GAMMA FUNCTION

Gamma function is the continuous analogue of the factorial function $n!$. Just as the factorial function $n!$ occurring naturally in the series expansion of $e^z$ and in the integral formula for derivatives of holomorphic functions because of differentiation, the Gamma function occurs naturally in the treatment of the Riemann zeta function which is the key function for the application of complex analysis to number theory such as the prime number theorem and the Riemann hypothesis. We will discuss the definition of the Gamma function and its important properties before we proceed to the topic of Dirichlet series and the Riemann zeta function.

**Definition of Gamma Function.** Gamma function is the continuous analogue of the factorial function $n!$. The factorial function $n!$ can be obtained from

$$ \frac{d^n}{dx^n}(x^n) = n!,$$

or by applying integration by parts to

$$ \int_0^\infty x^n e^{-x} \, dx$$

and integrate $e^{-x}$ first and do it $n$ times. To extend the definition of the factorial function $n!$ to the case of a continuous variable, we define

$$ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad \text{for} \quad x > 0,$$

where the condition of $x > 0$ is to make sure that the integral converges at $t = 0$.

In the definition there is a shift of the variable $x$ by 1 as a matter of convention so that $\Gamma(n) = (n - 1)!$, in order to move the singular point of the function from $x = -1$ to $x = 0$. When $x > 1$, by integration by parts we get

$$ \Gamma(x) = \left[ -t^{x-1} e^{-t} \right]_{t=0}^{t=\infty} + (x - 1) \int_0^\infty t^{x-2} e^{-t} \, dt = (x - 1) \Gamma(x - 1).$$

From $\Gamma(1) = \int_0^\infty e^{-t} = 1$ it follows that

$$ \Gamma(n) = (n - 1)!.$$
The defining formula
\[ \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \]
actually defines \( \Gamma(z) \) for \( z \in \mathbb{C} \) with \( \text{Re} \, z > 0 \).

The Gamma function \( \Gamma(z) \) can be extended to \( \text{Re} \, z > -1 \) by using
\[ \Gamma(z) = \frac{\Gamma(z + 1)}{z}, \]
which is meromorphic on \( \text{Re} \, z > -1 \) with a simple pole of residue 1 at \( z = 0 \), because \( \Gamma(1) = 1 \). We can repeat this technique to extend \( \Gamma(z) \) to all of \( \mathbb{C} \) by using
\[ \Gamma(z) = \frac{\Gamma(z + n)}{z(z - 1) \cdots (z - n + 1)}. \]

**Beta Function.** A similar analogue of the generalization of the binomial coefficient
\[ \binom{m+n}{m} = \frac{(m+n)!}{m! \, n!} \]
is the Beta function defined by
\[ B(x,y) = \frac{\Gamma(x) \, \Gamma(y)}{\Gamma(x + y)}. \]

We are going to derive the formula for the Beta function as a definite integral whose integrand depends on the variables \( x \) and \( y \). This is done by changing the order of integration of a double integral as follows. For \( x > 0 \) and \( y > 0 \) we have
\[ \Gamma(x)\Gamma(y) = \left( \int_0^\infty t^{x-1}e^{-t}dt \right) \left( \int_0^\infty u^{y-1}e^{-u}du \right). \]

Using the transformation \( u = tv \) (in order to get the factor \( t^{x+y-1} \) from \( t^{x-1}u^{y-1} \)) and then the transformation \( w = t (1 + v) \) (in order to express
\(\Gamma(x + y)\) as an integral in \(w\), we obtain

\[
\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty t^{x-1}u^{y-1}e^{-(t+u)}\,dudt
\]

\[
= \int_0^\infty \int_0^\infty t^{x+y-1}u^{y-1}e^{-t(1+u)}\,dvdt
\]

\[
= \int_0^\infty \int_0^\infty \frac{w^{x+y-1}}{(1+v)^{x+y-1}}v^{y-1}e^{-w}\,dv\,dw
\]

\[
= \left(\int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}\,dv\right)\left(\int_0^\infty w^{x+y-1}e^{-w}\,dw\right)
\]

\[
= \Gamma (x + y) \int_{v=0}^\infty w^{y-1} dw
\]

from which it follows that

\[
B(x, y) = \int_{v=0}^\infty \frac{v^{y-1}\,dv}{(1+v)^{x+y}}
\]

This integral representation of \(B(x, y)\) is not symmetric in \(x\) and \(y\). To transform it to a symmetric form, we use the linear fractional transformation \(v = \frac{\lambda}{1-\lambda}\) which changes the interval of integration from \((0, \infty)\) to \((0, 1)\). We obtain

\[
B(x, y) = \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}\,d\lambda,
\]

which is symmetric in \(x\) and \(y\).

