Math 213a Homework #9 Assigned November 5, 2024
due November 12, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 213a

Problem 1 *(Multiplication Theorem of Gauss and Legendre).* **(a)** Use the infinite product expansion of the Gamma function

$$(*) \qquad \frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

to verify the following statement in the multiplication theorem of Gauss and Legendre

$$\Gamma(z)\Gamma\left(z + \frac{1}{n}\right)\Gamma\left(z + \frac{2}{n}\right)\cdots\Gamma\left(z + \frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2} - nz}\Gamma(nz)$$

for any $n \in \mathbb{N}$.

Hint: Let

$$\phi(z) = \frac{n^{nz}\Gamma(z)\Gamma\left(z + \frac{1}{n}\right)\Gamma\left(z + \frac{2}{n}\right)\cdots\Gamma\left(z + \frac{n-1}{n}\right)}{n\Gamma(nz)}.$$

Use (*) to verify that

$$\phi(z) = \lim_{m \to \infty} \frac{\left((m-1)!\right)^n m^{\frac{n-1}{2}} n^{mn-1}}{(nm-1)!}$$

is independent of z. Use

$$\phi\left(\frac{1}{n}\right)^2 = \prod_{r=1}^{n-1}\left(\Gamma\left(\frac{r}{n}\right)\Gamma\left(1 - \frac{r}{n}\right)\right)$$

and the *Euler reflection formula* to obtain

$$\phi\left(\frac{1}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{-1}{2}}.$$

(b) Verify that

$$B(np, nq) = \frac{1}{n^{nq}} \frac{B(p,q)B\left(p+\frac{1}{n},\, q\right)\cdots B\left(p+\frac{n-1}{n},\, q\right)}{B(q,q)B(2q,q)\cdots B((n-1)q,q)}.$$

Problem 2 *(Examples of Mellin Transform – from Exercise #10 on p.176 of Stein-Shakarchi).* The Mellin transform $\mathcal{M}(f)$ of a function f is defined by

$$\left(\mathcal{M}(f)\right)(z) = \int_{t=0}^{\infty} f(t) t^{z-1} dt.$$

For example,

$$\Gamma(s) = \mathcal{M}\left(\frac{1}{e^s}\right) = \int_{t=0}^{\infty} \frac{t^{s-1}}{e^t}\, dt$$

and

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt.$$

(a) Prove that

$$\left(\mathcal{M}(\cos)\right)(z) = \int_{t=0}^{\infty} \cos(t) t^{z-1} dt = \Gamma(z) \cos\left(\frac{\pi z}{2}\right)$$

for $0 < \mathrm{Re}(z) < 1$ and

$$\left(\mathcal{M}(\sin)\right)(z) = \int_{t=0}^{\infty} \sin(t) t^{z-1} dt = \Gamma(z) \sin\left(\frac{\pi z}{2}\right).$$

for $0 < \mathrm{Re}(z) < 1$.

Hint: Consider the function $f(w) = e^{-w} w^{z-1}$ around the contour which is the boundary of

$$\left\{ z = re^{i\theta} \in \mathbb{C} \,\middle|\, \varepsilon < r < R,\ 0 < \theta < \frac{\pi}{2} \right\}.$$

(b) Show that the second of the above two identities is valid in the larger strip $-1 < \mathrm{Re}(z) < 1$, and that as a consequence, one has

$$\int_{x=0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2} \quad \text{and} \quad \int_{x=0}^{\infty} \frac{\sin x}{x^{\frac{3}{2}}}\, dx = \sqrt{2\pi}.$$

Problem 3 *(Tranformation between Dirichlet Series and Power Series via Mellin Tranform).* Consider the following Dirichlet series $F(z)$ and power series $f(z)$ with the same coefficients, namely,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$
$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Verify that the Mellin transform of $f(e^{-s})$ after division by $\Gamma(s)$ is $F(s)$, namely,

$$F(x) = \frac{1}{\Gamma(s)} \int_{x=0}^{\infty} x^{s-1} f(e^{-x}) dx.$$

Specify the the domains of f and F for the Mellin transform identity above to be valid.

