Elliptic Functions (Approach of Weierstrass)

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Three Basic Properties of General Elliptic Functions

- Before we discuss the approach of Weierstrass to elliptic functions, we first look at some basic properties of doubly periodic meromorphic functions. These functions are called *elliptic functions*.
- The Jacobian elliptic functions we have seen and the Weierstrass elliptic functions we are introducing are special cases of these general elliptic functions.
- The three basic properties of of general elliptic functions are as follows. We denote two primitive periods of an elliptic function $f$ by $\omega_1$ and $\omega_2$.
  
  (i) The sum of the residues of the function inside a fundamental parallelogram is zero.
  (ii) The number of zeroes of the function equals the number of poles inside a fundamental parallelogram.
  (iii) Inside a fundamental parallelogram the sum of the coordinates of the zeroes equals the sum of the coordinates of the poles modulo a period.
Figure: Fundamental Parallelogram
• To prove (i) we integrate $f(z)dz$ along the boundary of the fundamental parallelogram. By the residue theorem the integral is simply $2\pi i$ times the sum of the residues of $f$ inside the parallelogram. On the other hand the integral is zero, because by the periodicity of the integral over $[a, a + \omega_1]$ equals the integral over $[a + \omega_2, a + \omega_1 + \omega_2]$ and the integral over $[a, a + \omega_2]$ equals the integral over $[a + \omega_1, a + \omega_1 + \omega_2]$.

• Property (ii) follows from integrating

$$\frac{1}{2\pi i} \frac{f'(z)}{f(z)}dz$$

over the boundary of the fundamental parallelogram and from the argument principle.
• The proof of Property (iii) is slightly more complicated. One integrates
\[
\frac{1}{2\pi i} \int \frac{zf'(z)}{f(z)} \, dz
\]
over the boundary of the fundamental parallelogram, but in this case the integral may not be zero,

• because
\[
\frac{1}{2\pi i} \int_{[a, a+\omega_2]} \frac{zf'(z)}{f(z)} \, dz \quad \text{and} \quad \frac{1}{2\pi i} \int_{[a, a+\omega_2]} \frac{zf'(z)}{f(z)} \, dz
\]

\[
= \omega_1 \frac{1}{2\pi i} \int_{[a, a+\omega_2]} \frac{f'(z)}{f(z)} \, dz.
\]

• However,
\[
\frac{1}{2\pi i} \int_{[a, a+\omega_2]} \frac{f'(z)}{f(z)} \, dz
\]
equals \frac{1}{2\pi i} times the difference of the value of \( \log f(z) \) at \( a + \omega_2 \) and at \( a \) when \( z \) runs along \([a, a + \omega_2] \).
• Since \( f(z) \) has the same value at \( a \) as at \( a + \omega_2 \), the difference of the value of \( \log f(z) \) at \( a + \omega_2 \) and at \( a \) when \( z \) runs along \([a, a + \omega_2]\) must be \( 2\pi i \) times an integer.

• Therefore

\[
\frac{1}{2\pi i} \int_{[a, a + \omega_2]} \frac{f'(z)}{f(z)} \, dz
\]

is an integer and

\[
\frac{1}{2\pi i} \int_{[a + \omega_1, a + \omega_1 + \omega_2]} \frac{zf'(z)}{f(z)} \, dz - \frac{1}{2\pi i} \int_{[a, a + \omega_2]} \frac{zf'(z)}{f(z)} \, dz
\]

is a period of \( f \).

• Likewise

\[
\frac{1}{2\pi i} \int_{[a, a + \omega_1]} \frac{zf'(z)}{f(z)} \, dz - \frac{1}{2\pi i} \int_{[a + \omega_2, a + \omega_1 + \omega_2]} \frac{zf'(z)}{f(z)} \, dz
\]

is also a period of \( f \). This concludes the proof of (iii).
Weierstrass Elliptic Functions to be Introduced as Infinite Sum of Partial Fractions

- Though a Weierstrass elliptic function is to be introduced as an infinite sum of partial fractions which is doubly periodic as a meromorphic function on \( \mathbb{C} \),

- we will first discuss how the partial fractions in the infinite sum is motivated from the approach of Abel and Jacobi of introducing an elliptic function as the inverse function of an elliptic integral whose integrand is the reciprocal of a quartic polynomial.