**Relation Between Gamma Function and Sine Function (Euler’s Reflection Formula).** A very useful case for the Beta function is when \(x + y = 1\) in the above formula, in which case

\[
\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}\,dv}{1+v},
\]

which by residue calculus applied to the function

\[
\frac{z^{x-1}dz}{1+z}
\]

integrated over the contour integral of the boundary of the domain

\[
\{ r < |z| < R \} - \{ \text{Re } z \geq 0, -r \leq \text{Im } z \leq r \}
\]
as \( r \to 0^+ \) and \( R \to \infty \), yields
\[
\frac{\pi}{\sin \pi x}.
\]
Thus we have the following important formula relating the gamma function to the sine function
\[
\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x}
\]
for \( 0 < x < 1 \). By the identity theorem for meromorphic functions,
\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}
\]
on all of \( \mathbb{C} \), known as Euler’s reflection formula. The reflection refers to the reflection with respect to Re \( z = \frac{1}{2} \). The Euler reflection formula for the Gamma function gives us another way of extending the Gamma function to a meromorphic function on all of \( \mathbb{C} \) by defining
\[
\Gamma(z) = \frac{\pi}{\Gamma(1 - z)\sin(\pi z)} \quad \text{for} \quad \text{Re } z < \frac{1}{2}.
\]
Note that the graph of \( y = \sin \pi x \) is symmetric with respect to the vertical line \( x = \frac{1}{2} \). This symmetry with respect to the vertical line \( x = \frac{1}{2} \) has a special role in Riemann’s Zeta function and its application to the Prime Number Theorem and the Riemann Hypothesis.

**Duplication Formula for Gamma Function.** We now turn to the use of the symmetric form of the integral representation of the Beta function
\[
B(x, y) = \int_0^1 \lambda^{x-1} (1 - \lambda)^{y-1} \, d\lambda
\]
in the special case
\[
B(x, x) = \int_{\lambda=0}^1 (\lambda(1 - \lambda))^{x-1} \, d\lambda
\]
For the special situation \( x = \frac{1}{2} \), by using \( \lambda = \sin^2 \theta \), we get
\[
B\left(\frac{1}{2}, \frac{1}{2}\right) = \int_{\lambda=0}^1 \frac{d\lambda}{\sqrt{\lambda(1 - \lambda)}}
\]
\[
= \int_{\theta=0}^{\frac{\pi}{2}} \frac{d(\sin^2 \theta)}{\sqrt{\sin^2 \theta \cos^2 \theta}} = \int_{\theta=0}^{\frac{\pi}{2}} d\theta = \pi.
\]
Thus \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \).
For a general \( x \), we change the quadratic polynomial \( \lambda(1 - \lambda) \) to the degree-one polynomial \( \frac{1}{4}(1 - \mu) \) so that \( \lambda = \frac{1}{2} - \frac{1}{2}\sqrt{\mu} \) and \( d\lambda = -\frac{d\mu}{4\sqrt{\mu}} \) to get

\[
B(x, x) = 2 \int_{\lambda=0}^{\frac{1}{2}} (\lambda(1 - \lambda))^{x-1} d\lambda \\
= \frac{1}{2} \int_{0}^{1} \left( \frac{1}{4} - \frac{1}{4}\mu \right)^{x-1} \mu^{-\frac{1}{2}} d\mu \\
= 2^{1-2x} \int_{0}^{1} (1 - \mu)^{x-1} \mu^{-\frac{1}{2}} d\mu \\
= 2^{1-2x} B(x, \frac{1}{2}).
\]

From \( \frac{\Gamma(x)^2}{\Gamma(2x)} = B(x, x) = 2^{1-2x} B(x, \frac{1}{2}) = 2^{1-2x} \frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x + \frac{1}{2})} \)
we have the following duplication formula

\[
\Gamma(2x) \Gamma\left(\frac{1}{2}\right) = 2^{2x-1} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right).
\]

By the identity theorem for meromorphic functions, the duplication formula for the Gamma function \( \Gamma(2z) \Gamma\left(\frac{1}{2}\right) = 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \) holds on all of \( \mathbb{C} \).

