

THETA FUNCTIONS OF JACOBI

For factorization of an elliptic function into a quotient of products of translates of a theta functions with a single zero in the fundamental parallelogram (as the analogue of the factorization of rational functions from the fundamental theorem of algebra), we have already seen the Weierstrass sigma function $\sigma(z)$ for a given period lattice as such a theta function which is defined by an infinite product with Weierstrass factors (as the multiplicative analogue of the use, in the additive setting, of Taylor polynomials of appropriate degrees of partial fractions). We now look at the Jacobi approach. In the approach of Weierstrass, for a given period lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, only one basic elliptic function is considered, namely the Weierstrass \wp function. However, in the approach of Jacobi, (from the posted lecture notes on “Jacobi’s Approach to Elliptic Functions”) there are three basic Jacobian elliptic functions $\operatorname{sn} w$, $\operatorname{cn} w$, and $\operatorname{dn} w$, whose period lattices given in the figure below are all different.

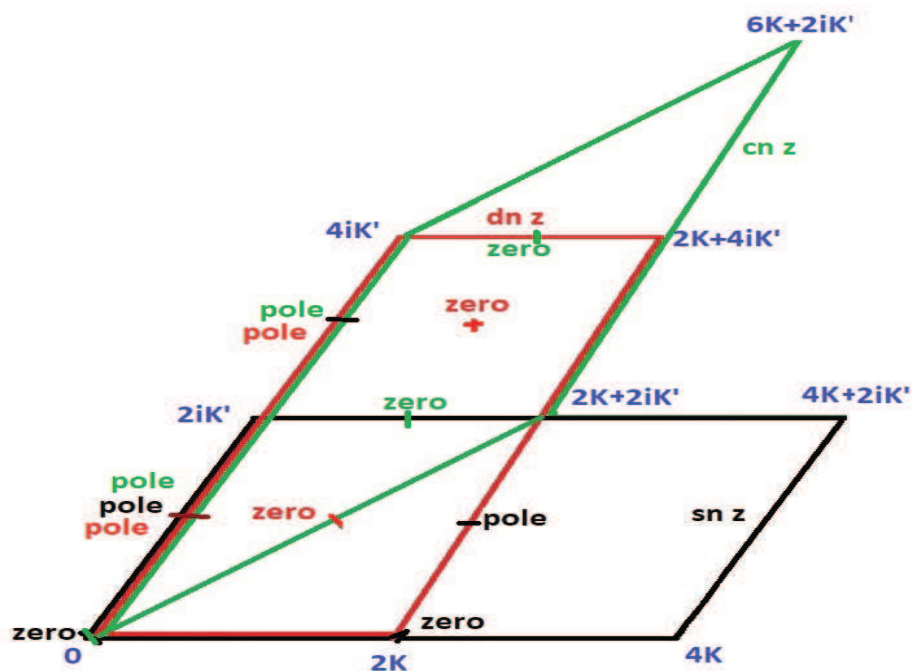


Figure 1: Three different period lattices for the three basic Jacobian elliptic functions

(Colored in Black) Period lattice of $\operatorname{sn} z$ is $\mathbb{Z}(4K) + \mathbb{Z}(2iK')$. Simple zeroes at 0 and $2K$. Simple poles at iK' and $2K + iK'$.

(Colored in Green) Period lattice of $\operatorname{cn} z$ is $\mathbb{Z}(2K + 2iK') + \mathbb{Z}(4iK')$. Simple zeroes at $K + 2iK'$ and $K + 4iK'$. Simple poles at iK' and $3iK'$.

(Colored in Red) Period lattice of $\operatorname{dn} z$ is $\mathbb{Z}(2K) + \mathbb{Z}(4iK')$. Simple zeroes at $K + iK'$ and $K + 3iK'$. Simple poles at iK' and $3iK'$.

