Math 213a Homework #6 Assigned October 15, 2024 due October 22, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 213a

Problem 1. *(Relation Between Modulus of Jacobian Elliptic Function and Coefficients of Cubic Polynomial for Weierstrass's $\wp$ Function).* Recall that the function $x = \operatorname{sn} w = \operatorname{sn}(w, k)$ is defined by the differential equation

$$\frac{d}{dw} \operatorname{sn} w = \sqrt{(1 - \operatorname{sn}^2 w)(1 - k^2 \operatorname{sn}^2 w)}$$

with the initial value of $\operatorname{sn} w = 0$ at $w = 0$ and the square root on the right-hand side chosen to be 1 at $\operatorname{sn} w = 0$. It is the inverse of the indefinite integral

$$\int \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

Let

$$P(w) = \frac{1}{\operatorname{sn}^2 w} - \frac{1 + k^2}{3}.$$

(a) By applying the fundamental theorem of calculus and the chain rule to the indefinite integral whose inverse is the function $\operatorname{sn} w$, show that

$$P(w) = \frac{1}{w^2} + \frac{1 - k^2 + k^4}{15} w^2 + \frac{2 - 3k^2 - 3k^4 + 2k^6}{189} w^4 + \cdots .$$

(b) Use (a) to prove that

$$P'(w)^2 = 4P(w)^3 - g_2 P(w) - g_3,$$

where

$$g_2 = \frac{4}{3}\left(1 - k^2 + k^4\right),$$

$$g_3 = \frac{4}{27}\left(1 + k^2\right)\left(1 - 2k^2\right)\left(2 - k^2\right).$$

Remark. This problem provides another link between the Jacobi elliptic sine function and the Weierstrass $\wp$ function.

Problem 2. *(First Period of Jacobian Elliptic Sine Function as Function of the Modulus – from #24 on p.254 of Stein-Shakarchi's Complex Analysis).* The elliptic integrals K and K' defined for $0 < k < 1$ by

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad K' = \int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$$

satisfy various interesting identities. For instance:

(a) Show that if $\tilde{k}^2 = 1 - k^2$ and $0 < \tilde{k} < 1$, then

$$K'(k) = K(\tilde{k}).$$

Hint: Change variables

$$x = \frac{1}{\sqrt{1 - \tilde{k}^2 y^2}}$$

in the integral defining $K'(k)$.

(b) Prove that if $\tilde{k}^2 = 1 - k^2$ and $0 < \tilde{k} < 1$, then

$$K(k) = \frac{2}{1+\tilde{k}}\, K\left(\frac{1-\tilde{k}}{1+\tilde{k}}\right).$$

Hint: Change variables

$$x = \frac{2t}{1 + \tilde{k} + (1-\tilde{k})t^2}.$$

(c) Show that for $0 < k < 1$ one has

$$K(k) = \frac{\pi}{2}\, F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right),$$

where F is the hypergeometric series

$$F\,(a,b,c;z) = \sum_{n=0}^{\infty} \frac{a\,(a+1)\cdots(a+n-1)\,b\,(b+1)\cdots(b+n-1)}{c\,(c+1)\cdots(c+n-1)} \frac{z^n}{n!},$$

which satisfies the Gauss hypergeometric differential equation

$$z(1-z)\frac{d^2F}{dz^2} + (c - (a+b+1)z)\,\frac{dF}{dz} - \alpha\beta F = 0.$$

Problem 3. *(Derivation of Jacobi's Fundamental Formulas for Jacobian Theta Functions from Special Orthogonal Matrix of Order* 4 *with* $\pm\frac{1}{2}$ *as Entries).* Let τ be a complex number with $\operatorname{Im}\tau > 0$ and $q = e^{i\pi\tau}$. Recall the following formulas for the four Jacobian therta functions.

$$\begin{aligned}
\vartheta_1(w) &= -i\, e^{iw+\frac{1}{4}\pi i\tau}\vartheta_4\left(w+\frac{1}{2}\pi\tau\right) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n e^{\frac{1}{4}(2n+1)^2 i\pi\tau}e^{(2n+1)iw} \\
&= 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)w, \\
\vartheta_2(w) &= \vartheta_1\left(w+\frac{\pi}{2}\right) \\
&= \sum_{n=-\infty}^{\infty} e^{\frac{1}{4}(2n+1)^2 i\pi\tau}e^{(2n+1)iw} = 2\sum_{n=0}^{\infty}q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)w, \\
\vartheta_3(w) &= \vartheta_4\left(w+\frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} e^{n^2 i\pi\tau}e^{2niw} = 1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nw. \\
\vartheta_4(w) &= \sum_{n=-\infty}^{\infty}(-1)^n e^{n^2 i\pi\tau}e^{2niw} = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nw.
\end{aligned}$$

Consider the following special orthogonal matrix A of order 4 with entries $\pm\frac{1}{2}$.

