

**Math 213a Homework #7 Assigned October 22, 2024
due October 29, 2024**

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to the CANVAS website for Math 213a**

Difference between Notations in the Posted Lecture Notes and in the Textbook by Stein-Shakarchi. For theta functions the lattice used in the posted lecture notes is

$$\mathbb{Z}\pi + \mathbb{Z}\pi\tau = \mathbb{Z}2K + \mathbb{Z}2iK'$$

which is the notation used by Jacobi and is compatible with the limiting case of the trigonometric functions when $\text{Im}\tau \rightarrow \infty$. On the other hand, the lattice in the book on *Complex Analysis* by Stein-Shakarchi is $\mathbb{Z} + \mathbb{Z}\tau$. The correspondence between the two kinds of notations is via the rescaling operation $z \mapsto \pi z$. In particular, the relation between the theta functions $\Pi(z|\tau)$ and $\Theta(z|\tau)$ in the book of Stein-Shakarchi and the third Jacobian theta function $\vartheta_3(z, \tau)$ is

$$\Pi(z|\tau) = \Theta(z|\tau) = \vartheta_3(\pi z, \tau).$$

Problem 1. (*Infinite Product Expansion of $\vartheta_3(z)$*). Prove that the third Jacobian theta function $\vartheta_3(z)$ which is defined by the following infinite Fourier series

$$\vartheta_3(z) = \sum_{n=-\infty}^{\infty} e^{n^2 i\pi\tau} e^{2niz}$$

can be expressed as the following infinite product

$$\vartheta_3(z) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}e^{2iz})(1 + q^{2n-1}e^{-2iz})$$

with $q = e^{2iz}$.

Hint: First check that the infinite product satisfies the same transformation rules for the periods π and $\pi\tau$ as $\vartheta_3(z)$. Then check that the constant $c(\tau)$ defined by the quotient of $\vartheta_3(z)$ by the infinite product must be 1 by verifying that $c(\tau) = c(4\tau)$ at $z = \frac{\pi}{2}$ and letting $k \rightarrow \infty$ in $c(\tau) = c(4^k\tau)$. Compare with the arguments on pp.284-289 of Stein-Shakarchi's *Complex Analysis*, where $\Pi(z|\tau) = \Theta(z|\tau) = \vartheta_3(\pi z, \tau)$.

Remark: Jacobian theta functions are related to Weierstrass sigma function via a Möbius transformation. Since Weierstrass sigma function is defined as an infinite product. The above infinite product expansion of the third Jacobian theta functions can also be derived from the definition of Weierstrass sigma function as an infinite product.

Problem 2. (*Triangular Number Identity and Septagonal Number Identity – from Exercise #7 on p.312 of Stein-Shakarchi's Complex Analysis*). Let $q = e^{\pi i\tau}$ with $\text{Im } \tau > 0$. Use the following product formula for $\Theta(z|\tau)$:

$$\Theta(z|\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}e^{2i\pi z})(1 + q^{2n-1}e^{-2i\pi z})$$

(and its Fourier series with period 1) to prove:

(a) The “triangular number” identity

$$\prod_{n=0}^{\infty} (1 + x^n)(1 - x^{2n+2}) = \sum_{n=-\infty}^{\infty} x^{\frac{n(n+1)}{2}}$$

which holds for $|x| < 1$.

(b) The “septagonal number” identity

$$\prod_{n=0}^{\infty} (1 - x^{5n+1})(1 - x^{5n+4})(1 - x^{5n+5}) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+3)}{2}}$$

which holds for $|x| < 1$.

Problem 3. (*Infinite Sum Expansion of Logarithmic Derivative of Jacobian Theta Functions*). Let $q = e^{\pi\tau i}$ with $\text{Im } \tau > 0$ and let the period lattice be $\mathbb{Z}\pi + \mathbb{Z}\pi\tau$.

(a) Show that

$$\frac{\vartheta_1'(z)}{\vartheta_1(z)} = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^{2n} \sin 2z}{1 - 2q^{2n} \cos 2z + q^{4n}}$$

by first showing that it is a doubly-periodic function of z with no singularities and then checking that the constant value of the difference between the left-hand side and the right-hand side is 0 at $z = 0$.

Hint: For $n \in \mathbb{N}$, the function

$$g_n(z) = 1 - 2q^{2n} \cos 2z + q^{4n}$$

can be factored as the product of $e^{2iz}q^{2n} - 1$ and $e^{-2iz}q^{2n} - 1$ and therefore has a simple zero at $z = n\pi\tau$. Note that

$$\frac{g_n'(z)}{g_n(z)} = 4 \frac{q^{2n} \sin 2z}{1 - 2q^{2n} \cos 2z + q^{4n}}.$$

(b) Prove similarly that

$$\begin{aligned} \frac{\vartheta_2'(z)}{\vartheta_2(z)} &= -\tan z - 4 \sum_{n=1}^{\infty} \frac{q^{2n} \sin 2z}{1 + 2q^{2n} \cos 2z + q^{4n}}, \\ \frac{\vartheta_3'(z)}{\vartheta_3(z)} &= -4 \sum_{n=1}^{\infty} \frac{q^{2n-1} \sin 2z}{1 + 2q^{2n-1} \cos 2z + q^{4n-2}}, \\ \frac{\vartheta_4'(z)}{\vartheta_4(z)} &= 4 \sum_{n=1}^{\infty} \frac{q^{2n-1} \sin 2z}{1 - 2q^{2n-1} \cos 2z + q^{4n-2}}. \end{aligned}$$

Problem 4. (*Infinite Product Expansions of Jacobian Theta Functions from Infinite Sums of Their Logarithmic Derivatives*). By integrating and then exponentiating both sides of the equations in Problem 3 above and using the effect of translation by half-periods on the Jacobi theta functions, prove that

there exists some constant G such that

$$\begin{aligned}\vartheta_1(z) &= 2Gq^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2z + q^{4n}), \\ \vartheta_2(z) &= 2Gq^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2z + q^{4n}), \\ \vartheta_3(z) &= G \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2z + q^{4n-2}), \\ \vartheta_4(z) &= G \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2z + q^{4n-2}).\end{aligned}$$

Then use the relation $\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$ and $G \rightarrow 1$ as $\text{Im } \tau \rightarrow \infty$ to show that

$$G = \prod_{n=1}^{\infty} (1 - q^{2n})$$

to give the following infinite product expansions of the four Jacobian theta functions:

$$\begin{aligned}\vartheta_1(z) &= 2q^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty} (1 - q^{2n}) (1 - 2q^{2n} \cos 2z + q^{4n}), \\ \vartheta_2(z) &= 2q^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty} (1 - q^{2n}) (1 + 2q^{2n} \cos 2z + q^{4n}), \\ \vartheta_3(z) &= \prod_{n=1}^{\infty} (1 - q^{2n}) (1 + 2q^{2n-1} \cos 2z + q^{4n-2}), \\ \vartheta_4(z) &= \prod_{n=1}^{\infty} (1 - q^{2n}) (1 - 2q^{2n-1} \cos 2z + q^{4n-2}).\end{aligned}$$