Common Half-Size Parallelogram. In order to have a common lattice for all three, we can consider the half-size fundamental parallelogram spanned by $2K$ and $2iK'$. When it is doubled up with its translation to the right, the result is the fundamental parallelogram for the Jacobian elliptic sine function $\operatorname{sn} w$, which is spanned by $4K, 2iK'$. When it is doubled up with its translation up, the result is the fundamental parallelogram for the Jacobian delta amplitude function $\operatorname{dn} w$, which is spanned by $2K, 4iK'$. For the Jacobian elliptic cosine function $\operatorname{cn} w$, the half-size fundamental parallelogram spanned by $2K$ and $2iK'$ is equivalent to the one spanned by $2K + 2iK'$ and $2iK'$, which can be doubled up by upward translation to the one spanned by $2K + 2iK'$ and $4iK'$ to form the period parallelogram for $\operatorname{cn} w$.

The Jacobian theta functions are constructed on this common half-size parallelogram spanned by $2K$ and $2iK'$. With respect to this common half-size parallelogram spanned by $2K$ and $2iK'$, the functions $\operatorname{sn} w$, $\operatorname{cn} w$, and $\operatorname{dn} w$ are not elliptic, but have ± 1 as the periodicity factors.

The analogous situation with the trigonometric function $\sin w$ with $2K = \pi$ and $K' = \infty$ is $\sin(w + 2K) = \sin(w + \pi) = -\sin w$ so that -1 is the periodicity for the period $\pi = 2K$. Here is the table for the periodicity factors for $\operatorname{sn} w$, $\operatorname{cn} w$, and $\operatorname{dn} w$ with respect to half-size parallelogram spanned by $2K$ and $2iK'$.

	$\operatorname{sn} z$	$\operatorname{cn} z$	$\operatorname{dn} z$
$2K$	-1	-1	1
$2iK'$	1	-1	-1

The entries for the table are read off from the following formulas which are consequences of the addition formula and special values of $\operatorname{sn} z$ at some half-

periods.

$$\begin{aligned} \operatorname{sn}(z + 2K) &= -\operatorname{sn} z & \text{and} & \operatorname{sn}(z + 2iK') = \operatorname{sn} z, \\ \operatorname{cn}(z + 2K) &= -\operatorname{cn} z & \text{and} & \operatorname{cn}(z + 2iK') = -\operatorname{cn} z, \\ \operatorname{dn}(z + 2K) &= \operatorname{dn} z & \text{and} & \operatorname{dn}(z + 2iK') = -\operatorname{dn} z. \end{aligned}$$

We are going to construct Jacobian theta functions on this common half-size fundamental parallelogram spanned by $2K$ and $2iK'$. We now relabel this parallelogram as π and $\pi\tau$ so that the degeneration of $\operatorname{sn} w$ to $\sin w$ is when $\operatorname{Im} \tau \rightarrow \infty$. It turns out that there are four basic Jacobian theta functions $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ (each one with a single zero in the fundamental domain of the lattice $\mathbb{Z}\pi + \mathbb{Z}\pi\tau$), related by translation by half-periods $\frac{\pi}{2}, \frac{\pi\tau}{2}, \frac{\pi+\pi\tau}{2}$. We will then discuss the algebraic relations (motivated by $\sin^2 w + \cos^2 w = 1$), the differential relations (motivated by $\frac{d}{dw} \sin w = \cos w$), and the additional formulas for the four Jacobian theta functions.

An important application is that arithmetic applications, like the sum-of-squares theorems and quadratic reciprocity and many identities of the kind in Ramanujan's notebook, come from the arithmetic properties of the Fourier coefficients. The one theta function ϑ_3 with single zero in the center of the fundamental parallelogram satisfies an important identity from a change of the pair of generating periods, corresponding to

$$(w, \tau) \mapsto \left(\frac{w}{\tau}, -\frac{1}{\tau} \right).$$

It is related to the Poisson summation formula for the sums of a function and its Fourier transform at integral points. Many arithmetic consequences, such as the sum-of-squares theorems and the quadratic reciprocity, can be derived from that identity. The functional equation for the Riemann zeta function is related to it via Mellin transform.

