Math 213a Homework #11 Assigned November 19, 2024
due November 26, 2024

Please submit the PDF file of your homework to the CANVAS website for Math 213a

Problem 1 *(from Exercise #12 on p.177 of Stein-Shakarchi).* **(a)** Show that $\frac{1}{|\Gamma(s)|}$ is not $O(e^{c|s|})$ for any $c > 0$.

Hint: If $s = -k - \frac{1}{2}$, where k is a positive integer, then

$$\frac{1}{|\Gamma(s)|} \geq \frac{k!}{\pi}.$$

(b) Show that there is no entire function $F(s)$ with $F(s) = O(e^{c|s|})$ that has simple zeroes at $s = 0, -1, -2, \cdots, -n, \cdots$, and that vanishes nowhere.

Hint: Use Part **(a)** and the fact that the pole-set of $\Gamma(z)$ consists of poles of order 1 at the nonpositive integers.

Problem 2 *(Infinite Number of Zeroes from Hadamard's Factorization Theorem – from Exercises 13 and 14 on p.155 of Stein-Shakarchi).* Use Hadamard's Factorization Theorem to prove the following two statements.

(a) The equation $e^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.

(b) If F is entire and of growth order ρ that is non-integral, then F has infinitely many zeroes.

Problem 3 *(Blaschke Product as Bounded Unit-Disk Analogue of Weierstrass Product for Entire Functions — from Problems 1 and 2 on p.156 of Stein's Book).* **(a)** Prove that if f is holomorphic on the open unit disk $\mathbb{D}$, bounded and not identically zero, and $z_1, z_2, \cdots, z_n, \cdots$ are its zeros, then

$$\sum_n (1 - |z_n|) < \infty.$$

Hint: Use Jensen's formula, which is the special case of the following Poisson-Jensen formula when $z = 0$ and f is holomorphic.

Poisson-Jensen formula. If $f(z)$ is a meromorphic function on $\{|z| \leqslant R\}$ whose zeroes $a_1, \cdots, a_M$ and poles $b_1, \cdots, b_N$ in $\{|z| \leqslant R\}$ do not lie on $\{|z| = R\}$, then for any $z = re^{i\theta}$ with $0 \leqslant r < R$ which is neither a pole nor a zero of $f(z)$, the Poisson-Jensen formula states that

$$\log|f(z)| = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} \log|f(Re^{i\varphi})| \frac{R^2 - r^2}{r^2 + R^2 - 2Rr\cos(\theta - \varphi)}\, d\varphi$$
$$+ \sum_{\mu=1}^{M} \log\left|\frac{R(z - a_\mu)}{R^2 - \bar{a}_\mu z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z - b_\nu)}{R^2 - \bar{b}_\nu z}\right|.$$

(b) Show that for $0 < |\alpha| < 1$ and $|z| \leq r < 1$ the inequality

$$\left|\frac{\alpha + |\alpha|z}{(1 - \bar{\alpha}z)\alpha}\right| < \frac{1 + r}{1 - r}.$$

holds.

(c) Let $\{\alpha_n\}$ be a sequence in the open unit disk $\mathbb{D}$ such that $\alpha_n \neq 0$ for all n and

$$\sum_{n=1}^{\infty}(1 - |\alpha_n|) < \infty.$$

Note that this will be the case if $\{\alpha_n\}$ are the zeroes of a bounded holomorphic function on $\mathbb{D}$ according to Part **(a)**. Show that the product

$$f(z) = \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \frac{|\alpha_n|}{\alpha_n}$$

converges uniformly for $|z| \leq r < 1$, and defines a holomorphic function on the unit disk $\mathbb{D}$ having precisely the zeroes α_n and no other zeroes. Show that $|f(z)| \leq 1$.

Problem 4 *(Meromorphic Function as Quotient of Weierstrass Canonical Products of Degree From Polynomial Growth Order of Nevanlinna Characteristic Function).* Let $q \in \mathbb{N}$ and f be a meromorphic function on $\mathbb{C}$ with zeroes at a_μ and poles at b_ν (with multiplicities represented by repeated occurrences) such that

$$\limsup_{r\to\infty} \frac{T(r, f)}{r^q} = 0,$$

where $T(r, f)$ is the Nevanlinna characteristic function for f. For $p \in \mathbb{N} \cup \{0\}$ let

$$E_p(z) = (1 - z)\, e^{z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}}$$

denote the *Weierstrass canonical factor* of degree p (with the convention that $E_0(z) = 1 - z$). Prove that

$$f(z) = z^p e^{P_{q-1}(z)} \lim_{r\to\infty} \frac{\prod_{|a_\mu|<r} E_{q-1}\left(\frac{z}{a_\mu}\right)}{\prod_{|b_\nu|<r} E_{q-1}\left(\frac{z}{b_\nu}\right)},$$

where $p \in \mathbb{Z}$ and $P_{q-1}(z)$ is a polynomial of degree $\leqslant q-1$, by following the steps outlined below in **(a)**, **(b)** and **(c)**.

