Math 213a Homework #1 Assigned September 10, 2024
due September 17, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 213a

Problem 1 *(Cauchy-Riemann Equations Insufficient for Complex Differentiability – from Exercise #12 on p.27 of Stein-Shakarchi).* The purpose of this problem is about the necessity of the differentiability of the real and imaginary parts as real-valued functions of two real variables in the question of the complex-differentiability of a complex-valued function of a complex variable which satisfies the Cauchy-Riemann equation at the point under consideration.

Consider the function

$$w = f(z) = f(x + iy) = \sqrt{|x||y|}$$

for $(x, y) \in \mathbb{R}^2$. Show that f satisfies the Cauchy-Riemann equation at the origin and yet the complex derivative $f'(z)$ of $f(z)$ does not exist at $z = 0$.

Problem 2 *(Cauchy-Riemann Equations in Polar Coordinates – from Exercise #9 on p.27 of Stein-Shakarchi).* Show that in polar coordinates (r, θ) of $\mathbb{C} = \mathbb{R}^2$, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where} \quad z = re^{i\theta} \quad \text{with} \quad -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Problem 3 *(Goursat's Technique Applied to Stokes's Theorem for Continuous Nonsmooth Forms).* Let R be a closed orthotope R in $\mathbb{R}^n$ (also known as a hyper-rectangle which is the n-dimensional version of a rectangle) and let Ω be an open neighborhood of R in $\mathbb{R}^n$. Let Ξ be a collection of C^∞ d-closed $(n-1)$-forms ξ on Ω. Let ω be a continuous $(n-1)$-form on Ω

with the property that for every point $c \in \Omega$ there is some $\xi^{(c)}$ in Ξ which approximates ω at c to an order higher than the first, in the sense that

$$\omega = \xi^{(c)} + E(x)$$

with

$$\lim_{x \to c} \frac{E(x)}{|x - c|} = 0$$

(where a form approaching 0 means all its coefficients approaching 0). Use Goursat's technique (from Goursat's contribution to the theorem of Cauchy-Goursat) to prove that $\int_{\partial R} \omega = 0$. Note that Stokes's theorem cannot be applied directly, because the coefficients of ω are not assumed to be continuously differentiable.

Hint: Derive a contradiction by assuming that the absolute value A of $\int_{\partial R} \omega$ is positive so that we can divide R equally into 2^n small closed orthotope with one small closed orthotope $R^{(1)}$ satisfying $\left| \int_{\partial R^{(1)}} \omega \right| \geq \frac{A}{2^n}$. Continue the process with R replaced by $R^{(1)}$ to get $R^{(2)}$ *et cetera* to end up with a sequence

$$\cdots \subset R^{(k-1)} \subset R^{(k)} \subset R^{(k+1)} \subset \cdots.$$

Problem 4 *(Quaternionic Differentiability in Terms of Complex Differentiability of Restrictions to Spaces of Complex Variables).* Denote by $\mathbb{H}$ the (noncommutative) quaternion division algebra $\mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ (over $\mathbb{R}$) with the multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

Denote by $\mathbb{S}$ the unit sphere

$$\{\mathbf{q} \in \mathbb{H} \,\big|\, \mathbf{q}^2 = -1\} = \{a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \,\big|\, (a, b, c) \in \mathbb{R}^3 \text{ with } a^2 + b^2 + c^2 = 1\}$$

in the purely imaginary part $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ of $\mathbb{H}$. For $\mathbf{I} \in \mathbb{S}$, let $L_{\mathbf{I}}$ denote the complex plane $\mathbb{R} + \mathbb{R}\mathbf{I}$ defined by $\mathbf{I}$.

For any open subset Ω of $\mathbb{H}$ and $\mathbf{I} \in \mathbb{S}$, let $\Omega_{\mathbf{I}}$ denote the restriction $\Omega \cap L_{\mathbf{I}}$ of Ω to the complex plane $L_{\mathbf{I}}$. For a $\mathbb{H}$-valued function f on an open subset Ω of $\mathbb{H}$, denote by $f_{\mathbf{I}}$ the restriction of f to $\Omega_{\mathbf{I}}$.

We call f quaterionic-differentiable on Ω if

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathbf{I}\frac{\partial}{\partial y}\right)f_{\mathbf{I}}(x+\mathbf{I}y)=0$$

at every point of $\Omega_{\mathbf{I}}$ and for all $\mathbf{I}\in\mathbb{S}$. Note that this equation corresponds to the vanishing of the differentiation $\frac{\partial}{\partial\bar{z}}$ for a function of the complex variable $z=x+iy$.

Let $\mathbf{J}\in\mathbb{S}$ be perpendicular to $\mathbf{I}$ in the sense that they are perpendicular as unit vectors in the purely imaginary part $\mathbb{R}^3$ of $\mathbb{H}$. Note that $\mathbb{H}=L_{\mathbf{I}}+L_{\mathbf{I}}\mathbf{J}$ when the two elements $\mathbf{I}$ and $\mathbf{J}$ of $\mathbb{S}$ are perpendicular.

Prove that f is quaterionic-differentiable on Ω if and only if for any two perpendicular elements $\mathbf{I}$ and $\mathbf{J}$ of $\mathbb{S}$, there are two functions F,G from $\Omega_{\mathbf{I}}$ to $L_{\mathbf{I}}$ satisfying

$$f_{\mathbf{I}}(x+\mathbf{I}y)=F(x+\mathbf{I}y)+G(x+\mathbf{I}y)\mathbf{J}$$

such that

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathbf{I}\frac{\partial}{\partial y}\right)F(x+\mathbf{I}y)=0$$

and

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathbf{I}\frac{\partial}{\partial y}\right)G(x+\mathbf{I}y)=0$$

at every point $x+\mathbf{I}y$ of $\Omega_{\mathbf{I}}$.