- Our discussion of the motivation starts with the replacement of the quartic polynomial by a cubic polynomial.
Polynomial in Elliptic Integral Reduced from Quartic to Cubic by Having One Root at Infinity to Control Pole Order of Inverse of Indefinite Elliptic Integral

- The approach of Abel and Jacobi inverts the indefinite integral
  \[ \int \frac{dz}{\sqrt{F(z)}} \]
  where \( F(z) \) is a quartic polynomial, especially of the form
  \( (1 - z^2)(1 - k^2 z^2) \).
- Actually the case of \( \deg F(x) = 3 \) and the case of \( \deg F(x) = 4 \) are the same.
- Suppose \( F(x) \) is of degree 4, which without loss of generality can be written as \( \prod_{\nu=1}^{4} (x - \gamma_\nu) \) with all four \( \gamma_\nu \in \mathbb{C} \) distinct.
- Apply the Möbius transformation
  \[ z = \frac{a\zeta + b}{c\zeta + d} \]
  with \( ad - bc \neq 0 \).
Then
\[ dz = \frac{ad - bc}{(c\zeta + d)^2} \]
and
\[ \int \frac{d\zeta}{\sqrt{F(\zeta)}} = (ad - bc) \int \frac{d\zeta}{\sqrt{Q(\zeta)}} d\zeta \]
with
\[ Q(t) = \prod_{\nu=1}^{4} ((a\zeta + b) - \gamma_{\nu}(c\zeta + d)). \]

We can choose \( a, b, c, d \) so that \( a - \gamma_1 c = 0 \). Then \( b - \gamma_1 d \neq 0 \) from \( ad - bc \neq 0 \). Then the degree of \( Q(\zeta) \) is 3 in \( \zeta \).

Geometrically this means that the Möbius transformation
\[ z = \frac{a\zeta + b}{c\zeta + d} \]
maps the point \( \zeta = \infty \) to the point \( z = \gamma_1 \) and with respect to the \( \zeta \) coordinate our integral comes from a polynomial of degree 3.

An affine variable change reduces the polynomial to \( 4\zeta^3 - g_2\zeta - g_3 \).
An indefinite elliptic integral of Weierstrass is of the form

$$\int_{\zeta=\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},$$

where $g_2, g_3$ are complex numbers and the three roots $e_1, e_2, e_3$ of the cubic polynomial $4\zeta^3 - g_2\zeta - g_3$ are distinct, which means that its discriminant

$$\prod_{1 \leq j < k \leq 3} (e_j - e_k)^2 = -16(g_2^3 - 27g_3^2)$$

is nonzero.

The initial point of integration is chosen to be $\infty$, because it is one of the roots of the quartic polynomial and is a branching point of the Riemann surface.
The choice of $\infty$ as the initial point of integration means that the inverse function of the indefinite elliptic integral

(i) has a double pole at the origin and
(ii) is an even function.

We would like to see what the principal part at the origin should be chosen when the elliptic function so that its translates by the periods can be used in the infinite sum of partial fractions to define the Weierstrass elliptic function.

By the fundamental theorem of calculus, the elliptic function $\wp(z)$ defined as the inverse function of

\[ z = \wp^{-1}(w) = \int_{\zeta=\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}} \]

satisfies

\[ \frac{dz}{dw} = \frac{1}{\sqrt{4w^3 - g_2w - g_3}} \]
• which is the same as

\[ \left( \frac{d\wp(z)}{dz} \right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \]

• If the principal part of \( \wp(z) \) at 0 is \( \sum_{\nu=-\ell}^{-1} \frac{c_{-\nu}}{z^\nu} \), equating the terms of most negative power on both sides yields

\[ \frac{\ell^2 c_{-\ell}^2}{z^{2(\ell+1)}} = 4 \frac{c_{-\ell}^3}{z^{3\ell}}, \]

which implies \( 2(\ell + 1) = 3\ell \) and \( \ell^2 c_{-\ell}^2 = 4c_{-\ell}^3 \), i.e., \( \ell = 2 \), and \( c_{-2} = 1 \).

• By the evenness of \( \wp(z) \) (from the initial point of integration being a branch-point of the Riemann surface), the principal part of \( \wp(z) \) at 0 must be \( \frac{1}{z^2} \).
Rigorous Definition of Weierstrass $\wp$ Function

- The above discussion involving the Riemann surface for the square root of a cubic polynomial with leading coefficient 4 and without second-highest degree term is just for the sake of giving the background motivation for choosing the principal part at the origin to be $\frac{1}{z^2}$.

