Math 213a Homework #8 Assigned October 29, 2024
due November 5, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 213a

Problem 1. *(Schwarz-Christoffel Transformations to Rectangles with One Vertex at ∞ – from Exercise #20 on p.252 of Stein-Shakarchi's Complex Analysis).* Other examples of elliptic integrals providing conformal maps from the upper half-plane to rectangles are given below.

(a) The function

$$\int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}}, \quad \text{with } \lambda \in \mathbb{R} \text{ and } \lambda \neq 1$$

maps the upper half-plane conformally to a rectangle, one of whose vertices is the image of the point at infinity.

(b) In the case $\lambda = -1$, the image of

$$\int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

is a square whose side lengths are

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}}.$$

Problem 2. *(Schwarz-Christoffel Transformations to Triangles – from Exercise #21 on p.253 of Stein-Shakarchi's Complex Analysis).* We consider conformal mappings to triangles.

(a) Show that

$$\int_0^z z^{-\beta_1}(1-z)^{-\beta_2}dz,$$

with $0 < \beta_1 < 1$, $0 < \beta_2 < 1$, and $1 < \beta_1 + \beta_2 < 2$, maps the open upper half-plane $\mathbb{H}$ to a triangle whose vertices are images of $0, 1, \infty$, and with angles $\alpha_1\pi$, $\alpha_2\pi$, and $\alpha_3\pi$, where $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

(b) What happens when $\beta_1 + \beta_2 = 1$?

(c) What happens when $0 < \beta_1 + \beta_2 < 1$?

(d) In **(a)**, the length of the side of the triangle opposite angle $\alpha_j \pi$ is

$$\frac{\sin(\alpha_j \pi)}{\pi} \, \Gamma(\alpha_1) \, \Gamma(\alpha_2) \, \Gamma(\alpha_3) \, .$$

Problem 3. *(Koebe's More Explicit Construction of Map in Riemann Mapping Theorem by Maximizing Inner Circle of Image Instead of Size of Derivative at the Origin – Problem #6 on p.258 of Stein-Shakarchi's Complex Analysis).* Applying ideas of Carathéodory, Koebe gave a proof of the Riemann mapping theorem by constructing (more explicitly) a sequence of functions that converges to the desired conformal map.

Starting with a *Koebe domain* (*i.e.,* a simply connected domain $\mathcal{K}_0 \subset \mathbb{D}$ that isn ot all of $\mathbb{D}$, and which contains the origin), the strategy is to find an injective function f_0 such that $f_0(\mathcal{K}_0) = \mathcal{K}_1$ is a Koebe domain "larger" than $\mathcal{K}_0$. Then, one iterates this process, finally obtaining functions $F_n = f_n \circ \cdots \circ f_0 : \mathcal{K}_0 \to \mathbb{D}$ such that $F(\mathcal{K}_0) = \mathcal{K}_{n+1}$ and $\lim F_n = F$ is a conformal map from $\mathcal{K}_0$ to $\mathbb{D}$.

The *inner radius* of a region $\mathcal{K} \subset \mathbb{D}$ that contains the origin is defined by

$$r_{\mathcal{K}} = \sup\{\rho \geq 0 \, : \, D(0, \rho) \subset \mathcal{K}\},$$

where $D(0, \rho)$ is the open disk of radius ρ centered at 0. Also, a holomorphic injection $Qf : \mathcal{K} \to \mathbb{D}$ is said to an *expansion* if $f(0)$ and $|f(z)| > |z|$ for all $z \in \mathcal{K} - \{0\}$.

(a) Prove that if f is an expansion, then $r_{f(\mathcal{K})} \geq r_{\mathcal{K}}$ and $|f'(0)| > 1$.

Hint: Write $f(z)zg(z)$ and use the maximum principle to prove that $|f'(0)| = |g(0)| > 1$.