One way to derive identities for elliptic functions is to eliminate principal parts additively or zeros and poles multiplicatively. The use of Fourier series provides a new way of establishing identities involving Jacobian theta functions, which are derived from the rather cumbersome method of equating Fourier coefficients. The factorization of the three basic Jacobian elliptic functions as quotients of four basic Jacobian theta functions is derived by verifying first-order differential equations by equating Fourier coefficients. An

important technique, known as the Jacobi fundamental formulas, provides identities of four terms, each of which is a product of four Jacobian theta functions.

Topics of Jacobian Theta Functions Covered. We will define the four Jacobian theta functions $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ for the lattice $\mathbb{Z}\pi + \mathbb{Z}\pi\tau$ by Fourier series. Their periodicity factors are summarized in the table of periodicity factors. The four identities which are analogues of $\sin^2 \theta + \cos^2 \theta = 1$ will be derived. They are identities involving the squares of Jacobian theta functions and their values at 0 which are called the Jacobian theta constants. The three basic Jacobian elliptic functions will be expressed as quotients of Jacobian theta functions. Their elliptic moduli will be expressed in terms of the Jacobian theta constants.

One crucial ingredient in expressing the three basic Jacobian elliptic functions as quotients of Jacobian theta functions is the identity

$$\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0),$$

which is the analogue of

$$\left(\frac{d}{d\theta} \sin \theta \right)_{\theta=0} = (\cos \theta)_{\theta=0},$$

and which is proved by using the heat-equation-type differential equation satisfied by the four Jacobian theta functions, proved by using their Fourier series.

Theta Functions as Multiplicative Building Blocks. Any meromorphic function f on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ can be written uniquely as

$$f(w) = \frac{\prod_{k=1}^p (w - a_k)}{\prod_{\ell=1}^q (w - b_\ell)}$$

when $\{a_1, \dots, a_p\}$ and $\{b_1, \dots, b_q\}$ are disjoint. We consider each of $w - a_j$ and $w - b_j$ a multiplicative building block which has a simple zero on \mathbb{C} and a pole at ∞ . An elliptic function is a meromorphic function on \mathbb{C} with two \mathbb{R} -linearly independent periods ω_1, ω_2 , which is the same as a meromorphic function on $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$. The question is whether there is a corresponding object of a multiplicative building block for elliptic functions.

It is not possible to have an elliptic function with a simple zero and a simple pole, because the derivative of its logarithm violates the condition that the sum of the residues of an elliptic function over a fundamental parallelogram must be zero. Since we are interested in taking the quotient of products of such multiplicative building blocks, we can relax the condition of periodicity to allow a nowhere zero holomorphic factor for each period. To make sure that the factor is nowhere zero holomorphic, the most natural way is to make it the exponential function of a holomorphic function. This function in the exponent of the periodicity factor cannot be a constant, otherwise the derivative of the logarithmic of the multiplicative building block would be an elliptic function for which the sum of its residues over a fundamental parallelogram is nonzero. The simplest situation is that the function in the exponent of the periodicity factor is a polynomial of degree ≤ 1 .

Construction of Theta Function by Recurrent Determination of the Coefficients of Its Fourier Series. Let us construct the simplest nontrivial example f of this kind. We use π and $\pi\tau$ with $\text{Im } \tau > 0$ as the two periods. Assume that our multiplicative building block has period π and a periodicity factor for $\pi\tau$ which is the exponent of a polynomial of degree ≤ 1 . Since f has period π , we can write $f(w)$ as a function of e^{2iw} .

$$f(w) = \sum_{n=-\infty}^{\infty} c_n e^{2niw}.$$

For the other period $\pi\tau$ we have a periodicity factor

$$f(w + \pi\tau) = e^{aw+b} f(w)$$

for some $a, b \in \mathbb{C}$.