Remark. This is a precise formulation, in terms of Millin transform, of the relation between $\sum_m \gamma_m w^m$ and $\sum_n \frac{a_n}{n^s}$ with $a_n = \gamma_m$, $n = e^m$, and $w = e^{-s}$.

Problem 4 *(Entire Extension of Mellin Transform (After Division by Gamma Function) of Function in Schwarz Space – from Exercise #17 on p.179 of Stein-Shakarchi).* Let f be an infinitely differentiable function on $\mathbb{R}$ that has compact support, or more generally, let f belong to the Schwartz space $\mathcal{S}$. Recall that a function f on $\mathbb{R}$ belongs to the Schwartz space $\mathcal{S}$ if and only if f and all its derivatives decay faster than any polynomials, *i.e.,*

$$\sum_{x \in \mathbb{R}} |x|^m |f^\ell(x)| < \infty$$

for all integers $m, \ell \geq 0$. Consider

$$I(s) = \frac{1}{\Gamma(s)} \int_{x=0}^{\infty} f(x) x^{-1+s} dx$$

which is the Mellin transform of $f(x)$ divided by the Gamma function.

(a) Observe that $I(s)$ is holomorphic for $\operatorname{Re} s > 0$. Prove that I has an analytic continuation as an entire function in the complex plane $\mathbb{C}$.

(b) Prove that $I(0) = 0$ and

$$I(-n) = (-1)^n f^{(n+1)}(0)$$

for all integers $n \geq 0$.

Hint: Integrate by parts k-times to show that

$$I(s) = \frac{(-1)^k}{\Gamma(s+k)} \int_{x=0}^{\infty} f^{(k)}(x) x^{s+k-1} dx.$$

Problem 5 *(Analytic Extension of Riemann Zeta Function by Applying Integration by Parts to Mellin Transform with Integration only over* $[1,\infty)$ *– from Problem #2 on p.180 of Stein-Shakarchi).* Let $Q(x) = \{x\} - \frac{1}{2}$, where $\{x\}$ means the fractional part of x, which means $\{x\}$ is equal to x minus the largest integer not exceeding x. Verify by explicitly computing the integral on the right-hand side that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is given by

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s \int_{x=1}^{\infty} \frac{Q(x)}{x^{s+1}} dx.$$

Let us construct $Q_k(x)$ recursively so that

$$\int_{x=0}^{1} Q_k(x) dx = 0, \quad \frac{dQ_{k+1}}{dx} = Q_k(x), \quad Q_0(x) = Q(x) \quad \text{and} \quad Q_k(x+1) = Q_k(x).$$

Then we can write

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s \int_{x=1}^{\infty} \left(\frac{d^k}{dx^k} Q_k(x) \right) x^{-s-1} dx,$$

and a k-fold integration by parts (for any positive integer k) gives the analytic continuation for $\zeta(s)$ as a holomorphic function on $\mathbb{C} - \{1\}$ with a simple pole at $s = 1$.

Problem 6 *(The Missing Case in Perron's Lemma – from Exercise #6 on p.201 of Stein-Shakarchi).* Show that for every $c > 0$,

$$\lim_{N \to \infty} \frac{1}{2\pi i} \int_{c-iN}^{c+iN} a^s \frac{ds}{s} = \begin{cases} 1 & \text{if} \quad a > 1, \\ \frac{1}{2} & \text{if} \quad a = 1, \\ 0 & \text{if} \quad 0 \leq a < 1, \end{cases}$$

The integral is taken over the vertical segment from $c - iN$ to $c + iN$.

Remark. The two cases of $a > 1$ and $0 \leq a < 1$ are given in the Perron Lemma in the posted lecture notes. The main point of this exercise concerns the missing case of $a = 1$.