

## GAMMA FUNCTION

Gamma function is the continuous analogue of the factorial function  $n!$ . Just as the factorial function  $n!$  occurring naturally in the series expansion of  $e^z$  and in the integral formula for derivatives of holomorphic functions because of differentiation, the Gamma function occurs naturally in the treatment of the Riemann zeta function which is the key function for the application of complex analysis to number theory such as the prime number theorem and the Riemann hypothesis. We will discuss the definition of the Gamma function and its important properties before we proceed to the topic of Dirichlet series and the Riemann zeta function.

*Definition of Gamma Function.* Gamma function is the continuous analogue of the factorial function  $n!$ . The factorial function  $n!$  can be obtained from

$$\frac{d^n}{dx^n}(x^n) = n!,$$

or by applying integration by parts to

$$\int_{x=0}^{\infty} x^n e^{-x} dx$$

and integrate  $e^{-x}$  first and do it  $n$  times. To extend the definition of the factorial function  $n!$  to the case of a continuous variable, we define

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad \text{for } x > 0,$$

where the condition of  $x > 0$  is to make sure that the integral converges at  $t = 0$ .

In the definition there is a shift of the variable  $x$  by 1 as a matter of convention so that  $\Gamma(n) = (n-1)!$ , in order to move the singular point of the function from  $x = -1$  to  $x = 0$ . When  $x > 1$ , by integration by parts we get

$$\Gamma(x) = \left[ -t^{x-1} e^{-t} \right]_{t=0}^{t=\infty} + (x-1) \int_0^{\infty} t^{x-2} e^{-t} dt = (x-1) \Gamma(x-1).$$

From  $\Gamma(1) = \int_0^{\infty} e^{-t} dt = 1$  it follows that

$$\Gamma(n) = (n-1)! .$$

The defining formula

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

actually defines  $\Gamma(z)$  for  $z \in \mathbb{C}$  with  $\operatorname{Re} z > 0$ .

The Gamma function  $\Gamma(z)$  can be extended to  $\operatorname{Re} z > -1$  by using

$$\Gamma(z) = \frac{\Gamma(z+1)}{z},$$

which is meromorphic on  $\operatorname{Re} z > -1$  with a simple pole of residue 1 at  $z = 0$ , because  $\Gamma(1) = 1$ . We can repeat this technique to extend  $\Gamma(z)$  to all of  $\mathbb{C}$  by using

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}.$$

The extended  $\Gamma(z)$  is a meromorphic function on all of  $\mathbb{C}$  whose only poles are all simple at the nonnegative integers  $0, -1, -2, \dots, -n, \dots$ . The residue at  $z = -n$  is equal to

$$\lim_{z \rightarrow -n} (z+n)\Gamma(z) = \lim_{z \rightarrow -n} \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)} = \frac{(-1)^n}{n!}.$$

*Beta Function.* A similar analogue of the generalization of the binomial coefficient

$$\binom{m+n}{m} = \frac{(m+n)!}{m!n!}$$

is the Beta function defined by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We are going to derive the formula for the Beta function as a definite integral whose integrand depends on the variables  $x$  and  $y$ . This is done by changing the order of integration of a double integral as follows. For  $x > 0$  and  $y > 0$  we have

$$\Gamma(x)\Gamma(y) = \left( \int_0^{\infty} t^{x-1} e^{-t} dt \right) \left( \int_0^{\infty} u^{y-1} e^{-u} du \right).$$

Using the transformation  $u = tv$  (in order to get the factor  $t^{x+y-1}$  from  $t^{x-1}u^{y-1}$ , needed to factor out  $\Gamma(x+y)$  later) and then the transformation

$w = t(1 + v)$  (in order to express  $\Gamma(x + y)$  as an integral in  $w$  when the two variables  $(v, w)$ , replacing  $(v, t)$ , are used to separate the integrand into two factors, each of which depends only on one variable), we obtain

$$\begin{aligned}
 \Gamma(x)\Gamma(y) &= \int_{t=0}^{\infty} \int_{u=0}^{\infty} t^{x-1} u^{y-1} e^{-(t+u)} du dt \\
 &= \int_{t=0}^{\infty} \int_{v=0}^{\infty} t^{x+y-1} v^{y-1} e^{-t(1+v)} dv dt \\
 &= \int_{v=0}^{\infty} \int_{w=0}^{\infty} \frac{w^{x+y-1}}{(1+v)^{x+y-1}} v^{y-1} e^{-w} dv \frac{dw}{1+v} \\
 &= \left( \int_{v=0}^{\infty} \frac{v^{y-1}}{(1+v)^{x+y}} dv \right) \left( \int_{w=0}^{\infty} w^{x+y-1} e^{-w} dw \right) \\
 &= \Gamma(x+y) \int_{v=0}^{\infty} \frac{v^{y-1} dv}{(1+v)^{x+y}},
 \end{aligned}$$

from which it follows that

$$B(x, y) = \int_{v=0}^{\infty} \frac{v^{y-1} dv}{(1+v)^{x+y}}.$$

This integral representation of  $B(x, y)$  is not symmetric in  $x$  and  $y$ . To transform it to a symmetric form, we use the linear fractional transformation  $v = \frac{\lambda}{1-\lambda}$  which changes the interval of integration from  $(0, \infty)$  to  $(0, 1)$ . We obtain

$$B(x, y) = \int_{\lambda=0}^1 \lambda^{x-1} (1-\lambda)^{y-1} d\lambda,$$

which is symmetric in  $x$  and  $y$ .