- The discussion helps to understand why the infinite sum of partial fractions to define the Weierstrass $\wp$ function is chosen to be of that particular form, but it is completely unnecessary for the rigorous definition of $\wp(z)$ which we now give.

- We can actually start our treatment of the Weierstrass elliptic function $\wp(z)$ from this point and forget about all the preceding material concerning elliptic functions.

- Let $\omega_1, \omega_2$ be two complex numbers which are $\mathbb{R}$-linearly independent. Denote by $L$ the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ generated by $\omega_1, \omega_2$. 
The Weierstrass $\wp(z)$ for the lattice $L$ is defined as the infinite series

$$\wp(z) = \frac{1}{z^2} + \sum_{n \in L - \{0\}} \left(\frac{1}{(z - \ell)^2} - \frac{1}{\ell^2}\right).$$

The reason for the infinite sum over $\ell \in L$ is the need to end up with a function for which every element of $L$ is a period.

The term $\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2}$ is used instead of just $\frac{1}{(z-\ell)^2}$, because without subtracting $\frac{1}{\ell^2}$ the infinite sum fails to converge for the following reason.

One is not able to conclude that for $z$ in a compact subset of $\mathbb{C}$ the Cauchy partial sum

$$\sum_{\ell \in L, \ p \leq |\ell| \leq q} \frac{1}{(z - \ell)^2}$$

to go to 0 as $p, q \to \infty$,.
because it is comparable to

\[
\sum_{k=p}^{q} \left( \sum_{k-1 \leq |\ell| \leq k+1} \frac{1}{|\ell|^2} \right) \sim \sum_{k=p}^{q} k \frac{1}{k^2} = \sum_{k=p}^{q} \frac{1}{k}.
\]

On the other hand,

\[
\sum_{\ell \in L, p \leq |\ell| \leq q} \left( \frac{1}{(z - \ell)^2} - \frac{1}{\ell^2} \right)
\]

is comparable to

\[
\sum_{k=p}^{q} \left( \sum_{k-1 \leq |\ell| \leq k+1} \frac{1}{|\ell|^3} \right) \sim \sum_{k=p}^{q} k \frac{1}{k^3} = \sum_{k=p}^{q} \frac{1}{k^2},
\]

which goes to 0 as \( p, q \to \infty \).
• Note that to determine the convergence of an infinite series of partial fractions uniformly on a compact subset $K$ of $\mathbb{C}$, one removes first the finite number of terms with poles in $K$.

• so that the convergence question is only for an infinite series of holomorphic functions instead of meromorphic functions.

• An unexpected difficulty in the verification of the periodicity property of $\wp(z)$ arises with the use of the term $\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2}$ for $\ell \in L - \{0\}$, because the term for the case of $\ell = 0$ is different due to the trouble of dividing 1 by 0.

• In fact, $\wp(z)$ so defined still has period $\ell$ for each $\ell \in L$. One can verify this in one of the following ways.

• The first way differentiates the infinite series of $\wp(z)$ term-by-term to get rid of $-\frac{1}{\ell^2}$ in each term for $\ell \in L - \{0\}$ and then integrate back. The vanishing of the constant of integration is guaranteed by the evenness of $\wp(z)$. 
• More precisely
\[ \wp'(z) = -2 \sum_{\ell \in L} \frac{1}{(z - \ell)^3} \]

which is justified from the convergence of the right-hand side absolutely and uniformly on compact subsets of \( \mathbb{C} \).

• Now clearly
\[ \wp'(z + \ell) = \wp'(z) \]

for \( \ell \in L \).

• Integrating with respect to \( z \) yields
\[ \wp(z + \ell) = \wp(z) + C_\ell \]

for some constant \( C_\ell \).

