Math 213a Homework #4 Assigned October 1, 2024
due October 8, 2024

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Notations and Formulas. The notations of the other nine Jacobian elliptic functions in terms of the three basic Jacobian elliptic functions (*i.e.,* the elliptic sine function, the elliptic cosine function, and the delta amplitude function) are as follows.

$$\operatorname{sc} z = \frac{\operatorname{sn} z}{\operatorname{cn} z}, \quad \operatorname{sd} z = \frac{\operatorname{sn} z}{\operatorname{dn} z},$$

$$\operatorname{cs} z = \frac{\operatorname{cn} z}{\operatorname{sn} z}, \quad \operatorname{cd} z = \frac{\operatorname{cn} z}{\operatorname{dn} z},$$

$$\operatorname{ds} z = \frac{\operatorname{dn} z}{\operatorname{sn} z}, \quad \operatorname{dc} z = \frac{\operatorname{dn} z}{\operatorname{cn} z},$$

$$\operatorname{ns} z = \frac{1}{\operatorname{sn} z}, \quad \operatorname{nc} z = \frac{1}{\operatorname{cn} z}, \quad \operatorname{nd} z = \frac{1}{\operatorname{dn} z}.$$

Defining Indefinite Integrals for the Inverses of the Three Basic Jacobian Elliptic Functions.

$$\begin{aligned}
\operatorname{sn}^{-1} w &= \int_{\zeta=0}^{w} \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \\
\operatorname{cn}^{-1} w &= \int_{\zeta=w}^{1} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2+k^2\zeta^2)}}, \\
\operatorname{dn}^{-1} w &= \int_{\zeta=w}^{1} \frac{d\zeta}{\sqrt{(\zeta^2-1)(1-k^2-\zeta^2)}}.
\end{aligned}$$

Differentiation Formulas for the Three Basic Jacobian Elliptic Functions.

$$\frac{d}{dz}\,\mathrm{sn}\,z = \mathrm{cn}\,z\,\mathrm{dn}\,z,$$
$$\frac{d}{dz}\,\mathrm{cn}\,z = -\mathrm{sn}\,z\,\mathrm{dn}\,z,$$
$$\frac{d}{dz}\,\mathrm{dn}\,z = -k^2\mathrm{sn}\,z\,\mathrm{cn}\,z.$$

Addition Formulas for the Three Basic Jacobian Elliptic Functions.

$$\mathrm{sn}(z+w) = \frac{\mathrm{sn}\,z\,\mathrm{cn}\,w\,\mathrm{dn}\,w + \mathrm{sn}\,w\,\mathrm{cn}\,z\,\mathrm{dn}\,z}{1-k^2\mathrm{sn}^2 z\,\mathrm{sn}^2\,w},$$

$$\mathrm{cn}(z+w) = \frac{\mathrm{cn}\,z\,\mathrm{cn}\,w - \mathrm{sn}\,z\,\mathrm{sn}\,w\,\mathrm{dn}\,z\,\mathrm{dn}\,w}{1-k^2\mathrm{sn}^2 z\,\mathrm{sn}^2\,w},$$

$$\mathrm{dn}(z+w) = \frac{\mathrm{dn}\,z\,\mathrm{dn}\,w - k^2\mathrm{sn}\,z\,\mathrm{sn}\,w\,\mathrm{cn}\,z\,\mathrm{cn}\,w}{1-k^2\mathrm{sn}^2 z\,\mathrm{sn}^2\,w}.$$

Problem 1. *(Integration of Jacobian Elliptic Functions).* Verify the following formulas for the integration of the three basic Jacobian elliptic functions.

$$\int \mathrm{sn}\,z dz = \frac{1}{k}\log\left(\mathrm{dn}\,z - k\mathrm{cn}\,z\right),$$
$$\int \mathrm{cn}\,z dz = \frac{1}{k}\sin^{-1}(k\,\mathrm{sn}\,z),$$
$$\int \mathrm{dn}\,z dz = \sin^{-1}(\mathrm{sn}\,z).$$