*Relation Between Gamma Function and Sine Function (Euler's Reflection Formula).* A very useful case for the Beta function is when  $x + y = 1$  in the above formula, in which case

$$\Gamma(x)\Gamma(1-x) = \int_0^{\infty} \frac{v^{x-1} dv}{1+v},$$

which by residue calculus applied to the function

$$\frac{z^{x-1} dz}{1+z}$$

integrated over the contour integral of the boundary of the domain

$$\{r < |z| < R\} - \{\operatorname{Re} z \geq 0, -r \leq \operatorname{Im} z \leq r\}$$

as  $r \rightarrow 0+$  and  $R \rightarrow \infty$ , yields

$$\frac{\pi}{\sin \pi x}.$$

Thus we have the following important formula relating the gamma function to the sine function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

for  $0 < x < 1$ . By the identity theorem for meromorphic functions,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

on all of  $\mathbb{C}$ , known as *Euler's reflection formula*. The reflection refers to the reflection with respect to  $\operatorname{Re} z = \frac{1}{2}$ . The Euler reflection formula for the Gamma function gives us another way of extending the Gamma function to a meromorphic function on all of  $\mathbb{C}$  by defining

$$\Gamma(z) = \frac{\pi}{\Gamma(1-z)\sin(\pi z)} \quad \text{for } \operatorname{Re} z < \frac{1}{2}.$$

Note that the graph of  $y = \sin \pi x$  is symmetric with respect to the vertical line  $x = \frac{1}{2}$ . This symmetry with respect to the vertical line  $x = \frac{1}{2}$  has a special role in Riemann's Zeta function and its application to the Prime Number Theorem and the Riemann Hypothesis.

From Euler's reflection formula and the zero-divisor of  $\sin \pi z$ , one can conclude that  $\frac{1}{\Gamma(z)}$  is entire on  $\mathbb{C}$  whose only zeroes are at the nonpositive integers, all simple.

*Duplication Formula for Gamma Function.* We now turn to the use of the symmetric form of the integral representation of the Beta function

$$B(x, y) = \int_0^1 \lambda^{x-1} (1-\lambda)^{y-1} d\lambda$$

in the special case

$$B(x, x) = \int_{\lambda=0}^1 (\lambda(1-\lambda))^{x-1} d\lambda$$

For the special situation  $x = \frac{1}{2}$ , by using  $\lambda = \sin^2 \theta$ , we get

$$\begin{aligned} B\left(\frac{1}{2}, \frac{1}{2}\right) &= \int_{\lambda=0}^1 \frac{d\lambda}{\sqrt{\lambda(1-\lambda)}} \\ &= \int_{\theta=0}^{\frac{\pi}{2}} \frac{d(\sin^2 \theta)}{\sqrt{\sin^2 \theta \cos^2 \theta}} = 2 \int_{\theta=0}^{\frac{\pi}{2}} d\theta = \pi. \end{aligned}$$

Thus  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ .

For a general  $x$ , we change the quadratic polynomial  $\lambda(1-\lambda)$  to the degree-one polynomial  $\frac{1}{4}(1-\mu)$  so that  $\lambda = \frac{1}{2} - \frac{1}{2}\sqrt{\mu}$  and  $d\lambda = -\frac{d\mu}{4\sqrt{\mu}}$  to get

$$\begin{aligned} B(x, x) &= 2 \int_{\lambda=0}^{\frac{1}{2}} (\lambda(1-\lambda))^{x-1} d\lambda \\ &= \frac{1}{2} \int_0^1 \left(\frac{1}{4} - \frac{1}{4}\mu\right)^{x-1} \mu^{-\frac{1}{2}} \\ &= 2^{1-2x} \int_0^1 (1-\mu)^{x-1} \mu^{-\frac{1}{2}} d\mu \\ &= 2^{1-2x} B\left(x, \frac{1}{2}\right). \end{aligned}$$

From

$$\frac{\Gamma(x)^2}{\Gamma(2x)} = B(x, x) = 2^{1-2x} B\left(x, \frac{1}{2}\right) = 2^{1-2x} \frac{\Gamma(x)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(x + \frac{1}{2}\right)}$$

we have the following *duplication formula*

$$\Gamma(2x) \Gamma\left(\frac{1}{2}\right) = 2^{2x-1} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right).$$

By the identity theorem for meromorphic functions, the *duplication formula for the Gamma function*

$$\Gamma(2z) \Gamma\left(\frac{1}{2}\right) = 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$$

holds on all of  $\mathbb{C}$ .