**Meromorphic Extension of Gamma Function by Representation by Contour Integral.** We now look at another way of extending the Gamma function to all of \( \mathbb{C} \) by using by representing the gamma function as a contour integral. The definition

\[
\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} dt
\]

is restricted to \( z \in \mathbb{C} \) with \( \text{Re} \, z > 0 \), because of the local integrability of \( t^{z-1} e^{-t} \) at \( t = 0 \). This difficulty can be circumvented by replacing the interval
of integration \((0, \infty)\) by another contour \(C\) which is defined as follows. The contour \(C\) starts from positive infinity and goes toward the origin above the real axis and circles the origin once in the positive direction and then goes to positive infinity below the real axis. For \(\Re z > 0\) consider

\[
\int_C e^{-w}(-w)^{z-1} dw.
\]

We can evaluate the integral as follows. We use the branch-cut \((0, \infty)\) to choose a branch of the multi-valued function \((-w)^{z-1}\) so that

\[
(-w)^{z-1} = e^{(z-1)\log(-w)}
\]

with

\[
\log(-w) = \log \rho + i(\phi - \pi)
\]

when \(w = \rho e^{i\phi}\) and \(0 < \phi < 2\pi\). So

\[
\int_C e^{-w}(-w)^{z-1} dw
\]

\[
= \int_{\rho=\infty}^{0} e^{-\rho} e^{(z-1)(\log \rho - i\pi)} d\rho + \int_{\rho=0}^{\infty} e^{-\rho} e^{(z-1)(\log \rho + i\pi)} d\rho
\]

\[
= (e^{(z-1)\pi i} - e^{-(z-1)\pi i}) \int_{\rho=0}^{\infty} e^{-\rho} e^{(z-1)\log \rho} d\rho
\]

\[
= 2i \sin ((z - 1)\pi) \int_{\rho=0}^{\infty} e^{-\rho} \rho^{z-1} d\rho = -2i \sin \pi z \Gamma(z).
\]

Note that the condition \(\Re z > 0\) is used to make sure that the integral around a small circle centered at the origin goes to zero as the radius of the circle goes to zero. When the condition \(\Re z > 0\) is not assumed, we can define the Gamma function by

\[
\Gamma(z) = \frac{i}{2 \sin \pi z} \int_C e^{-w}(-w)^{z-1} dw.
\]

The only possible poles of \(\Gamma(z)\) are the zeroes of \(\sin \pi z\), namely \(z \in \mathbb{Z}\). We already know that \(\Gamma(z)\) is regular at points of \(\Re z > 0\). Thus the only possible poles of \(\Gamma(z)\) are at the non-positive integers. The residues at a non-positive integer \(z = -n\) is computed as follows. Since the principal part of

\[
e^{-w}(-w)^{n-1} = (-w)^{-n-1} \sum_{k=0}^{\infty} \frac{(-1)^k w^k}{k!}
\]
at $w = 0$ is $-\frac{1}{n! w}$, it follows that
\[
\int_C e^{-w}(-w)^{-n-1}dw = -\frac{2\pi i}{n!}
\]
and the residue of $\Gamma(z)$ at $z = -n$ is equal to
\[
\lim_{z\to -n} -\frac{2\pi i}{n!} \frac{i(z + n)}{2 \sin \pi z} = \frac{(-1)^n}{n!}.
\]

**Infinite Product Expansion of Gamma Function.** We now discuss the infinite product expansion of $\frac{1}{\Gamma(z)}$ by looking at the Euler reflection formula
\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}
\]
from a multiplicative point of view. We have an infinite product expansion
\[
\sin \pi z = \pi z \prod_{n \in \mathbb{Z} - \{0\}} \left(1 - \frac{z}{n}\right) e^{\pi} \left(1 - \frac{z}{n}\right)
\]
which is obtained from the partial fraction expansion of its logarithmic derivative
\[
\frac{d}{dz} \left(\log \frac{\sin \pi z}{\pi z}\right) = \pi \cot \pi z = \sum_{n \in \mathbb{Z} - \{0\}} \left(\frac{1}{z - n} + \frac{1}{n}\right)
\]
by the technique of applying the Cauchy integral formula with a modified Cauchy kernel to a meromorphic function.

