Math 213a Homework #10 Assigned November 12, 2024
due November 19, 2024

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Problem 1 *(Average of Absolute-Value Square of Dirichlet Series Along a Vertical Line – from Problem #1 on p.203 of Stein-Shakarchi).* Let

$$F(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

where $|a_n|\leq M$ for all n.

(a) Prove that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{t=-T}^{T}\left|F(\sigma+it)\right|^2dt=\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}\quad\text{if}\quad\sigma>1.$$

Hint: Use the fact that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{t=-T}^{T}(nm)^{-\sigma}n^{-it}m^{it}dt=\begin{cases}n^{-2\sigma} & \text{if } n=m,\\ \\ 0 & \text{if } n\neq m.\end{cases}$$

(b) Show as a consequence the uniqueness of Dirichlet series: If

$$F(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

where the coefficients are assumed to satisfy $|a_n|\leq cn^k$ for some k, and $F(s)\equiv 0$, then $a_n=0$ for all n.

Remark. The statement in Part (a) can be interpreted in the context of a $\mathbb{C}$-vector space E with an inner product whose norm is $\|\cdot\|_E$. The norm square $\|f\|_E^2$ of an element $f=\sum_{n=1}^{\infty}c_ng_n$ of E can be written as

$$\|f\|_E^2=\sum_{n=1}^{\infty}\frac{|c_n|^2}{\|g_n\|^2}$$

if g_n is orthogonal to g_m for $n \neq m$. In the case at hand, the vector space E depends on σ and for fixed σ the inner product of $\varphi(s)$ and $\psi(s)$ as functions of $t = \operatorname{Im} s$ is

$$\lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} \varphi(\sigma + it)\, \overline{\psi(\sigma + it)}\, dt.$$

This is Dirichlet-series analogue of the situation of the L^2 norm of the restriction of a holomorphic function f to a circle of radius r centered at z_0 when

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is expressible as a convergent power series with radius of convergence $> r$ so that

$$\frac{1}{2\pi} \int_{\theta=-\pi}^{\pi} |f(z_0 + re^{i\theta}|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Problem 2 *(Logarithmic Growth Order for Riemann's Zeta Function, its Derivative, and its Reciprocal on the Vertical Line with Abscissa* 1 *– from Exercise 9 on p.202 of Stein-Shakarchi).*

(a) For any real number x, denote by $\lfloor x \rfloor$ the integral part of x in the sense that $\lfloor x \rfloor$ is the integer such that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. By applying integration by parts to the Stieltjes integral

$$\sum_{n=N+1}^{\infty} \frac{1}{n^s} = \int_{x=N}^{\infty} \frac{1}{x^s}\, d\lfloor x \rfloor$$

and using the explicit evaluation of the integral

$$\int_{x=N}^{\infty} \frac{x - \frac{1}{2}}{x^{s+1}}\, dx$$

to verify that

$$(*) \qquad \zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s} = s \int_{x=N}^{\infty} \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}}\, dx + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2}$$

first for $\sigma = \operatorname{Re} s > 1$ and by analytic continuation for $\sigma = \operatorname{Re} s > 0$.

(b) Use (*) in Part(**a**) with the choice of $N = \lfloor t \rfloor$ to verify that there exists a positive constant A such that

$$|\zeta(1+it)| \leq A \log |t|$$

for $|t| \geq 2$.

(c) By differentiating (*) in Part(**a**) to get

$$\zeta'(s) = -\sum_{n=2}^{N} \frac{\log n}{n^s} + \int_{x=N}^{\infty} \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} (1 - s \log s)\, dx - \frac{N^{1-s} \log N}{s-1} - \frac{N^{-s}}{(s-1)^2} + \frac{N^{-s} \log N}{2},$$

with the choice of $N = \lfloor t \rfloor$, verify that there exists a positive constant A such that

$$|\zeta'(1+it)| \leq A (\log |t|)^2$$

for $|t| \geq 2$.

(d) For $t \geq 2$, by using

$$\left| \frac{1}{\zeta(\sigma+it)} \right| \leq |\zeta(\sigma)|^{\frac{3}{4}} \left| \zeta(\sigma+2it) \right|^{\frac{1}{4}} = O\left(\frac{(\log t)^{\frac{1}{4}}}{(\sigma-1)^{\frac{3}{4}}} \right) \quad \text{for} \quad \sigma > 1$$

and

$$\zeta(1+it) - \zeta(\sigma+it) = -\int_{u=1}^{\sigma} \zeta'(u+it)\, du = O\left((\sigma-1)(\log t)^2\right)$$

and with the choice of

$$\sigma - 1 = \frac{\varepsilon}{(\log t)^9}$$

for $\varepsilon > 0$ sufficiently small, verify that there exists a positive constant A such that

$$\left| \frac{1}{\zeta(1+it)} \right| \leq A (\log |t|)^7$$

for $|t| \geq 2$.