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

(a) Verify that

$$\sum_{j=1}^{4}\left(n_j'\right)^2 = \sum_{j=1}^{4}\left(n_j\right)^2$$

holds for integers n_1, n_2, n_3, n_4 and n_1', n_2', n_3', n_4' related by

$$\begin{pmatrix} n_1' \\ n_2' \\ n_3' \\ n_4' \end{pmatrix} = A\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}.$$

Moreover, if n_1, n_2, n_3, n_4 are either all even integers or all odd integers, then also n_1', n_2', n_3', n_4' are all even integers or all odd integers.

(b) Use Part(**a**) and the formulas above for ϑ_1, ϑ_2, ϑ_3, ϑ_4 to verify that

$$\prod_{\nu=1}^{4} \vartheta_2(w_\nu) + \prod_{\nu=1}^{4} \vartheta_3(w_\nu) = \prod_{\nu=1}^{4} \vartheta_2(w'_\nu) + \prod_{\nu=1}^{4} \vartheta_3(w'_\nu)$$

holds if

$$\begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix},$$

by expanding $\prod_{\nu=1}^{4} \vartheta_2(w_\nu)$ and $\prod_{\nu=1}^{4} \vartheta_3(w_\nu)$ as the product of the defining infinite series for $\vartheta_2(w_\nu)$ and $\vartheta_3(w_\nu)$.

(c) Use Part(**b**) to verify the following two special addition formulas for Jacobian theta functions

$$\vartheta_1(y+z)\vartheta_1(y-z)\vartheta_4(0)^2 = \vartheta_1(y)^2\vartheta_4(z)^2 - \vartheta_4(y)^2\vartheta_1(z)^2, \tag{1}$$

$$\vartheta_4(y+z)\vartheta_4(y-z)\vartheta_4(0)^2 = \vartheta_4(y)^2\vartheta_4(z)^2 - \vartheta_1(y)^2\vartheta_1(z)^2 \tag{2}$$

by appropriately expressing w_1, w_2, w_3, w_4 as linear functions of y, z, $\frac{\pi}{2}$, $\frac{\pi\tau}{2}$.

Hint: Compare (2) with (A3) on p.21 of the appended list of theta function identities in the posted lecture notes "Theta Functions of Jacobi".

(d) Assume as known the following formula for the Jacobian elliptic sine function as a quotient of two Jacobian theta functions.

$$\mathrm{sn}(w,k) = \frac{\vartheta_3(0)}{\vartheta_2(0)} \frac{\vartheta_1\left(\frac{w}{\vartheta_3(0)^2}\right)}{\vartheta_4\left(\frac{w}{\vartheta_3(0)^2}\right)},$$

where

$$k = \left(\frac{\vartheta_2(0)}{\vartheta_3(0)}\right)^2.$$

By dividing Equation (1) by Equation (2) of Part(**c**) and with the choice of

$$y = \frac{w}{\vartheta_3(0)^2} \quad \text{and} \quad z = \frac{\zeta}{\vartheta_3(0)^2},$$

derive the following addition formula for the Jacobian elliptic sine function

$$\operatorname{sn}(w+\zeta)\,\operatorname{sn}(w-\zeta) = \frac{\operatorname{sn}^2 w - \operatorname{sn}^2 \zeta}{1 - k^2 \operatorname{sn}^2 w\, \operatorname{sn}^2 \zeta}.$$

Check that this agrees with the product of the usual addition formula for the Jacobian elliptic sine function

$$\operatorname{sn}(w+\zeta) = \frac{\operatorname{sn} w\, \operatorname{cn} \zeta\, \operatorname{dn} \zeta + \operatorname{sn} \zeta\, \operatorname{cn} w\, \operatorname{dn} w}{1 - k^2 \operatorname{sn}^2 w\, \operatorname{sn}^2 \zeta}$$

and the one for $\operatorname{sn}(w-\zeta)$.