In order to make the induction process of the determination of the unknown coefficients c_n as simple as possible, we choose $a = -2i$ and $b = -\pi\tau i + \pi i$ so that $e^{aw+b} = -e^{-2iw-i\pi\tau}$ and

$$f(w + \pi\tau) = -e^{-2iw-i\pi\tau} f(w)$$

to solve for the undetermined coefficients c_n . From

$$\begin{aligned} \sum_{n=-\infty}^{\infty} c_n e^{2ni(w+\pi\tau)} &= -e^{-2iw-\pi\tau i} \sum_{n=-\infty}^{\infty} c_n e^{2niw} \\ &= \sum_{n=-\infty}^{\infty} -e^{-\pi\tau i} c_n e^{2i(n-1)w} = \sum_{n=-\infty}^{\infty} -e^{-\pi\tau i} c_{n+1} e^{2niw} \end{aligned}$$

we get the recurrent formula $c_{n+1} = -e^{(2n+1)\pi\tau i} c_n$ for the coefficients c_n . We choose the simplest initial condition $c_0 = 1$. Let $q = e^{\pi\tau i}$. Since the sum of $1, 3, \dots, 2n-1$ is n^2 , it follows that $c_n = (-1)^n q^{n^2}$. The reason for the choice of $a = -2i$ is to shift the exponent from e^{2niw} to $e^{2i(n-1)w}$ so that after changing the dummy index from $n-1$ to n we end up with a formula which relates c_{n+1} to c_n . The choice of having $-\pi i\tau$ in $b = -\pi i\tau + \pi$ is to change the factor $e^{2ni\pi\tau i}$ due to $w \mapsto w + \pi\tau$ to $e^{(2n+1)\pi\tau i}$ so that the sum $1, 3, \dots, 2n-1$ from the recurrent formula becomes the simpler expression n^2 . The choice of having π in $b = -\pi i\tau + \pi$ simply adds the factor $(-1)^n$.

This way we arrive at the function

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niw} = \sum_{n=-\infty}^{\infty} (-1)^n e^{n^2\pi i\tau + 2niw}$$

and we denote it by $\vartheta(w)$ (also by $\vartheta(w|\tau)$ or $\vartheta(w, \tau)$ to emphasize its dependence on τ). It is the simplest Jacobi theta function. The transformation equations for ϑ by translation by π and $\pi\tau$ are

$$\vartheta(w + \pi) = \vartheta(w), \quad \vartheta(w + \pi\tau) = -e^{-2iw - i\pi\tau} \vartheta(w).$$

We would like to remark why we use w and not z as the independent variable. The reason is that the elliptic functions are defined as inverse functions and so w which is frequently chosen as the dependent variable is used as the independent variable.

Convergence and Number of Zeros of Constructed Jacobian Theta Function. First of all we observe that the infinite series converges, because

$$|q| = |e^{\pi\tau i}| = e^{-\pi \operatorname{Im} \tau} < 1$$

and its exponent n^2 grows fast enough as a function of n to handle the growth of e^{2niw} whose exponent depends only linearly on n . We now determine the

number of zeroes of this entire function ϑ in a fundamental parallelogram by using the argument principle for a fundamental parallelogram and integrating $d \log \vartheta$ along its boundary. Since π is actually a period for ϑ , the integral of $d \log \vartheta$ along the line segments $-[0, \pi\tau]$ and $[\pi, \pi + \pi\tau]$ cancel out, because of the difference in orientation.

The integral of $d \log \vartheta$ along the line segments $[0, \pi]$ and $-\pi\tau, \pi + \pi\tau]$ is equal to the length of $[0, \pi]$ times $2i$ which is the negative of the derivative of the exponent of the periodicity factor for $\pi\tau$. Thus the net result of integrating $d \log \vartheta$ along the boundary of the fundamental parallelogram with vertices $0, \pi, \pi + \pi\tau, \pi\tau$ is $2\pi i$. According to the argument principle, we have precisely one zero for ϑ in a fundamental parallelogram. However, at this point we do not know where the zero is. The zero can be located in the following way.