*Meromorphic Extension of Gamma Function by Representation by Contour Integral.* We now look at another way of extending the Gamma function to

all of  $\mathbb{C}$  by using by representing the gamma function as a contour integral. The definition

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

is restricted to  $z \in \mathbb{C}$  with  $\operatorname{Re} z > 0$ , because of the local integrability of  $t^{z-1} e^{-t}$  at  $t = 0$ . This difficulty can be circumvented by replacing the interval of integration  $(0, \infty)$  by another contour  $C$  which is defined as follows. The contour  $C$  starts from positive infinity and goes toward the origin above the real axis and circles the origin once in the positive direction and then goes to positive infinity below the real axis. For  $\operatorname{Re} z > 0$  consider

$$\int_C e^{-w} (-w)^{z-1} dw.$$

We can evaluate the integral as follows. We use the branch-cut  $(0, \infty)$  to choose a branch of the multi-valued function  $(-w)^{z-1}$  so that

$$(-w)^{z-1} = e^{(z-1) \log(-w)}$$

with

$$\log(-w) = \log \rho + i(\phi - \pi)$$

when  $w = \rho e^{i\phi}$  and  $0 < \phi < 2\pi$ . So

$$\begin{aligned} & \int_C e^{-w} (-w)^{z-1} dw \\ &= \int_{\rho=\infty}^0 e^{-\rho} e^{(z-1)(\log \rho - i\pi)} d\rho + \int_{\rho=0}^\infty e^{-\rho} e^{(z-1)(\log \rho + i\pi)} d\rho \\ &= (e^{(z-1)\pi i} - e^{-(z-1)\pi i}) \int_{\rho=0}^\infty e^{-\rho} e^{(z-1) \log \rho} d\rho \\ &= 2i \sin((z-1)\pi) \int_{\rho=0}^\infty e^{-\rho} \rho^{z-1} d\rho = -2i \sin \pi z \Gamma(z). \end{aligned}$$

Note that the condition  $\operatorname{Re} z > 0$  is used to make sure that the integral around a small circle centered at the origin goes to zero as the radius of the circle goes to zero. When the condition  $\operatorname{Re} z > 0$  is not assumed, we can define the Gamma function by

$$\Gamma(z) = \frac{i}{2 \sin \pi z} \int_C e^{-w} (-w)^{z-1} dw.$$

The only possible poles of  $\Gamma(z)$  are the zeroes of  $\sin \pi z$ , namely  $z \in \mathbb{Z}$ . We already know that  $\Gamma(z)$  is regular at points of  $\operatorname{Re} z > 0$ . Thus the only possible poles of  $\Gamma(z)$  are at the non-positive integers. The residues at a non-positive integer  $z = -n$  is computed as follows. Since the principal part of

$$e^{-w}(-w)^{n-1} = (-w)^{-n-1} \sum_{k=0}^{\infty} \frac{(-1)^k w^k}{k!}$$

at  $w = 0$  is  $-\frac{1}{n!w}$ , it follows that

$$\int_C e^{-w}(-w)^{-n-1} dw = -\frac{2\pi i}{n!}$$

and the residue of  $\Gamma(z)$  at  $z = -n$  is equal to

$$\lim_{z \rightarrow -n} -\frac{2\pi i}{n!} \frac{i(z+n)}{2 \sin \pi z} = \frac{(-1)^n}{n!}.$$

*Infinite Product Expansion of Gamma Function.* We now discuss the infinite product expansion of  $\frac{1}{\Gamma(z)}$  by looking at the Euler reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

from a multiplicative point of view. We have an infinite product expansion

$$\sin \pi z = \pi z \prod_{n \in \mathbb{Z} - \{0\}} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}$$

which is obtained from the partial fraction expansion of its logarithmic derivative

$$\frac{d}{dz} \left( \log \frac{\sin \pi z}{\pi z} \right) = \pi \cot \pi z = \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{1}{z-n} + \frac{1}{n} \right)$$

by the technique of applying the Cauchy integral formula with a modified Cauchy kernel to a meromorphic function.

The Euler reflection formula

$$\frac{1}{\Gamma(z)} \frac{1}{\Gamma(1-z)} = \frac{\sin(\pi z)}{\pi}$$

can be interpreted as saying that, multiplicatively speaking,  $\frac{1}{\Gamma(z)}$  represents the part of  $\sin(\pi z)$  on the half-plane to the left of  $\operatorname{Re} z = \frac{1}{2}$ . It is possible to use Stirling's formula and Hadamard's factorization theorem to get an infinite product expansion of the reciprocal of the Gamma function.

We will not follow that route here. Instead, we will follow the same path as the derivation of the infinite product expansion of  $\sin(\pi z)$  by taking the logarithmic derivative  $(\log \Gamma(z))'$  of  $\Gamma$  and get its partial fraction expansion. Unlike the case of  $\sin(\pi z)$  where we have a way of getting the partial fraction expansion of

$$\frac{d}{dz} \left( \log \frac{\sin \pi z}{\pi z} \right) = \pi \cot \pi z,$$

we do not have any such tool. The tool we use will be the (symmetric) integral representation of the Beta function. Here is how it is done. The information for  $\Gamma(z)$  comes from integral representations, which give us the only tool we have to work with. Instead of using the difference quotient

$$\frac{\log \Gamma(z+h) - \log \Gamma(z)}{h},$$

we consider the exponentiation of the numerator and use the symmetric version of the integral representation of the Beta function. This motivates us to consider

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \int_0^1 (1-t)^{z-h-1} t^{h-1} dt.$$

From

$$\Gamma(z+1) = \frac{\Gamma(z)}{z}$$

we know that the principal part of  $\Gamma(z)$  at  $z = 0$  is  $\frac{1}{z}$ .