The Euler reflection formula
\[
\frac{1}{\Gamma(z)} \frac{1}{\Gamma(1 - z)} = \frac{\sin(\pi z)}{\pi}
\]
can be interpreted as saying that, multiplicatively speaking, $\frac{1}{\Gamma(z)}$ represents the part of $\sin(\pi z)$ on the half-plane to the left of $\text{Re}z = \frac{1}{2}$. To follow the same path as the derivation of the infinite product expansion of $\sin(\pi z)$, we should take the logarithmic derivative $(\log \Gamma(z))'$ of $\Gamma$ and get its partial fraction expansion. Unlike the case of $\sin(\pi z)$ where we have a way of getting the partial fraction expansion of
\[
\frac{d}{dz} \left(\log \frac{\sin \pi z}{\pi z}\right) = \pi \cot \pi z,
\]
we do not have any such tool. The information for $\Gamma(z)$ comes from integral representations, which give us the only tool we have to work with. Instead of using the difference quotient
\[
\frac{\log \Gamma(z + h) - \log \Gamma(z)}{h},
\]
we consider the exponentiation of the numerator and use the symmetric version of the integral representation of the Beta function. This motivates us to consider
\[
\frac{\Gamma(z - h)\Gamma(h)}{\Gamma(z)} = \int_0^1 (1 - t)^{z-h-1} t^{-1} dt.
\]
From
\[
\Gamma(z + 1) = \frac{\Gamma(z)}{z}
\]
we know that the principal part of $\Gamma(z)$ at $z = 0$ is $\frac{1}{z}$.

For $\Re z > h > 0$ with $z$ fixed and $h$ varying, the principal part of
\[
\frac{\Gamma(z - h)\Gamma(h)}{\Gamma(z)}
\]
is $\frac{1}{h}$ as we can see by multiplying it $h$ and let $h \to 0$. We would like to take it out so that what is left will be holomorphic for $h$ in a small open neighborhood of 0 in $\mathbb{C}$. For that purpose, we use
\[
\frac{1}{h} = \int_0^1 t^{h-1} dt
\]
to get
\[
\frac{\Gamma(z - h)\Gamma(h)}{\Gamma(z)} = \frac{1}{h} + \int_0^1 ((1 - t)^{z-h-1} - 1) t^{h-1} dt.
\]
The integral on the right-hand side now is holomorphic for $h$ in a small open neighborhood of 0 in $\mathbb{C}$ and we can write it as its value at $h = 0$ plus a term of the order $o(h)$ as $h \to 0$. Thus
\[
\frac{\Gamma(z - h)\Gamma(h)}{\Gamma(z)} = \frac{1}{h} + \int_0^1 ((1 - t)^{z-1} - 1) t^{-1} dt + o(h).
\]
This is the Laurent expansion of the Beta function $B(z - h, z)$ in the variable $h$ at $h = 0$. We compare this to the Laurent series expansion of

$$\frac{\Gamma(z - h)\Gamma(h)}{\Gamma(z)}$$

in $h$ and get

$$\frac{\Gamma(z - h)\Gamma(h)}{\Gamma(z)} = \frac{1}{\Gamma(z)} (\Gamma(z) - h \Gamma'(z) + \cdots) \left( \frac{1}{h} + A + \cdots \right),$$

where $A$ is a constant. Equating the constant terms of

$$\frac{1}{\Gamma(z)} (\Gamma(z) - h \Gamma'(z) + \cdots) \left( \frac{1}{h} + A + \cdots \right) = \frac{1}{h} + \int_0^1 ((1 - t)^{z-1} - 1) t^{-1} dt + o(h),$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^1 (1 - (1 - t)^{z-1}) t^{-1} dt + A$$