(a) Use

$$\sum_{|b_\nu|<r} \left| \frac{(\bar{b}_\nu)^q}{(r^2 - \bar{b}_\nu z)^q} \right| < \frac{2^q\, \boldsymbol{n}(r, f, \infty)}{r^q}$$

for $|z| < \frac{r}{2}$ (where $\boldsymbol{n}(r, f, \infty)$ is the number of poles, with multiplicities counted, for f in the disk of radius r centered at 0) and

$$T(2r, f) \geqslant \int_{t=r}^{2r} \frac{\boldsymbol{n}(r, f, \infty)}{t} dt$$

to show that

$$\sum_{|b_\nu|<r} \frac{(\bar{b}_\nu)^q}{(r^2 - \bar{b}_\nu z)^q}$$

converges uniformly on compact subsets of $\mathbb{C}$ to 0 as $r \to \infty$. Likewise, use

$$T\left(r, \frac{1}{f}\right) = T(r, f) + O(1)$$

to show that

$$\sum_{|a_\mu|<r} \frac{(\bar{a}_\mu)^q}{(r^2 - \bar{a}_\mu z)^q}$$

converges uniformly on compact subsets of $\mathbb{C}$ to 0 as $r \to \infty$.

(b) By apply $\left(\frac{\partial}{\partial z}\right)^q$ to the Poisson-Jensen formula, verify that

$$\begin{aligned} \frac{d^q}{dz^q} \log f(z) = {} & \frac{q!}{\pi} \int_{\theta=0}^{2\pi} \frac{\log|f(re^{i\theta})|\, re^{i\theta}\, d\theta}{(re^{i\theta} - z)^{q+1}} \\ & + (q-1)! \sum_{|b_\nu|<r} \left(\frac{1}{(b_\nu - z)^q} - \frac{(\bar{b}_\nu)^q}{(r^2 - \bar{b}_\nu z)^q} \right) \\ & - (q-1)! \sum_{|a_\mu|<r} \left(\frac{1}{(a_\mu - z)^q} - \frac{(\bar{a}_\mu)^q}{(r^2 - \bar{a}_\mu z)^q} \right) \end{aligned}$$

if $f(z)$ is holomorphic and nonzero at $z = 0$.

(c) By integrating q times the equation in Step **(b)**, applied to $f(z)z^{-p}$ for some appropriate $p \in \mathbb{Z}$, and then exponentiating and using Step **(a)** to show that

$$f(z) = z^p e^{P_{q-1}(z)} \lim_{r\to\infty} \frac{\prod_{|a_\mu|<r} E_{q-1}\left(\frac{z}{a_\mu}\right)}{\prod_{|b_\nu|<r} E_{q-1}\left(\frac{z}{b_\nu}\right)}$$

for some $p \in \mathbb{Z}$ and some polynomial $P_{q-1}(z)$ of degree $\leqslant q-1$.

Problem 5 *(Alternative Proof of Hadamard's Factorization Theorem by Iterated Differentiation and Integration of the Poisson-Jensen Formula).*

(a) Verify that if $f(z)$ is an entire function on $\mathbb{C}$ such that for any $\varepsilon > 0$ the inequality

$$|f(z)| \leq A_\varepsilon e^{B_\varepsilon |z|^{\rho+\varepsilon}}$$

holds on $\mathbb{C}$ for some positive numbers A_ε and B_ε, then

$$\limsup_{r\to\infty} \frac{T(r,f)}{r^{k+1}} = 0,$$

where $k = \lfloor \rho \rfloor$ is the integral part of ρ.

(b) Use the result in Part **(a)** to provide a proof of Hadamard's factorization theorem that every entire function $f(z)$ of finite order ρ admits an infinity factorization of the form

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

for some nonnegative integer m and some polynomial $P(z)$ of degree $\leq k$, where $k = \lfloor \rho \rfloor$ is the integral part of ρ.

Problem 6 *(Rationality of Meromorphic Function on $\mathbb{C}$ with Finite Zero-Set and Pole-Set and Nevanlinna Characteristic Function of Growth Order < 1).*

(a) Let $q \in \mathbb{N}$ and f be a meromorphic function on $\mathbb{C}$ with zeroes at a_μ and poles at b_ν such that

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^q} = 0.$$

By slightly modifying the proof in Problem 5, show that there exists an increasing sequence r_k of positive numbers with $\lim_{k\to\infty} r_k = \infty$ such that

$$f(z) = z^p e^{P_{q-1}(z)} \lim_{k\to\infty} \frac{\prod_{|a_\mu|<r_k} E_{q-1}\left(\frac{z}{a_\mu}\right)}{\prod_{|b_\nu|<r_k} E_{q-1}\left(\frac{z}{b_\nu}\right)}$$

for some $p \in \mathbb{Z}$ and some polynomial $P_{q-1}(z)$ of degree $\leqslant q-1$.

(b) Prove that if $f(z)$ is a meromorphic function on $\mathbb{C}$ with only a finite number of zeroes and poles and satisfies

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0,$$

then $f(z)$ is a rational function.