For  $\operatorname{Re} z > h > 0$  with  $z$  fixed and  $h$  varying, the principal part of

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}$$

is  $\frac{1}{h}$  as we can see by multiplying it  $h$  and let  $h \rightarrow 0$ . We would like to take it out so that what is left will be holomorphic for  $h$  in a small open neighborhood of 0 in  $\mathbb{C}$ . For that purpose, we use

$$\frac{1}{h} = \int_0^1 t^{h-1} dt$$

to get

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \frac{1}{h} + \int_0^1 ((1-t)^{z-h-1} - 1) t^{h-1} dt.$$

The integral on the right-hand side now is holomorphic for  $h$  in a small open neighborhood of 0 in  $\mathbb{C}$  and we can write it as its value at  $h = 0$  plus a term of the order  $o(h)$  as  $h \rightarrow 0$ . Thus

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \frac{1}{h} + \int_0^1 ((1-t)^{z-1} - 1) t^{-1} dt + o(h).$$

This is the Laurent expansion of the Beta function  $B(z-h, z)$  in the variable  $h$  at  $h = 0$ . We compare this to the Laurent series expansion of

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}$$

in  $h$  and get

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \frac{1}{\Gamma(z)} (\Gamma(z) - h\Gamma'(z) + \dots) \left( \frac{1}{h} + A + \dots \right),$$

where  $A$  is a constant. Equating the constant terms of

$$\begin{aligned} & \frac{1}{\Gamma(z)} (\Gamma(z) - h\Gamma'(z) + \dots) \left( \frac{1}{h} + A + \dots \right) \\ &= \frac{1}{h} + \int_0^1 ((1-t)^{z-1} - 1) t^{-1} dt + o(h), \end{aligned}$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^1 (1 - (1-t)^{z-1}) t^{-1} dt + A$$

for  $\operatorname{Re} z > 0$ . Using

$$\frac{1}{t} = \frac{1}{1 - (1-t)} = \sum_{n=0}^{\infty} (1-t)^n,$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = A + \int_0^1 (1 - (1-t)^{z-1}) \left( \sum_{n=0}^{\infty} (1-t)^n \right) dt$$

$$\begin{aligned}
&= A + \int_0^1 \left( \sum_{n=0}^{\infty} ((1-t)^n - (1-t)^{n+z-1}) \right) dt \\
&= A + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right).
\end{aligned}$$

We can rewrite it as

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+z} \right) + A.$$

To determine the constant  $A$ , we integrate and take exponents of both sides and get

$$\frac{1}{\Gamma(z)} = e^{Az} z \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}.$$

Setting  $z = 1$ , we get

$$1 = e^A \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-\frac{1}{n}}.$$

Hence

$$A = -\log \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-\frac{1}{n}} = \lim_{N \rightarrow \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \log N \right)$$

which is equal to the Euler constant  $\gamma$ . We have finally the following infinite product decomposition for  $\Gamma(z)$ .

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}.$$

*Stirling's Formula.* Stirling's formula gives the asymptotic behavior of  $\Gamma(z)$  for large  $z$ . Its formulation is

$$\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{|z|}\right)$$

for  $z$  in the the set

$$-\pi + \delta \leq \arg z \leq \pi - \delta$$

with  $\delta > 0$ .

One reason for the use of the sector is to exclude the poles of  $\Gamma(z)$  at nonpositive integer values of  $z$ .

We now prove Stirling's formula.

*Four Ingredients Needed for the Proof.* The proof puts together the following four ingredients.

(i) Stirling's formula for integral value  $N$  of the variable  $z$  of  $\Gamma(z)$

$$\log((N-1)!) = \left(N - \frac{1}{2}\right) \log N - N + C + o(1),$$

for some constant  $C$  as  $N \rightarrow \infty$ , by comparing  $\log \nu$  with

$$\int_{t=\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t \, dt$$

and summing up from  $\nu = 1$  to  $\nu = N - 1$  to get an estimate of the error term

(ii) Apply logarithm to both sides of the infinite product formula

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

to get

$$\log \Gamma(z) = -\gamma z - \log z + \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right).$$

(iii) Compute the following error integral

$$\int_{u=0}^{\infty} \frac{[u] - u + \frac{1}{2}}{u+z} \, du = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \int_{u=n}^{n+1} \frac{[u] - u + \frac{1}{2}}{u+z} \, du$$

in two ways. First, by applying integration by parts to the left-hand side to integrate  $[u] - u + \frac{1}{2}$  first to get  $O\left(\frac{1}{r}\right)$  (with  $r = |z|$ ) as its error estimate. Second, the summation

$$\sum_{n=0}^{N-1} \int_{u=n}^{n+1} \frac{[u] - u + \frac{1}{2}}{u+z} \, du$$

can be performed by a telescopic argument to end up with

$$= -N + \left(N - \frac{1}{2} + z\right) \log(N + z) - \left(\frac{1}{2} + z\right) \log z - \sum_{n=1}^{N-1} \log(n + z)$$

which can then put into the following form to prepare for the use of Ingredients (i) and (ii).

$$\begin{aligned} & -N + \left(N - \frac{1}{2} + z\right) \log(N + z) - \left(\frac{1}{2} + z\right) \log z \\ & + \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - z \sum_{n=1}^{N-1} \frac{1}{n} - \log((N-1)!). \end{aligned}$$

This, together with Ingredients (i) and (ii), yields

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + C + O\left(\frac{1}{|z|}\right)$$

for some constant  $C$ .