• Evaluating at \( z = -\frac{\ell}{2} \) yields
\[ \wp \left( \frac{\ell}{2} \right) = \wp \left( -\frac{\ell}{2} \right) + C_\ell, \]

which from the evenness of \( \wp(z) \) implies that \( C_\ell = 0 \).
• The second way of verifying the periodicity of $\wp(z)$ is simply to rewrite the infinite sum as

$$\wp(z) = \sum_{n_1 \in \mathbb{Z}} \frac{1}{(z - n_1 \omega_1)^2} - \sum_{n_1 \in \mathbb{Z} - \{0\}} \frac{1}{n_1^2 \omega_1^2}$$

$$+ \sum_{n_2 \in \mathbb{Z} - \{0\}} \sum_{n_1 \in \mathbb{Z}} \left( \frac{1}{(z - n_1 \omega_1 - n_2 \omega_2)^2} - \frac{1}{(n_1 \omega_1 - n_2 \omega_2)^2} \right)$$

• which clearly satisfies $\wp(z + \omega_1) = \wp(z)$. For the same reason $\wp(z + \omega_2) = \wp(z)$ holds.

• The key point of this second way of verification is that the two series of single summation

$$\sum_{n_1 \in \mathbb{Z}} \frac{1}{(z - n_1 \omega_1)^2} \quad \text{and} \quad \sum_{n_1 \in \mathbb{Z} - \{0\}} \frac{1}{n_1^2 \omega_1^2}$$

converge and the trouble only involves single summation along the one period for which periodicity is checked.
We now derive the first order differential equation for $\wp(z)$ by eliminating the principal part of the only pole in a fundamental parallelogram for an elliptic function constructed as a polynomial of $\wp(z)$ and $\wp'(z)$.

Since $\wp'(z)$ as the derivative of an even function is odd, to work with even elliptic functions to minimize the number of terms in the principal part at the origin, we use $\wp'(z)^2$ whose principal part at 0 is $\frac{4}{z^6} + \frac{a}{w^4} + \frac{b}{w^2}$ for some $a, b \in \mathbb{C}$ due to its evenness.

To cancel the term $\frac{4}{z^6}$ in the principal part at 0 with the most negative power, we should use $4\wp(z)^3$, but the process still leaves a principal part $\frac{c}{z^4} + \frac{d}{z^2}$ for some $c, d \in \mathbb{C}$.

Thus, we know that $\wp'(z)^2 = 4\wp(z)^3 + \alpha_2 \wp'(x)^2 + \alpha_1 \wp(x) + \alpha_0$ for some constants $\alpha_0, \alpha_1, \alpha_2$. 

Derivation of Differential Equation for $\wp(z)$ by Eliminating Principal Part of Only Pole in Fundamental Parallelogram for Elliptic Function
To determine the constants $\alpha_0, \alpha_1, \alpha_2$, we have to explicitly write down a few terms in the Laurent series expansion of $\wp(z)$ at 0.

We now derive the first order differential equation for $\wp(z)$ by eliminating the principal part of the only pole in a fundamental parallelogram for an elliptic function constructed as a polynomial of $\wp(z)$ and $\wp'(z)$.

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To cancel the term $\frac{4}{z^6}$ in the principal part at 0 with the most negative power, we should use $4\wp(z)^3$, but the process still leaves a principal part $\frac{c}{z^4} + \frac{d}{z^2}$ for some $c, d \in \mathbb{C}$.

Thus, we know that $\wp'(z)^2 = 4\wp(z)^3 + \alpha_2\wp(z)^2 + \alpha_1\wp(z) + \alpha_0$ for some constants $\alpha_0, \alpha_1, \alpha_2$.

To determine the constants $\alpha_0, \alpha_1, \alpha_2$, we have to explicitly write down a few terms in the Laurent series expansion of $\wp(z)$ at 0.
First Few Terms in Laurent Series Expansion of $\wp(z)$ at 0

- For $n \geq 3$ Let
  \[ s_n = \sum_{\ell \in L - \{0\}} \frac{1}{\ell^n}. \]

- Since
  \[
  \frac{1}{(z - \ell)^2} = \frac{1}{\ell^2 (1 - \frac{z}{\ell})^2} = \frac{1}{\ell^2} + 2 \frac{z}{\ell^3} + 3 \frac{z^2}{\ell^4} + \cdots \text{ for } |z| < |\ell|,
  \]

- it follows that
  \[
  \wp(z) = z^{-2} + 3s_4 z^2 + 5s_6 z^4 + \cdots, \\
  \wp'(z) = -2z^{-3} + 6s_4 z + 20s_6 z^3 + \cdots, \\
  \wp'(z)^2 = 4z^{-6} - 24s_4 z^{-2} - 80s_6 + \cdots, \\
  \wp(z)^3 = z^{-6} + 9s_4 z^{-2} + 15s_6 + \cdots,
  \]
and
\[ \varphi'(z)^2 - 4\varphi(z)^3 + 60s_4\varphi(z) = -140s_6 + \cdots. \]