for $\text{Re} \ z > 0$. Using

$$\frac{1}{t} = \frac{1}{1 - (1 - t)} = \sum_{n=0}^{\infty} (1 - t)^n,$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = A + \int_0^1 (1 - (1 - t)^{z-1}) \left( \sum_{n=0}^{\infty} (1 - t)^n \right) dt$$

$$= A + \int_0^1 \left( \sum_{n=0}^{\infty} ((1 - t)^n - (1 - t)^{n+z-1}) \right) dt$$

$$= A + \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} - \frac{1}{n + z} \right).$$

We can rewrite it as

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + z} \right) + A.$$
To determine the constant $A$, we integrate and take exponents of both sides and get
\[
\frac{1}{\Gamma(z)} = e^{Az} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.
\]
Setting $z = 1$, we get
\[
1 = e^{A} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}}.
\]
Hence
\[
A = -\log \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}} = \lim_{N \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \log N\right)
\]
which is equal to the Euler constant $\gamma$. We have finally the following infinite product decomposition for $\Gamma(z)$.
\[
\frac{1}{\Gamma(z)} = e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.
\]

**Stirling’s Formula.** Stirling’s formula gives the asymptotic behavior of $\Gamma(z)$ for large $z$. Its formulation is
\[
\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log (2\pi) + O\left(\frac{1}{|z|}\right)
\]
for $z$ in the set
\[-\pi + \delta \leq \arg z \leq \pi - \delta
\]
with $\delta > 0$.

One reason for the use of the sector is to exclude the poles of $\Gamma(z)$ at nonpositive integer values of $z$.

We now prove Stirling’s formula. The idea of the proof is to get first the special discrete version when $z$ is an integer $N$ (so that $\Gamma(N) = (N - 1)!$) and then estimate the difference between the continuous version of a general $z$ and the special case of $z = N$. 
The dominant term in Stirling’s formula
\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log (2\pi) + O\left( \frac{1}{|z|} \right)
\]
is \(z \log z\). When \(z\) is an integer \(N\), \(\Gamma(z)\) becomes the factorial \((N - 1)!\) which means that the dominant term for \(\log((N - 1)!))\) should be \(N \log N\). We know that the indefinite integral for \(\log x\) is \(x \log x - x\). To express \(\log((N - 1)!))\) in terms of \(N\) and \(\log N\), we are led to consider the integration of \(\log t\) from \(t = 1\) to an upper limit of order \(N\). We do this by breaking up the integral from 1 to an upper limit of order \(N\) into \(N\) parts. So we consider the integral
\[
\int_{\nu - \frac{1}{2}}^{\nu + \frac{1}{2}} \log t \, dt
\]
and sum over \(\nu\) from \(\nu = 1\) to \(N\) to study the asymptotic behavior of the sum as \(N \to \infty\). We have
\[
\int_{\nu - \frac{1}{2}}^{\nu + \frac{1}{2}} \log t \, dt = \int_0^1 \left( \log(\nu + t) + \log(\nu - t) \right) \, dt
\]
\[
= \int_0^1 \left( \log(\nu^2) + \log\left(1 - \frac{t^2}{\nu^2}\right) \right) \, dt = \log \nu + C_\nu,
\]
where \(|C_\nu| \leq \frac{A}{\nu^2}\) for some constant \(A\) independent of \(\nu\). Then
\[
\log((N - 1)!)) = \int_{\frac{1}{2}}^{N - \frac{1}{2}} \log t \, dt - \sum_{\nu = 1}^{N - 1} C_\nu = \left( N - \frac{1}{2} \right) \log N - N + C + o(1)
\]
for some constant \(C\) as \(N \to \infty\).

