Math 213a Homework #2 Assigned September 17, 2024
due September 24, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 213a

Problem 1 *(Gaussian Error Function Unchanged Under Fourier Transform, from Evaluation of Integral by Cauchy's Theory – from Stein-Shakarchi p.65, Exercise #4)* Prove that for all $\xi \in \mathbb{C}$,

$$e^{-\pi\xi^2} = \int_{x=-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx$$

holds.

Hint: Apply Cauchy's theorem to the holomorphic function

$$z \mapsto e^{\pi z^2}$$

on the rectangle with vertices at $\pm x, \pm x + iy$ and let $x \to \infty$.

Problem 2 *Second Category Property of $\mathbb{C}$ and Identity Theorem for Holomorphic Function – from Stein-Shakarchi p.67, Exercise #13)* Suppose f is a holomorphic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expression

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

Hint: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument and the second-category property of $\mathbb{C}$.

Problem 3 *(Radius of Convergence Computed from Distance to Nearest Pole and from Ratio Test – from Stein-Shakarchi p.67, Exercise #14)* Suppose that f is holomorphic in an open set containing the closed unit disk, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disk, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Hint: Write

$$f(z) = \sum_{j=1}^{\ell} \frac{b_j}{(z-z_0)^j} + \sum_{k=0}^{\infty} c_k z^k$$

(where b_j and c_k are complex numbers and $b_\ell \neq 0$) with the radius of convergence of the power series $\sum_{k=0}^{\infty} c_k z^k$ being > 1. Expand the principal part

$$\sum_{j=1}^{\ell} \frac{b_j}{(z-z_0)^j}$$

as a power series in z and use the ratio test for the convergence of a series.

Problem 4 *(Bergman Kernel as Reproducing Kernel for Square Integrable Holomorphic Functions, Constructed from Sub-Mean Value Property of Absolute-Value Square of Holomorphic Functions).* Let Ω be a bounded domain in $\mathbb{C}$. Let $f_\nu(z)$ $(1 \leq \nu < \infty)$ be an orthonormal basis of the Hilbert space of all square integrable holomorphic functions on Ω.

(a) Show that $\sum_{\nu=1}^{\infty} |f_\nu(z)|^2$ converges uniformly on compact subsets of Ω. (Hint: for any relatively compact disk D in Ω, use the fact that for any holomorphic function g on Ω the value $|g|^2$ at the center c of D is no more than the average of $|g|^2$ on D and apply this to the case $g = \sum_{\nu=1}^{n} a_\nu f_\nu(z)$ with $a_\nu = \overline{f_\nu(c)}$.)

(b) Define $K(z, \bar{w}) = \sum_{\nu=1}^{\infty} f_\nu(z)\overline{f_\nu(w)}$ and show that the series converges on uniformly on compact subsets of $\Omega \times \Omega$ and $K(z, \bar{w})$ is holomorphic in z and anti-holomorphic in w. The function $K(z, \bar{w})$ is known as the Bergman kernel function.

(c) Show that the Bergman kernel function has the following reproducing kernel property for square integrable holomorphic function $f(z)$ on Ω.

$$f(z) = \int_{\Omega} K(z, w) f(w) dw.$$

Problem 5 *(Finite Dimensionality of Space of Automorphic Forms by Application of Schwarz's Lemma).* Let Γ be a discrete subgroup of the biholomorphism group of the open unit disk $\mathbb{D}$ with a relative compact fundamental domain Ω. That is, Ω is a relatively compact subset of $\mathbb{D}$ such that the quotient map $\mathbb{D} \to \mathbb{D}/\Gamma$ maps Ω bijectively onto $\mathbb{D}/\Gamma$. Let $\{\sigma_\gamma \,\big|\, \gamma \in \Gamma\}$ be a collection of factors of automorphy in the sense that each σ_γ is a nowhere zero holomorphic function on Ω such that $\sigma_{\gamma\circ\rho)}(z) = \sigma_\gamma(\rho(z))\sigma_\rho(z)$ for $z \in \Omega$ and $\gamma, \rho \in \Gamma$. Let $\mathcal{F}$ be the $\mathbb{C}$-vector space of all automorphic forms f for the given set $\{\sigma_\gamma \,\big|\, \gamma \in \Gamma\}$ of factors of automorphy. That is,

$$f(\gamma z) = \sigma_\gamma(z) f(z)$$

for $z \in \mathbb{D}$ and $\gamma \in \Gamma$ and $f \in \mathcal{F}$.

Use Schwarz's lemma to prove that the $\mathbb{C}$-vector space $\mathcal{F}$ is finite-dimensional by following the steps outlined in the Hint below.

Hint: Choose $0 < R_1 < R_2 < 1$ such that the closure of Ω is contained in the open disk $\mathbb{D}_{R_1}$, where $\mathbb{D}_R$ means the open disk of radius R centered at the origin. By using

$$f(\gamma z) = \sigma_\gamma(z) f(z)$$

for $z \in \mathbb{D}$ and $\gamma \in \Gamma$ and $f \in \mathcal{F}$, one can find a positive constant C such that

$$\sup_{\mathbb{D}_{R_2}} |f| \leq C \sup_{\mathbb{D}_{R_1}} |f|$$

for $f \in \mathcal{F}$. If $\dim_{\mathbb{C}} \mathcal{F} > m$, there exists a non identically zero element $\hat{f}$ of $\mathcal{F}$ which vanishes at 0 to order $\geq m$. When m is too large, the Schwarz lemma yields a contradiction from

$$\sup_{\mathbb{D}_{R_1}} |\hat{f}| \leq \left(\frac{R_1}{R_2}\right)^m \sup_{\mathbb{D}_{R_2}} |\hat{f}|.$$