Remark on Problem 2. The difference between the statement in Part(**b**) and the weaker estimate $|\zeta(1+it)| \leq c_\varepsilon |t|^\varepsilon$ for $\varepsilon > 0$ and $|t| \geq 1$ obtained by using

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_{x=n}^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx$$

is that in Part(**b**) when the error between $\frac{1}{n^s}$ and $\frac{1}{x^s}$ is considered, the subtler partial sum from 1 to N with N equal to the integral part of t is used instead of the full infinite sum.

Problem 3 *(Meromorphic Continuation of Dirichlet L-Series from the Mellin Transform Representation of its Product with the Gamma Function – from Exercise #4 on p.200 of Stein-Shakarchi).* Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $a_n = a_m$ if $n \equiv m \mod q$ for some positive integer q. Define the *Dirichlet L-series* associated to $\{a_n\}$ by

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{for} \quad \operatorname{Re} s > 1.$$

Let $a_0 = a_q$ and

$$Q(x) = \sum_{m=0}^{q-1} a_{q-m} e^{mx}.$$

As a generalization of the special case of $\zeta(s) = L(s)$ where $a_n = 1$ for all n, show that

$$L(s) = \frac{1}{\Gamma(s)} \int_{x=0}^{\infty} \frac{Q(x) x^{s-1}}{e^{qx} - 1} \, dx \quad \text{for} \quad \operatorname{Re} s > 1.$$

That is, $\Gamma(s) L(s)$ is the Mellin transform of the function

$$\frac{Q(x)}{e^{qx} - 1}.$$

Prove as a result that $L(s)$ is continuable into the complex plane, with the only possible singularity a pole at $s = 1$. In fact, $L(s)$ is regular at $s = 1$ if and only if $\sum_{m=0}^{q-1} a_m = 0$.

Problem 4 *(Size of the n-th Prime – from Exercises 12 on p.203 of Stein-Shakarchi).* Let p_n denote the n-th prime. Prove that the prime number theorem implies that $p_n \sim n \log n$ as $n \to \infty$, by following the steps given in Part(**a**) and Part(**b**) below.

(a) Show that $\pi(x) \sim \frac{x}{\log s}$ implies that

$$\log \pi(x) + \log \log x \sim \log x.$$

(b) As a consequence, prove that $\log \pi(x) \sim \log x$, and take $x = p_n$ to conclude that $p_n \sim n \log n$ as $n \to \infty$.

Problem 5 *(Interpolation Entire Functions – from Exercises 17 on p.156 of Stein-Shakarchi).* Given two countably infinite sequences of complex numbers $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ with $\lim_{n\to\infty} a_n = \infty$, it is always possible to find an entire function $F(z)$ that satisfies $F(a_k) = b_k$ for all k.

(a) (*Lagrange Interpolation Polynomial*). Given n distinct complex numbers $a_1, \cdots, a_n$ and another n complex numbers $b_1, \cdots, b_n$, construct a polynomial of degree $\leq n-1$ with

$$P(a_j) = b_j \quad \text{for} \quad j = 1, 2, \cdots, n.$$

Hint: Consider the polynomial

$$P(x) = \sum_{j=1}^{n} \left(\prod_{\substack{1 \leq i \leq n \\ i \neq j}} \frac{x - a_i}{a_j - a_i} \right) b_j,$$

or alternatively use Cramer's rule and the Vandermonde matrix.

(b) (*Pringsheim Interpolation Formula*). Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of distinct terms in $\mathbb{C}$ with $a_0 = 0$ and

$$\lim_{k\to\infty} a_k = \infty.$$

Let $E(z)$ denote a Weierstrass product associated with $\{a_k\}_{k=0}^{\infty}$. That is,

$$E(z) = z^m \prod_{k=1}^{\infty} \left(\left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{\nu_k}\left(\frac{z}{a_k}\right)^{\nu_k}} \right)$$

for some $m \in \mathbb{N}$ and for some sequence $\{\nu_k\}_{k=1}^{\infty}$ in $\mathbb{N} \cup \{0\}$ with

$$\sum_{n=1}^{\infty} \frac{1}{|a_k|^{\nu_k+1}} < \infty.$$

Given complex numbers $\{b_k\}_{k=0}^{\infty}$, show that there exists $m_k \in \mathbb{N}$ such that the series

$$F(z) = \frac{b_0}{E'(0)} \frac{E(z)}{z} + \sum_{k=1}^{\infty} \frac{b_k}{E'(a_k)} \frac{E(z)}{z - a_k} \left(\frac{z}{a_k}\right)^{m_k}$$

defines an entire function that satisfies

$$F(a_k) = b_k \quad \text{for all} \quad k \geq 0.$$