To get back from the discrete version of \(\log((N - 1)!))\) to the continuous version of \(\Gamma(z)\), we consider the function \(\psi(x) = \lfloor x \rfloor - x + \frac{1}{2}\). The first part \(\lfloor x \rfloor - x\) is just the negative of the fractional part of \(x\), because \(\lfloor x \rfloor\) is the integral part of \(x\) which is the largest integer not exceeding \(x\). The second part of adding \(\frac{1}{2}\) is to make the jump from \(-\frac{1}{2}\) to \(\frac{1}{2}\) instead of from \(-1\) to 0 when the negative of the fractional part \(\lfloor x \rfloor - x\) crosses an integer point. The function \(\psi(x)\) has period 1 so that \(\psi(x + n) = \psi(x)\) for \(n \in \mathbb{Z}\). We are interested in \(\log((N - 1)!)\) which is the sum of \(\log \nu\) as \(\nu\) goes from 1
to $N - 1$. The discrepancy between the discrete version to the continuous version is estimated by

$$\int_{u=1}^{N} \frac{\psi(u)du}{u + z},$$

as we will see later when we explicitly integrate it out. We now estimate the discrepancy by

$$\int_{u=1}^{\infty} \frac{\psi(u)du}{u + z}$$

by integration by parts which integrates first the numerator so that the end-result increases the power of the denominator by 1 and makes it possible to get a convergent and estimable integral. So we introduce

$$\phi(u) = \int_{0}^{u} \psi(v)dv$$

which on the interval $[0, 1]$ is the same as the function $y = \frac{1}{2} (x - x^2)$ and whose graph is the inverted parabola which starts at the origin and increases to its maximum of $\frac{1}{8}$ at the point $x = \frac{1}{2}$ and then decreases to 0 at $x = 1$. 
Let $r = |z|$. With

$$|u + z| = (u + r \cos \theta)^2 + (r \sin \theta)^2 \geq (u - r \cos \delta)^2 + (r \sin \delta)^2,$$
the estimation of a bound for the discrepancy is
\[
\int_{u=0}^{\infty} \frac{\psi(u)du}{u + z} = \int_{u=0}^{\infty} \frac{\phi(u)du}{(u + z)^2}
\]
\[
= O \left( \int_{u=0}^{\infty} \frac{du}{(u - r \cos \delta)^2 + (r \sin \delta)^2} \right)
\]
\[
= O \left( \int_{u=0}^{\infty} \frac{du}{u + z} \right)
\]
\[
= O \left( \frac{1}{r \sin \delta} \left( \frac{\pi}{2} + \cot \delta \right) \right) = O \left( \frac{1}{r} \right).
\]

We now explicitly evaluate the integral which is expected as the discrepancy between the discrete version \( \log((N - 1)!) \) and the continuous version \( \Gamma(z) \) so that we can convince ourselves that indeed it is the discrepancy. We have
\[
\int_{u=0}^{N} \frac{\psi(u)du}{u + z} = \int_{u=0}^{N} \frac{|u| - u + \frac{1}{2}}{u + z} \, du = \sum_{n=0}^{N-1} \int_{u=n}^{n+1} \frac{|u| - u + \frac{1}{2}}{u + z} \, du
\]
\[
= \sum_{n=0}^{N-1} \left( n + \frac{1}{2} + z \right) \left( \log (u + z) \right)_{u=n}^{n+1}
\]
\[
= \sum_{n=0}^{N-1} \left( n + \frac{1}{2} + z \right) \left( \log (n + 1 + z) - \left( n + \frac{1}{2} + z \right) \log (n + z) \right).
\]