(iv) Finally, the constant  $C$  is determined by the duplication formula

$$\Gamma(2z) \Gamma\left(\frac{1}{2}\right) = 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

of  $\Gamma(z)$ .

*Derivation of Ingredient (i) to Estimate the Growth Order of  $\log(N-1)!$  in Terms of  $N$ .* To express

$$\log((N-1)!) = \sum_{\nu=1}^{N-1} \log \nu$$

in terms of  $N$ , we compare the sum

$$\sum_{\nu=1}^{N-1} \log \nu$$

to its approximating integral version

$$\int_{t=1}^x \log t \, dt = \left[t \log t - t\right]_{t=1}^{t=x} = x \log x - x + 1$$

which is rewritten as a sum (when  $x = N + \frac{1}{2}$ )

$$\int_{t=1}^{N+\frac{1}{2}} \log t \, dt = \sum_{\nu=1}^N \int_{t=\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t \, dt.$$

For the comparison of the corresponding summands, we use

$$\begin{aligned} \int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t \, dt &= \int_0^{\frac{1}{2}} (\log(\nu+t) + \log(\nu-t)) \, dt \\ &= \int_0^{\frac{1}{2}} \left( \log(\nu^2) + \log\left(1 - \frac{t^2}{\nu^2}\right) \right) dt = \log \nu + C_\nu \end{aligned}$$

to arrive at the error  $C_\nu$ , where  $|C_\nu| \leq \frac{A}{\nu^2}$  for some constant  $A$  independent of  $\nu$ . Then

$$\log((N-1)!) = \int_{\frac{1}{2}}^{N-\frac{1}{2}} \log t \, dt - \sum_{\nu=1}^{N-1} C_\nu = \left(N - \frac{1}{2}\right) \log N - N + C + o(1)$$

for some constant  $C$  as  $N \rightarrow \infty$ .

*Alternative Way of Using Telescopic Cancellation of Terms to Derive Error Term in Ingredient (i) as an Integral with Natural Continuous Version.* This section is purely for the purpose of explaining why the expression in Ingredient (iii) is used and has no effect in the logical rigorous argument for the proof of Stirling's formula. The inequality

$$\log((N-1)!) = \int_{\frac{1}{2}}^{N-\frac{1}{2}} \log t \, dt - \sum_{\nu=1}^{N-1} C_\nu = \left(N - \frac{1}{2}\right) \log N - N + C + o(1)$$

in Ingredient(i) is the discrete version of the Stirling's formula (without information on the constant  $C$ ) which states that

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{|z|}\right).$$

To go more easily from the discrete version to Stirling's formula, we redo the inequality in Ingredient (i) in order to be able to write the error term in an integral form which can naturally be changed to its continuous version. This

redoing of the inequality involves the method of telescopic cancellation of terms in a sum. In the derivation of Ingredient (i) the error is a sum whose summand is

$$\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t \, dt - \log \nu = \int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} (\log t - \log \nu) \, dt.$$

We now consider

$$\int_{t=0}^1 \left( \log(u+t) - \log\left(u + \frac{1}{2}\right) \right) dt$$

for  $u = \nu - \frac{1}{2}$ , which by integration by parts applied to integrating the factor 1 of the integrand becomes

$$\begin{aligned} & \left[ t \left( \log(u+t) - \log\left(u + \frac{1}{2}\right) \right) \right]_{t=0}^{t=1} - \int_{t=0}^1 \frac{t}{u+t} dt \\ &= \log(u+1) - \log\left(u + \frac{1}{2}\right) - \int_{t=0}^1 \left( 1 - \frac{u}{u+t} \right) dt \\ &= \log(u+1) - \log\left(u + \frac{1}{2}\right) - 1 + \int_{t=0}^1 \frac{u}{u+t} dt. \end{aligned}$$

With  $u = \nu - \frac{1}{2}$  and summation from  $\nu = 1$  to  $\nu = N - 1$ , the contribution of

$$\begin{aligned} & \log(u+1) - \log\left(u + \frac{1}{2}\right) - 1 \\ &= \log\left(1 - \frac{1}{2u+1}\right) - 1 \\ &= -1 + \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{1}{2u+1} \right)^k \end{aligned}$$

will be at most of order  $O(N)$ , from our knowledge of

$$\log((N-1)!) = \left(N - \frac{1}{2}\right) \log N - N + C + o(1).$$

This suggests that we should use

$$\int_{t=0}^1 \frac{u}{u+t} dt$$

for  $u = \nu - \frac{1}{2}$  as the error term in an integral form for the summation from  $\nu = 1$  to  $\nu = N - 1$  in the estimate for  $\log((N - 1)!)$  as  $N \rightarrow \infty$  if we tolerate the total error to be at most  $O(N)$ . This choice has the double advantage that there is a telescopic cancellation in the summation and it can be easily changed to its continuous version.

