**Math 213a Homework #5 Assigned October 8, 2024**
**due October 15, 2024**

**Please submit the PDF file of your homework**
**to the CANVAS website for Math 213a**

**Problem 1.** *(Derivation of Double Periodicity of Weierstrass $\wp$ Function without Using its Derivative – from #4 on p.279 of Stein-Shakarchi's Complex Analysis).* Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1, \omega_2$ are two complex numbers which are $\mathbb{R}$-linearly independent. By rearranging the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} \left( \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right),$$

show directly, without differentiation, that $\wp(z+\omega) = \wp(z)$ whenever $\omega \in L$.

*Hint:* For $R$ sufficiently large, note that

$$\wp(z) = \wp^R(z) + O\left(\frac{1}{R}\right)$$

as $R \to \infty$, where

$$\wp^R(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in L, \\ 0<|\omega|<R}} \left( \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right).$$

Next, for any $c > 0$ observe that

$$\wp^R(z+\omega) - \wp^R(z) = O\left( \sum_{\substack{\omega \in L, \\ R-c<|\omega|<R+c}} \frac{1}{\omega^2} \right) = O\left(\frac{1}{R}\right)$$

as $R \to \infty$.

**Problem 2.** *(Symmetric Form of Addition Theorem for Weierstrass $\wp$ Function).* Prove that, if $u, v, w \in \mathbb{C} - \{0\}$ such that $u + v + w = 0$, then

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{vmatrix} = 0.$$

*Hint:* Choose complex constants $A, B, C$ not all zero such that $A\wp(z) + B\wp'(z) + C$ vanishes at $u$ and $v$.

**Problem 3.** *(Identities of Elliptic Functions from Elimination of Principal Parts).* For any pair of distinct complex numbers $a, b$, show by Liouville's theorem that

$$\begin{aligned}&\frac{d}{dz}\left(\wp(z-a)\wp(z-b)\right)\\&=\wp(a-b)\left(\wp'(z-a)+\wp'(z-b)\right)-\wp'(a-b)\left(\wp(z-a)-\wp(z-b)\right).\end{aligned}$$

**Problem 4.** *(Alternative Proof of the Addition Theorem of the Weierstrass $\wp$ Function by Using its Principal Part at 0).* Let $\omega_1, \omega_2$ be complex numbers which are linearly independent over $\mathbb{R}$ and $\wp(z)$ be the Weierstrass $\wp$ whose period lattice is $L$. Use

$$\wp(z) = \frac{1}{z^2} + 3s_4 z^2 + \cdots,$$

where

$$s_4 = \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4},$$

to prove the addition theorem

$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 - \wp(z_1) - \wp(z_2)$$

by verifying that for fixed $z_2$ the doubly periodic function

$$\varphi(z) := \frac{1}{4}\left(\frac{\wp'(z) - \wp'(z_2)}{\wp(z) - \wp(z_2)}\right)^2 - \wp(z) - \wp(z_2) - \wp(z + z_2)$$

of $z$ is entire on $\mathbb{C}$ and vanishes at 0.

*Hint:* Treat $s_4$ simply as a constant and ignore its relation to the lattice.

**Problem 5.** *(Equation involving $\wp(z)$, $\wp'(z)$ and $\sigma(z)$ obtained by Checking Double Periodicity and Location of Zeros and Poles).* For a period lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (where $\omega_1, \omega_2$ are two complex numbers which are $\mathbb{R}$-linearly independent), the *Weierstrass sigma function* $\sigma(z)$ for the lattice $L$ is defined by

$$\sigma(z) = z \prod_{\ell \in L - \{0\}} \left(1 - \frac{z}{\ell}\right) e^{\frac{z}{\ell} + \frac{1}{2}\left(\frac{z}{\ell}\right)^2},$$

which is a *theta function* in the sense that

$$\sigma(z + \omega_\nu) = -e^{\eta_\nu\left(z + \frac{1}{2}\omega_\nu\right)} \sigma(z) \quad \text{for} \quad \nu = 1, 2$$

for some complex constants $\eta_1, \eta_2$.

**(a)** For $u, v \in \mathbb{C}$, prove that

$$\wp(u) - \wp(v) = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}.$$

**(b)** For $u, v, w \in \mathbb{C}$, prove that

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{vmatrix} = \frac{2\sigma(u+v+w)\sigma(u-v)\sigma(v-w)\sigma(w-u)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3}.$$