For the summation on the right-hand side, we remove the first term and the last term and regroup the remaining terms in consecutive pairs so that the expression of each consecutive pair is simple enough for us to sum (like in the process of telescopic collapse) to explicitly get
\[
\int_{u=0}^{N} \frac{\psi(u)du}{u + z} = -N + \left( N - \frac{1}{2} + z \right) \log (N + z) - \left( \frac{1}{2} + z \right) \log z
\]
\[
+ \sum_{n=1}^{N-1} \left( - \left( n + \frac{1}{2} + z \right) \log (n + z) + \left( n - 1 + \frac{1}{2} + z \right) \log (n + z) \right)
\]
\[
= -N + \left( N - \frac{1}{2} + z \right) \log (N + z) - \left( \frac{1}{2} + z \right) \log z - \sum_{n=1}^{N-1} \log(n + z).
\]
In order to link it to \( \log((N-1)!) \) we change \( \log(n+z) \) to \( \log\left(1 + \frac{z}{n}\right) + \log n \) inside the summation. Moreover, in order to link it to \( \log \Gamma(z) \) by taking the logarithmic derivative of the infinite product expansion of \( \Gamma(z) \), we replace \\
\[-\log\left(1 + \frac{z}{n}\right)\] with \\
\(\left(z - \log\left(1 + \frac{z}{n}\right)\right) + \frac{z}{n}\) inside the summation and get \\
\[\int_{u=0}^{N} \frac{\psi(u)du}{u+z} = -N + \left(N - \frac{1}{2} + z\right) \log(N+z) - \left(\frac{1}{2} + z\right) \log z \]
\[+ \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - z \sum_{n=1}^{N-1} \frac{1}{n} - \log((N-1)!)
\]
\[= -N + \left(N - \frac{1}{2} + z\right) \log(N+z) - \left(\frac{1}{2} + z\right) \log z \]
\[+ \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - \gamma z - z \log N - \log((N-1)!)+o(1),
\]
where the term \(o(1)\) means having limit 0 as \(N \to \infty\) for fixed \(z\) and occurs when we replace \(\sum_{n=1}^{N-1} \frac{1}{n}\) by \(\log N + \gamma + o(1)\). We now put in \\
\[\log((N-1)!) = \left(N - \frac{1}{2}\right) \log N - N + C + o(1)
\]
and write \\
\[\log(N+z) = \log N + \log\left(1 + \frac{z}{N}\right) = \log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right)\quad \text{as} \quad N \to \infty
\]
to get \\
\[\int_{u=0}^{N} \frac{\psi(u)du}{u+z} = -N + \left(N - \frac{1}{2} + z\right) \left(\log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right)\right) - \left(\frac{1}{2} + z\right) \log z \]
\[+ \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - \gamma z - z \log N - \left(N - \frac{1}{2}\right) \log N + N - C + o(1)
\]
\[= z - \left(\frac{1}{2} + z\right) \log z + \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - \gamma z - C + o(1).
\]
We now put in the formula \\
\[\log \Gamma(z) = -\gamma z - \log z + \lim_{N \to \infty} \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right)\]
from the logarithm of the product formula of $\Gamma(z)$ to get
\[ \int_{u=0}^{N} \frac{\psi(u)\,du}{u+z} = z - \left( \frac{1}{2} + z \right) \log z + \log \Gamma(z) + \gamma z + \log z - \gamma z - C + o(1) \]
\[ = z - \left( z - \frac{1}{2} \right) \log z + \log \Gamma(z) - C + o(1). \]

Letting $N \to \infty$, we end up with
\[ O \left( \frac{1}{|z|} \right) = z - \left( z - \frac{1}{2} \right) \log z + \log \Gamma(z) - C \]
or
\[ \log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + C + O \left( \frac{1}{|z|} \right). \]

We now use the duplication formula
\[ \Gamma (2z) \Gamma \left( \frac{1}{2} \right) = 2^{2z-1} \Gamma (z) \Gamma \left( z + \frac{1}{2} \right). \]
of $\Gamma(z)$ to determine the constant $C$. Taking logarithm of the duplication formula and using $\Gamma \left( \frac{1}{2} \right) = \sqrt{\pi}$, we obtain
\[ \left( 2z - \frac{1}{2} \right) \log (2z) - 2z + C + O \left( \frac{1}{|z|} \right) + \log \sqrt{\pi} \]
\[ = (2z - 1) \log 2 + \left( z - \frac{1}{2} \right) \log z - z + C + z \log z - z - \frac{1}{2} + C \]
and conclude that $C = \frac{1}{2} \log (2\pi)$. This gives us Stirling’s formula
\[ \log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log (2\pi) + O \left( \frac{1}{|z|} \right) \]
for $z$ in the the set
\[ -\pi + \delta \leq \arg z \leq \pi - \delta \]
with $\delta > 0$. The reason for restricting to the domain
\[ -\pi + \delta \leq \arg z \leq \pi - \delta \]
is to be able to obtain a branch of each of $\log \left( 1 + \frac{z}{n} \right)$ at the same time for all of $n \in \mathbb{N}$ so that we can justify convergence and the error estimate as $z \to \infty$. 