Let us now look at how the telescopic cancellation in the summation from  $\nu = 1$  to  $\nu = N - 1$  works. To make the notations simpler, we now use  $u = \nu$  instead of  $u = \nu - \frac{1}{2}$ . The shifting of the variable by  $\frac{1}{2}$  does not affect the relevant terms in the estimate. With

$$\left[ \int_{t=0}^1 \frac{u}{u+t} dt \right]_{u=\nu} = \int_{t=0}^1 \frac{\nu}{\nu+t} dt = \nu \log(\nu+1) - \nu \log \nu,$$

the summation from  $\nu = 1$  to  $\nu = N - 1$  yields

$$\begin{aligned} \sum_{\nu=1}^{N-1} \left[ \int_{t=0}^1 \frac{u}{u+t} dt \right]_{u=\nu} &= \sum_{\nu=1}^{N-1} (\nu \log(\nu+1) - \nu \log \nu) \\ &= (N-1) \log N + \sum_{\nu=2}^{N-1} (-\nu \log \nu + (\nu-1) \log \nu). \end{aligned}$$

Here the first term in the bracketed term with  $\nu = N - 1$  in the summation from  $\nu = 1$  to  $\nu = N - 1$  is separated out and the remaining terms are put into pairs so that the second term of the  $\nu$ -th bracketed term is paired with the first of the  $(\nu - 1)$ -th term. This rearrangement allows the simple computation

$$\begin{aligned} \sum_{\nu=1}^{N-1} \left[ \int_{t=0}^1 \frac{u}{u+t} dt \right]_{u=\nu} &= (N-1) \log N - \sum_{\nu=2}^{N-1} \log \nu \\ &= (N-1) \log N - \log((N-1)!). \end{aligned}$$

Now we discuss how to change the error expression from the discrete version to the continuous version. We simply use

$$\int_{t=\nu}^{\nu+1} \frac{u}{u+t} dt$$

for  $\nu \leq u \leq \nu + 1$  as the continuous version of

$$\left[ \int_{t=\nu}^{\nu+1} \frac{u}{u+t} dt \right]_{u=\nu}$$

In our discussion of Ingredient (iii) below we will use the following difference between the discrete version (after summation) and the continuous version

$$\int_{u=0}^N \frac{\lfloor u \rfloor - u + \frac{1}{2}}{u + z} du$$

when we estimate  $\Gamma(z)$  in the proof of Stirling's formula. The shifting of the variable  $u$  by  $\frac{1}{2}$  does not affect the relevant terms in the estimate. Note that when  $z$  is an integer, the replacement of  $u$  by  $u + z$  does not change  $\lfloor u \rfloor - u$ .

*No Need to Comment on Ingredient (ii).* There is no need to say anything special about Ingredient (ii), because it is simply taking logarithm of both sides of the infinite product expansion of  $\frac{1}{\Gamma(z)}$ . Now we come to Ingredient (iii).

*Derivation of Ingredient (iii) Involving the Error Integral.* Consider the function  $\psi(x) = \lfloor x \rfloor - x + \frac{1}{2}$ . The first part  $\lfloor x \rfloor - x$  is just the negative of the fractional part of  $x$ , because  $\lfloor x \rfloor$  is the integral part of  $x$  which is the largest integer not exceeding  $x$ . The second part of adding  $\frac{1}{2}$  is to make the jump from  $-\frac{1}{2}$  to  $\frac{1}{2}$  instead of from  $-1$  to  $0$  when the negative of the fractional part  $\lfloor x \rfloor - x$  crosses an integer point. The function  $\psi(x)$  has period 1 so that  $\psi(x + n) = \psi(x)$  for  $n \in \mathbb{Z}$ . We are interested in  $\log((N - 1)!)$  which is the sum of  $\log \nu$  as  $\nu$  goes from 1 to  $N - 1$ . The discrepancy between the discrete version to the continuous version is estimated by

$$\int_{u=1}^N \frac{\psi(u) du}{u + z}.$$

We discuss the motivation for it earlier. Later we will see how the estimates of  $\log((N - 1)!)$  and  $\Gamma(z)$  are related it when we explicitly integrate it out. We now estimate the discrepancy by

$$\int_{u=1}^{\infty} \frac{\psi(u) du}{u + z}$$

by integration by parts which integrates first the numerator so that the end-result increases the power of the denominator by 1 and makes it possible to get a convergent and estimable integral. So we introduce

$$\phi(u) = \int_0^u \psi(v) dv$$

which on the interval  $[0, 1]$  is the same as the function  $y = \frac{1}{2}(x - x^2)$  and whose graph is the inverted parabola which starts at the origin and increases to its maximum of  $\frac{1}{8}$  at the point  $x = \frac{1}{2}$  and then decreases to 0 at  $x = 1$ .