In terms of trigonometric functions $\vartheta(w)$ is given by

$$\vartheta(w) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nw,$$

because both $(-1)^n$ and q^{n^2} are unchanged under $n \mapsto -n$. It indicates that perhaps by changing in some way the cosine function to the sine function, one may say something about where the zero is from the vanishing of the sine function at 0. The sine function can be obtained by translating the cosine function by $\frac{\pi}{2}$, which is a half-period in the lattice $\mathbb{Z}\pi + \mathbb{Z}\pi\tau$ of periods. This motivates the consideration of applying half-period translations to $\vartheta(w)$. There are three ways of applying half-period translations, by translating by $\frac{\pi}{2}$, $\frac{\pi\tau}{2}$ and $\frac{\pi}{2} + \frac{\pi\tau}{2}$.

Translations by Half-Periods. By translating the Jacobi theta function $\vartheta(w)$ by one of the three half-periods in a fundamental parallelogram we get three other functions $\vartheta_1, \vartheta_2, \vartheta_3$ after we use a factor which is the exponential of a polynomial of degree ≤ 1 to simplify the coefficient of the nonzero term e^{2niw} with the smallest $|n|$. We obtain the following three new theta functions $\vartheta_1(w), \vartheta_2(w), \vartheta_3(w)$ in addition to $\vartheta(w)$ which we now denote also by $\vartheta_4(w)$.

$$\begin{aligned}
\vartheta_1(w) &= -i e^{iw + \frac{1}{4}\pi i \tau} \vartheta_4\left(w + \frac{1}{2}\pi\tau\right) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n e^{\frac{1}{4}(2n+1)^2 i \pi \tau} e^{(2n+1)iw} \\
&= 2 \sum_{n=0}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} \sin(2n+1)w, \\
\vartheta_2(w) &= \vartheta_1\left(w + \frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} e^{\frac{1}{4}(2n+1)^2 i \pi \tau} e^{(2n+1)iw} = 2 \sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} \cos(2n+1)w, \\
\vartheta_3(w) &= \vartheta_4\left(w + \frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} e^{n^2 i \pi \tau} e^{2niw} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nw. \\
\vartheta_4(w) &= \sum_{n=-\infty}^{\infty} (-1)^n e^{n^2 i \pi \tau} e^{2niw} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nw.
\end{aligned}$$

A fractional power q^α means $e^{\alpha\pi i \tau}$. Here we use trigonometric functions instead of the exponential function in the series expansion to help us see the origin as the zero in $\vartheta_1(w)$.

We denote our original function $\vartheta(w)$ by $\vartheta_4(w)$, because the zero of $\vartheta_1(w)$ is at $w = 0$ and the numbering corresponds to the zero assuming the half-period points in the counter-clockwise sense. These four functions are all called the *Jacobian theta functions*.

Degenerate Theta Functions as Modulus Goes to Infinity. One advantage of using the trigonometric functions in the series expansion of $\vartheta_\nu(w)$ is that we can more readily see what the degenerate case $\tau = +\infty i$ corresponds to. In the degenerate case $\tau = +\infty i$ the limit

$$\lim_{\tau \rightarrow +\infty i} \frac{1}{2} q^{-1/4} \vartheta_\nu(w)$$

is simply $\sin w$ when $\nu = 1$ and is simply $\cos w$ when $\nu = 2$. When $\tau = +\infty i$, the two other Jacobian theta functions $\vartheta_3(w)$ and $\vartheta_4(w)$ become identically 1.

The sine function plays two roles when the period lattice degenerates to rank one. One role is that the sine function is a degenerate elliptic function

for the lattice $\mathbb{Z}2\pi$. The other role is that the sine function is the limit of a constant multiple of the first Jacobian theta function for the period lattice degenerates to the lattice $\mathbb{Z}\pi$ when $\tau = +\infty i$. Note that the lattices in both rôles are different. The reason for the difference of the two lattices is that the nonzero factor

$$-i \exp\left(iw + \frac{1}{4}\pi i\tau\right)$$

has been added to simplify the expression for $\vartheta_1(w)$ (see the table of periodicity factors below).