Problem 2 *(Addition Formula for Elliptic Cosine Function from Vanishing of Logarithmic Derivative Proved by Using Formulas for Its First-Order and Second-Order Derivatives).* Prove the addition formula

$$\mathrm{cn}(z+w) = \frac{\mathrm{cn}\,z\,\mathrm{cn}\,w - \mathrm{sn}\,z\,\mathrm{sn}\,w\,\mathrm{dn}\,z\,\mathrm{dn}\,w}{1-k^2\mathrm{sn}^2 z\,\mathrm{sn}^2\,w}$$

for the elliptic cosine function after rewriting it as

$$\operatorname{cn}(z+w)=\frac{\operatorname{cn} z \operatorname{cn} w-\operatorname{cn}' z \operatorname{cn}' w}{1-k^2(1-\operatorname{cn}^2 z)(1-\operatorname{cn}^2 w)}$$

(where $\operatorname{cn}' z$ means the derivative of $\operatorname{cn} z$ with respect to z) and verifying the vanishing of the logarithmic derivative of the right-hand side with respect to z after introducing

$$\xi=\operatorname{cn} z, \quad \eta=\operatorname{cn} w$$

and setting $z+w$ equal to a complex constant C. (Note that this is the same procedure as the one used in the lecture notes for the proof of the addition formula for the elliptic sine function.)

Problem 3. *(Alternative Elliptic Function Analogues of Addition Formulas for Trigonometric Sine and Cosine Functions).* Verify the following alternative elliptic function analogues of the addition formulas for the trigonometric sine and cosine functions.

$$\begin{aligned}
\operatorname{dn} z \operatorname{sn}(z+w) &= \operatorname{cn} z \operatorname{sn} w+\operatorname{sn} z \operatorname{cn} w \operatorname{dn}(z+w), \\
\operatorname{cn}(z+w) &= \operatorname{cn} z \operatorname{cn} w-\operatorname{sn} z \operatorname{sn} w \operatorname{dn}(z+w), \\
\operatorname{dn} z \operatorname{dn}(z+w) &= \operatorname{dn} w-k^2 \operatorname{sn} z \operatorname{cn} w \operatorname{sn}(z+w).
\end{aligned}$$

Remark. When the (elliptic) modulus k degenerates to 0, the function $\operatorname{dn} z$ degenerates to the constant function 1 and the first two addition formulas become the addition formulas for $\sin(z+w)$ and $\cos(z+w)$ respectively.

Problem 4. *(Jacobi's Imaginary Transformation).* **(a)** For trigonometric functions the following identity holds

$$\sin(iz)=\frac{1}{2i}\left(e^{i(iz)}-e^{-i(iz)}\right)=\frac{1}{2}\left(e^{z}-e^{-z}\right)=i\sinh z$$

when the variable of the sine function is multiplied by i.

Prove its analogue

$$\operatorname{sn}(iz,k)=i\operatorname{sc}(z,k')$$

for the elliptic sine function with (elliptic) modulus $0<k<1$, where k' is the complementary modulus of k which is defined to be $\sqrt{1-k^2}$ (chosen with the plus sign). Here the notation $\operatorname{sn}(z,k)$ is used, instead of $\operatorname{sn} z$, to highlight its dependence on its (elliptic) modulus k.

Hint: Need only check it when $z = x$ is real. Let $\xi = \mathrm{sn}(iz, k)$ and $\eta = \mathrm{sn}(x, k')$ so that

$$\mathrm{sc}(x, k') = \frac{\eta}{\sqrt{1-\eta^2}}.$$

The identity to be proved is

$$\xi = i\frac{\eta}{\sqrt{1-\eta^2}},$$

which in terms of indefinite integrals means

$$ix = \int_0^{i\frac{\eta}{\sqrt{1-\eta^2}}} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

under the assumption that

$$x = \int_0^{\eta} \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}.$$

Use the substitution

$$s = i\frac{t}{\sqrt{1-t^2}}$$

to transform the first integral. Need only check the case $0 < \eta < 1$.

(b) Similarly, prove

$$\mathrm{cn}(iz, k) = \mathrm{nc}(z, k')$$

and

$$\mathrm{dn}(iz, k) = \mathrm{dc}(z, k').$$