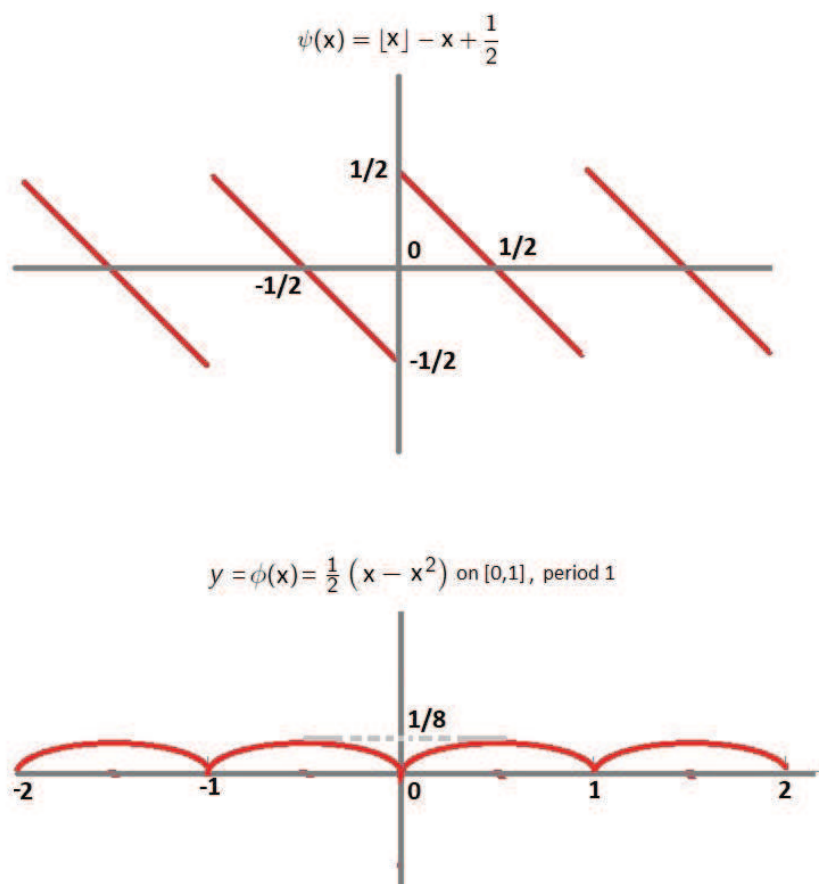


Figure 1: Graphs of  $\psi(x)$  and  $\phi(x)$

Let  $r = |z|$ . With

$$|u + z| = (u + r \cos \theta)^2 + (r \sin \theta)^2 \geq (u - r \cos \delta)^2 + (r \sin \delta)^2,$$

the estimation of a bound for the discrepancy is

$$\begin{aligned}
\int_{u=0}^{\infty} \frac{\psi(u)du}{u+z} &= \int_{u=0}^{\infty} \frac{\phi(u)du}{(u+z)^2} \\
&= O\left(\int_{u=0}^{\infty} \frac{du}{(u-r\cos\delta)^2 + (r\sin\delta)^2}\right) \\
&= O\left(\left[\frac{1}{r\sin\delta} \arctan\left(\frac{u-r\cos\delta}{r\sin\delta}\right)\right]_{u=0}^{u=\infty}\right) \\
&= O\left(\frac{1}{r\sin\delta} \left(\frac{\pi}{2} + \cot\delta\right)\right) = O\left(\frac{1}{r}\right).
\end{aligned}$$

The purpose of integration by parts is to increase the degree of the denominator by 1 as a polynomial in  $u+z$  so that when we integrate out with respect  $u$  we end up with the order no more than  $\frac{1}{|z|}$  for the error.

We now explicitly evaluate the integral

$$\int_{u=0}^{\infty} \frac{\psi(u)du}{u+z}$$

just from  $u=0$  to  $u=N-1$  without going to the limit as  $N \rightarrow \infty$ . We have

$$\begin{aligned}
\int_{u=0}^N \frac{\psi(u)du}{u+z} &= \int_{u=0}^N \frac{\lfloor u \rfloor - u + \frac{1}{2}}{u+z} du = \sum_{n=0}^{N-1} \int_{u=n}^{n+1} \frac{\lfloor u \rfloor - u + \frac{1}{2}}{u+z} du \\
&= \sum_{n=0}^{N-1} \int_{u=n}^{n+1} \left(\frac{n + \frac{1}{2} + z}{u+z} - 1\right) du = -N + \sum_{n=0}^{N-1} \int_{u=n}^{n+1} \frac{n + \frac{1}{2} + z}{u+z} du \\
&= -N + \sum_{n=0}^{N-1} \left(n + \frac{1}{2} + z\right) [\log(u+z)]_{u=n}^{n+1} \\
(\natural) \quad &= -N + \sum_{n=0}^{N-1} \left(\left(n + \frac{1}{2} + z\right) \log(n+1+z) - \left(n + \frac{1}{2} + z\right) \log(n+z)\right).
\end{aligned}$$

For the summation on the right-hand side  $(\natural)$ , we remove the first term and the last term and regroup the remaining terms in consecutive pairs so that the expression of each consecutive pair is simple enough for us to sum (like

in the process of telescopic collapse) to explicitly get

$$\begin{aligned} \int_{u=0}^N \frac{\psi(u)du}{u+z} &= -N + \left(N - \frac{1}{2} + z\right) \log(N+z) - \left(\frac{1}{2} + z\right) \log z \\ &\quad + \sum_{n=1}^{N-1} \left(-\left(n + \frac{1}{2} + z\right) \log(n+z) + \left(n - 1 + \frac{1}{2} + z\right) \log(n+z)\right) \\ (\ddagger) &= -N + \left(N - \frac{1}{2} + z\right) \log(N+z) - \left(\frac{1}{2} + z\right) \log z - \sum_{n=1}^{N-1} \log(n+z). \end{aligned}$$