Table of Periodicity Factors of Four Jacobian Theta Functions. The periodicity factors of the four Jacobian theta functions are given in the following table.

	$\vartheta_1(w)$	$\vartheta_2(w)$	$\vartheta_3(w)$	$\vartheta_4(w)$
π	-1	-1	1	1
$\pi\tau$	$-N$	N	N	$-N$

Here

$$N = \frac{1}{q} e^{-2iw} = e^{-2iw - \pi i\tau}.$$

The reason why π is not a period (without periodicity factors) for $\vartheta_1(w)$ and $\vartheta_2(w)$ (though π is a period for $\vartheta_3(w)$ and $\vartheta_4(w)$) is again because of the addition of the nonzero factor

$$-i \exp\left(iw + \frac{1}{4}\pi i\tau\right)$$

to simplify the expression for $\vartheta_1(w)$. One use of the table of periodicity factors is that we can readily see from it how to take the square of the quotient of two Jacobian theta functions to get a doubly periodic function with respect to the lattice $\mathbb{Z}\pi + \mathbb{Z}\pi\tau$.

Terminology of Periodicity Factor. We are referring to the factor $e^{a_\nu w + b_\nu}$ in

$$f(w + \omega_\nu) = e^{a_\nu w + b_\nu} f(w)$$

as a periodicity factor when $f(w)$ is a function on \mathbb{C} and ω_1, ω_2 are \mathbb{R} -linearly independent complex numbers. In the literature such a factor is also called a *factor of automorphy*. We would like to comment on the terminology.

When we study a function f with two primitive periods ω_1, ω_2 , it can be considered as a function on the compact Riemann surface $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ of genus 1. The question arises about what happens to functions with more symmetry properties than having two primitive periods and how they should correspondingly be regarded as functions on a compact Riemann surface of genus > 1 .

According to the uniformization theorem which we will later discuss, unlike the case of a compact Riemann surface of genus 1 whose universal cover is \mathbb{C} , the universal cover of a compact Riemann surface of genus > 1 is the open unit disk \mathbb{D} . Unlike the case of a compact Riemann surface of genus 1 which is the quotient \mathbb{C}/L of \mathbb{C} by a period lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, a compact Riemann surface of genus > 1 is the quotient \mathbb{D}/Γ of the open unit disk \mathbb{D} by the action of a discrete subgroup Γ of the biholomorphism group $\text{Aut}(\mathbb{D})$. For a formulation which works for compact Riemann surfaces of all genus, we re-interpret a period ω for a function f as invariance of the function f under the action of translation $w \mapsto w + \omega$ in \mathbb{C} . The higher-genus analogue of a doubly periodic function is a function f on \mathbb{D} which is invariant under the action of every element of the discrete subgroup Γ of $\text{Aut}(\mathbb{D})$. Such a function is called an *automorphic function* with respect to Γ . correspondingly regarded as functions on a compact Riemann surface of genus > 1 .

When the situation is like a theta function requiring a factor in the formula for invariance, we call such a function f on \mathbb{D} (or more generally a complex manifold X) an *automorphic form* with respect to a discrete subgroup Γ of the biholomorphism group of \mathbb{D} (or of X) if $f(\gamma(w)) = \chi_\gamma(w)f(w)$ for $\gamma \in \Gamma$, where χ_γ is a nowhere zero holomorphic function on \mathbb{D} (or on X) and is called a *factor of automorphy*. That is why in the literature a factor such as $e^{a_\nu w + b_\nu}$ in

$$f(w + \omega_\nu) = e^{a_\nu w + b_\nu} f(w)$$

is also called a *factor of automorphy*. We follow the practice of some older textbooks such as the book *A Course of Modern Analysis* by Whittaker and Watson (Cambridge University Press 1973) and use the older terminology of *periodicity factor* (line 5 from the bottom of p.463 of the book of Whittaker-Watson).

TO BE CONTINUED ...