'_{\nu} \to r_{\infty}$ also as $\nu \to \infty$. Taking the limit of

$$r'_{\nu} - r_{\nu} = \frac{1}{T(r_{\nu})}$$

as $\nu \to \infty$ would give us the contradiction

$$0 = \frac{1}{T(r_{\infty})} > 0$$

from the continuity of $T(r)$. This verifies the claim.
The choice of $r_\nu$ implies the failure of the inequality $(\ast)_r$ for a sequence of $r$ approaching $r_\nu$ from the right. In particular, it means that the inequality $(\ast)_r$ fails at $r = r_\nu$ and

$$T \left( r_\nu + \frac{1}{T(r_\nu)} \right) \geq 2T(r_\nu),$$

which, by the definition of $r'_\nu$, means that

$$T(r'_\nu) \geq 2T(r_\nu).$$

From the fact that $T(r)$ is an increasing function of $r$ it follows from $r_{\nu+1} \geq r'_\nu$ that

$$T(r_{\nu+1}) \geq 2T(r_\nu).$$

Hence the linear measure of $Z$ (which is contained in $\cup_{\nu=1}^\infty [r_\nu, r'_\nu]$) is bounded above by

$$\sum_{\nu=1}^\infty (r'_\nu - r_\nu) = \sum_{\nu=1}^\infty \frac{1}{T(r_\nu)} \leq \sum_{\nu=1}^\infty \frac{1}{2^{\nu-1}T(r_1)} \leq \sum_{\nu=1}^\infty \frac{1}{2^{\nu-1}} = 2.$$

This finishes the proof of Borel’s lemma.

The set referred to in the notation $\|$ in the statement of the logarithmic derivative lemma is the set $Z$ in Borel’s lemma when $T(r)$ is set to be the Nevanlinna characteristic function $T(r, f)$. We choose $R = r + \frac{1}{T(r,f)}$. By Borel’s lemma

$$T(R, f) < 2T(r, f) \|$$

so that

$$\log^+ T(R, f) < 2T(r, f) \leq \log^+ T(r, f) + O(1) \|.$$

Moreover, $R - r = \frac{1}{T(r,f)}$ implies that

$$\log \frac{1}{R - r} = \log T(r, f) \leq \log^+ T(r, f)$$

and

$$\log R = \log(r + T(r, f)) \leq \log r + \log^+ T(r, f) + O(1).$$

This finishes the proof of the logarithmic derivative lemma

$$m \left( r, \frac{f'}{f}, \infty \right) \leq O \left( \log T(r, f) + \log r \right),$$
Remark on the Exceptional Set in the Statement of the Logarithmic Derivative Lemma. The notation $\| \|$ in the statement of the logarithmic derivative lemma occurs because of the use of Borel’s lemma in its proof. The exceptional set in the proof of Borel’s lemma is actually of Euclidean measure $\leq 2$. In the statement of the logarithmic lemma we require only the much weaker statement that the exceptional set is of finite measure with respect to $\frac{dt}{t}$. The difference is not relevant to the application of the logarithmic derivative lemma to the proof of the second main theorem, where the actual condition used that outside a finite number of such exceptional sets there is a sequence of points going to infinity. We have seen a similar requirement of the existence of an appropriate sequence of points going to infinity when we estimate the growth rate of the reciprocal of an infinite product in the proof of Hadamard’s factorization theorem. As long as the exceptional set is specified in such a way that $(A, \infty)$ is not contained in the union of a finite number of such exceptional sets no matter how large $A$ is. That is, the reason why the condition for exceptional sets to have finite measure with respect to $\frac{dt}{t}$ is good enough, because $\int_{t=A}^{\infty} \frac{dt}{t} = \infty$ for any $A > 0$ no matter how large $A$ is.

TO BE CONTINUED ...