More precisely, the process of going from (†) to (‡) involves pairing the second term

$$-\left(n + \frac{1}{2} + z\right) \log(n+z)$$

of the parenthetical term

$$\left(n + \frac{1}{2} + z\right) \log(n+1+z) - \left(n + \frac{1}{2} + z\right) \log(n+z)$$

inside the summation (†) indexed by  $n$  with the first term

$$\left(n - 1 + \frac{1}{2} + z\right) \log(n+z)$$

of the parenthetical term

$$\left(n - 1 + \frac{1}{2} + z\right) \log(n+z) - \left(n - 1 + \frac{1}{2} + z\right) \log(n-1+z)$$

inside the summation (†) indexed by  $n-1$  to get

$$-\left(n + \frac{1}{2} + z\right) \log(n+z) + \left(n - 1 + \frac{1}{2} + z\right) \log(n+z) = -\log(n+z)$$

so that the first term inside the parenthetical term indexed by  $n = N-1$  and the second term inside the parenthetical term indexed by  $n = 0$  are left over to yield

$$\left(N - \frac{1}{2} + z\right) \log(N+z) - \left(\frac{1}{2} + z\right) \log z.$$

In order to link (‡‡) to  $\log((N-1)!)$ , we change  $\log(n+z)$  to

$$\log\left(1 + \frac{z}{n}\right) + \log n$$

inside the summation (‡‡). Moreover, in order to link it to  $\log \Gamma(z)$  by taking the logarithmic derivative of the infinite product expansion of  $\Gamma(z)$ , we replace

$$-\log\left(1 + \frac{z}{n}\right)$$

by

$$\left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) + \frac{z}{n}$$

inside the summation (‡‡) and get

$$\begin{aligned} \int_{u=0}^N \frac{\psi(u)du}{u+z} &= -N + \left(N - \frac{1}{2} + z\right) \log(N+z) - \left(\frac{1}{2} + z\right) \log z \\ &+ \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - z \sum_{n=1}^{N-1} \frac{1}{n} - \log((N-1)!) \\ &= -N + \left(N - \frac{1}{2} + z\right) \log(N+z) - \left(\frac{1}{2} + z\right) \log z \\ &+ \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - \gamma z - z \log N - \log((N-1)!) + o(1), \end{aligned}$$

where the term  $o(1)$  means having limit 0 as  $N \rightarrow \infty$  for fixed  $z$  and occurs when we replace  $\sum_{n=1}^{N-1} \frac{1}{n}$  by  $\log N + \gamma + o(1)$ . We now put in

$$\log((N-1)!) = \left(N - \frac{1}{2}\right) \log N - N + C + o(1)$$

and write

$$\log(N+z) = \log N + \log\left(1 + \frac{z}{N}\right) = \log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right) \quad \text{as } N \rightarrow \infty$$

to get

$$\begin{aligned} \int_{u=0}^N \frac{\psi(u)du}{u+z} &= -N + \left(N - \frac{1}{2} + z\right) \left(\log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right)\right) - \left(\frac{1}{2} + z\right) \log z \\ &+ \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - \gamma z - z \log N - \left(N - \frac{1}{2}\right) \log N + N - C + o(1) \\ &= z - \left(\frac{1}{2} + z\right) \log z + \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - \gamma z - C + o(1). \end{aligned}$$

The details of getting the last equality is to break up the second term

$$\left(N - \frac{1}{2} + z\right) \left(\log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right)\right)$$

into

$$\left(N - \frac{1}{2}\right) \log N + z \log N + z + O\left(\frac{1}{N}\right),$$

which cancels with

$$-z \log N - \left(N - \frac{1}{2}\right) \log N$$

to yield

$$z + O\left(\frac{1}{N}\right).$$

We now put in the formula

$$\log \Gamma(z) = -\gamma z - \log z + \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right)$$

from the logarithm of the product formula of  $\Gamma(z)$  to get

$$\begin{aligned} \int_{u=0}^N \frac{\psi(u)du}{u+z} &= z - \left(\frac{1}{2} + z\right) \log z + \log \Gamma(z) + \gamma z + \log z - \gamma z - C + o(1) \\ &= z - \left(z - \frac{1}{2}\right) \log z + \log \Gamma(z) - C + o(1). \end{aligned}$$

Letting  $N \rightarrow \infty$ , we end up with

$$O\left(\frac{1}{|z|}\right) = z - \left(z - \frac{1}{2}\right) \log z + \log \Gamma(z) - C$$

or

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + C + O\left(\frac{1}{|z|}\right).$$

We now come to the use of Ingredient (iv). We use the duplication formula

$$\Gamma(2z) \Gamma\left(\frac{1}{2}\right) = 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

of  $\Gamma(z)$  to determine the constant  $C$ . Taking logarithm of the duplication formula and using  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ , we obtain

$$\begin{aligned} & \left(2z - \frac{1}{2}\right) \log(2z) - 2z + C + O\left(\frac{1}{|z|}\right) + \log \sqrt{\pi} \\ &= (2z - 1) \log 2 + \left(z - \frac{1}{2}\right) \log z - z + C + z \log z - z - \frac{1}{2} + C \end{aligned}$$

and conclude that  $C = \frac{1}{2} \log(2\pi)$ . This gives us Stirling's formula

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{|z|}\right)$$

for  $z$  in the the set

$$-\pi + \delta \leq \arg z \leq \pi - \delta$$

with  $\delta > 0$ . The reason for restricting to the domain

$$-\pi + \delta \leq \arg z \leq \pi - \delta$$

is to be able to obtain a branch of each of  $\log\left(1 + \frac{z}{n}\right)$  at the same time for all of  $n \in \mathbb{N}$  so that we can justify convergence and the error estimate as  $z \rightarrow \infty$ .