Math 213a Homework #3 Assigned September 24, 2024
due October 1, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 213a

Problem 1 *(Evaluation of a Definite Integral by Contour Integration of a Branch of a Holomorphic Function).* Let $0 < \alpha < 1$. Verify that

$$\int_{x=0}^{1} \frac{dx}{x^\alpha (1-x)^{1-\alpha}} = \frac{\pi}{\sin \alpha\pi}$$

by using the following choice of a branch of the multi-valued holomorphic function $z^\alpha (1-z)^{1-\alpha}$ and the following contour of integration.

The branch of $z^\alpha (1-z)^{1-\alpha}$ on $\mathbb{C} - [0,1]$ is defined as the product of

(i) the branch of $z^\alpha = r^\alpha e^{i\alpha\theta}$ on $\mathbb{C} - [0,\infty)$ with $z = re^{i\theta}$ and $0 < \theta < 2\pi$ and

(ii) the branch of $(1-z)^{1-\alpha} = \rho^{1-\alpha} e^{i(1-\alpha)\varphi}$ on $\mathbb{C} - [1,\infty)$ with $z = 1 - \rho e^{i\varphi}$ and $-\pi < \varphi < \pi$,

where the discrepancies of the values of z^α and $(1-z)^{1-\alpha}$ across the common part $(1,\infty)$ of the branch-cuts $[0,\infty)$ and $[1,\infty)$ cancel out each other.

Let G be the domain in $\mathbb{C}$ which is $\mathbb{D}_R(0)$ minus the closed set

$$\overline{\mathbb{D}_\varepsilon(0)} \cup \overline{\mathbb{D}_\varepsilon(1)} \cup \left\{ z = x + iy \in \mathbb{C} \,\middle|\, 0 \le x \le 1,\ -\varepsilon \le y \le \varepsilon \right\},$$

where $R > 0$ and $0 < \varepsilon < R - 1$. The contour of integration is the boundary ∂G of G.

Remark. This integral occurs in the proof of *Euler's reflection formula* for the Gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Problem 2 *(Partial Fraction Expansion by Applying Modified Cauchy Integral Formula to Meromorphic Function).* Let $f(z) = \operatorname{cosec} z - \frac{1}{z}$ for $z \neq 0$ and $f(0) = 0$. Use

$$\int_{C_n} \frac{f(\zeta)d\zeta}{\zeta(\zeta - z)}$$

for some appropriate sequence of contours C_n to determine the constants a_n in the partial fraction expansion

$$\operatorname{cosec} z - \frac{1}{z} = \sum_{n \in \mathbb{Z} - \{0\}} (-1)^n \left(\frac{a_n}{z - n\pi} + \frac{a_n}{n\pi} \right).$$

Remark. The corresponding formula for the partial fraction expansion of $\cot z$ is given in the posted lecture notes.

Problem 3 *(Application of Argument Principle).* Let ω_1, ω_2 be two complex numbers such that the imaginary part of $\frac{\omega_2}{\omega_1}$ is positive. Let $f(z)$ be a non-identically-zero entire function and a_1, a_2, b_1, b_2 be complex numbers such that

$$f(z + \omega_j) = e^{a_j z + b_j} f(z)$$

for $z \in \mathbb{C}$ and $j = 1, 2$. Let

$$\Omega = \left\{ \alpha_1 \omega_1 + \alpha_2 \omega_2 \,\middle|\, 0 < \alpha_1 < 1,\ 0 < \alpha_2 < 1 \right\}.$$

Assume that $f(z)$ does not vanish at any point of the boundary $\partial\Omega$ of Ω. Apply the argument principle to f on Ω to express the total number n_f of zeroes (with multiplicities counted) of f on Ω as a polynomial in $a_1, a_2, b_1, b_2, \omega_1, \omega_2$ whose coefficients are in the field $\mathbb{Q}(\pi i)$.

Hint: Compare the two increases of the argument of f along two parallel sides of the parallelogram $\partial\Omega$.

Remark. The result of Problem 3 will be used in computing the number of zeroes of a theta function inside a fundamental parallelogram.

Problem 4 *(Infinite Product Expansion from Partial Fraction Expansion of Logarithmic Derivative – Modified from Exercise 10 on p.155 of Stein's Book).* Find an infinite product expansion of the following two functions by using the partial fraction expansions of their logarithmic derivatives.

(a) $e^z - 1$.

(b) $\cos \pi z$.

Hint: The answers are

$$e^z - 1 = e^{\frac{z}{2}} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4n^2\pi^2}\right)$$

and

$$\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right).$$

Obtain the partial fraction expansion of the logarithmic derivative as a meromorphic function by applying the modified Cauchy integral formula to the meromorphic function, as in Problem 2.

Problem 5 *(Application of Rouché's Theorem).* Let $a > e$. Prove that the equation $e^z = az^n$ has n roots inside the unit circle.

Problem 6 *(Partial Fraction Expansion for Meromorphic Functions with Pole Orders ≤ 2)* Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$ whose poles are $\{a_n\}_{1 \leq n < \infty}$ with

$$0 < |a_1| \leq |a_2| \leq \cdots$$

so that the principal part of $f(z)$ at a_n is

$$\frac{b_n}{z - a_n} + \frac{c_n}{(z - a_n)^2}$$

with $(b_n, c_n) \neq (0, 0)$. Suppose that there is a sequence of closed contours C_n such that C_n includes $a_1, \cdots, a_n$ but no other poles. Assume that the distance R_n from C_n to the origin goes to infinity as $n \to \infty$ and the length L_n of C_n is of the order $O(R_n)$. Assume that on C_n we have $f(z) = o(R_n^{p+1})$. Prove that

$$f(z) = \sum_{\nu=0}^{p} \frac{z^\nu}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + P_n(z)\right) + \sum_{n=1}^{\infty} c_n \left(\frac{1}{(z - a_n)^n} + Q_n(z)\right)$$

for some polynomials $P_n(z)$ and $Q_n(z)$ of degree $\leq p$. Write down explicitly the polynomials $P_n(z), Q_n(z)$ in terms of a_n, b_n, c_n.

Remark. The special case of $c_n = 0$ is proved in the posted lecture notes.