(b) Suppose we begin with a Koebe domain $\mathcal{K}_0$ and a sequence of expansions $\{f_0, f_1, \cdots, f_n, \cdots\}$, so that $\mathcal{K}_{n+1} = f_n(\mathcal{K}_n)$ are also Koebe domains. We then define holomorphic maps $F_n : \mathcal{K}_0 \to \mathbb{D}$ by $F_n = f_n \circ \cdots \circ f_0$. Prove that for each n, the function F_n is an expansion. Moreover, $F_n'(0) = \prod_{k=0}^n f_k'(0)$, and conclude that $\lim_{n\to\infty} |f_n'(0)| = 1$.

Hint: Prove that the sequence $\{|F_n'(0)|\}$ has a limit by showing that it is bounded above and monotone increasing. Use the Schwarz lemma.

(c) Show that if the sequence is osculating, that is, $r_{\mathcal{K}_n} \to 1$ as $n \to \infty$, then $\{F_n\}$ converges uniformly on compact subsets of $\mathcal{K}_0$ to a conformal map $F : \mathcal{K}_0 \to \mathbb{D}$.

Hint: If $r_{F(\mathcal{K}_0)} \geq 1$, then F is surjective.

(d) To construct the desired osculating sequence we shall use the automorphisms

$$\psi_\alpha = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

Given a Koebe domain $\mathcal{K}$¡ choose a point $\alpha \in \mathbb{D}$ on the boundary of $\mathcal{K}$ such that $|\alpha| = r_{\mathcal{K}}$, and also choose $\beta \in \mathbb{D}$ such that $\beta^2 = \alpha$. Let S denote the square root of ψ_α such that $S(0) = 0$. Why is such a function well defined? Prove that the function $f : \mathcal{K} \to \mathbb{D}$ defined by $f(z) = \psi_\beta \circ S \circ \psi_\alpha$ is an expansion. Moreover, show that

$$|f'(0)| = \frac{1 + r_{\mathcal{K}}}{2\sqrt{r_{\mathcal{K}}}}.$$

Hint: To prove that $|f(z)| > |z|$ on $\mathcal{K} - \{0\}$ apply the Schwarz lemma to the inverse function., namely $\psi_\alpha \circ g \circ \psi_\beta$ where $g(z) = z^2$.

(e) Use **(d)** to construct the desired sequence.

Problem 4. *(Elliptic Integral as Hypergeometric Series in Terms of Limit of Iterated Arithmetic and Geometric Means – Problem #9 on p.259 of Stein-Shakarchi's Complex Analysis).* Gauss found a connection between elliptic integrals and the familiar operations of forming arithmetic and geometric means.

We start with any pair (a, b) of numbers that satisfy $a \geq b > 0$, and form the arithmetic and geometric means of a and b, that is,

$$a = \frac{a + b}{2} \quad \text{and} \quad b_1 = \sqrt{ab}.$$

we then repeat these operations with a and b replaced by a_1 and b_1. Iterating this process provides two sequences $\{a_n\}$ and $\{b_n\}$ where a_{n+1} and b_{n+1} are the arithmetic and geometric means of a_n and b_n, respectively.

(a) Prove that the two sequences $\{a_n\}$ and $\{b_n\}$ have a common limit. This limit, which we denote by $M(z, g)$, is called the *arithmetic-geometric mean* of a and b.

Hint: Show that

$$a \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq b_n \geq \cdots \geq b_1 \geq b$$

and

$$a_n - b_n \leq \frac{a-b}{2^n}.$$

(b) Gauss's identity states that

$$\frac{1}{M(a,b)} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}.$$

To prove this relation, show that if $I(a, b)$ denotes the integral on the right-hand side, then it suffices to establish the invariance of I, namely

$$I(a,b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

Then observe that the connection with elliptic integrals takes the form

$$(*) \qquad I(a,b) = \frac{1}{a} K(k) = \frac{1}{a} \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

where $k^2 = 1 - \frac{b^2}{a^2}$, and that the relation (*) is a consequence of the identity

$$K(k) = \frac{2}{1+\tilde{k}} K\left(\frac{1-\tilde{k}}{1+\tilde{k}}\right)$$

if $\tilde{k}^2 = 1 - k^2$ and $10 < \tilde{k} < 1$ (which is the identity in Problem 2(b) in Homework #6).