Thus we have the differential equation
\[ \varphi'(z)^2 = 4\varphi(z)^3 - g_2\varphi(z) - g_3, \]

where
\[ g_2 = 60s_4 = 60 \sum_{\ell \in L - \{0\}}^{1} \frac{1}{\ell^4} \quad \text{and} \quad g_3 = 140s_6 = 140 \sum_{\ell \in L - \{0\}}^{1} \frac{1}{\ell^6}. \]

From the differential equation it follows that \( w = \varphi(z) \) is the inverse function of the indefinite integral
\[ w \mapsto z = \int_{\zeta = \infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}. \]

The inversion step of using the fundamental group of the Riemann surface of the denominator of the integrand has been replaced by the elimination of the principal part at 0 of a polynomial in the doubly periodic functions \( \varphi(z) \) and \( \varphi'(z) \) when \( \varphi(z) \) is defined as the infinite sum of partial fractions.
The addition theorem for the Weierstrass $\wp$ function is obtained by using Property (iii) of doubly periodic functions.

Let $x = \wp(w)$ and $y = \wp'(w)$ and, for some complex numbers $a \neq 0$ and $b$ to be determined later, we consider the doubly periodic function $y + ax + b$.

This doubly periodic function has a pole of order 3 at the origin and no other poles inside a fundamental parallelogram. So the sum of its three zeroes must be zero modulo a period.

We are free to choose $a$ and $b$. We can choose $a$ and $b$ so that the doubly periodic function $y + ax + b$ vanishes at $w_1$ and $w_2$. Then $y + ax + b$ must also vanish at $-(w_1 + w_2)$.

On the other hand we have the equation

$$y^2 = 4x^3 - g_2x - g_3.$$
• So by solving the two equations

\[ y + ax + b = 0 \]

and

\[ y^2 = 4x^3 - g_2x - g_3, \]

we would get the values of \( x \) and \( y \) at \(- (w_1 + w_2)\).

• Since one equation is a linear equation and the second one is a cubic equation, we expect to get 3 solutions for \((x, y)\).

• The other two solutions are the values of \((x, y)\) at \(w_1\) and \(w_2\). Knowing these two solutions makes getting the third solution very easy,

• because one can use the fact that for a cubic equation with unit leading coefficient the sum of the three roots is the negative of the second coefficient.
• We now carry out the details to get our addition theorem. From

\[ \wp'(w_1) + a\wp(w_1) + b = 0 \]
\[ \wp'(w_2) + a\wp(w_2) + b = 0 \]

we get

\[ a = -\frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)}. \]

(As we see later we do not need to solve for \( b \).)

• From the equation

\[ (ax + b)^2 = 4x^3 - g_2x - g_3 \]

we obtain (by the evenness of \( \wp(w) \))

\[ \wp(w_1) + \wp(w_2) + \wp(w_1 + w_2) = \frac{a^2}{4} = \frac{1}{4} \left( \frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)} \right)^2. \]

• Thus we have the addition formula

\[ \wp(w_1 + w_2) = -\wp(w_1) - \wp(w_2) + \frac{1}{4} \left( \frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)} \right)^2. \]
The addition theorem for the Weierstrass $\wp$ function can be formulated in terms of indefinite integrals as

$$
\int_{\zeta=\infty}^{\xi} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}} + \int_{\zeta=\infty}^{\eta} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}} = \int_{\zeta=\infty}^{-\xi - \eta + \frac{1}{4} \left( \frac{\sqrt{4\xi^3 - g_2\xi - g_3} - \sqrt{4\eta^3 - g_2\eta - g_3}}{\xi - \eta} \right)^2} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},
$$

because when we let $\xi = \wp(w_1)$ and $\eta = \wp(w_2)$, we have

$$
\wp^{-1}(\xi) + \wp^{-1}(\eta) = w_1 + w_2 = \wp^{-1}(\wp(w_1 + w_2))
$$

$$
= \wp^{-1} \left( -\wp(w_1) - \wp(w_2) + \frac{1}{4} \left( \frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)} \right)^2 \right)
$$

$$
= \wp^{-1} \left( -\xi - \eta + \frac{1}{4} \left( \frac{\sqrt{4\xi^3 - g_2\xi - g_3} - \sqrt{4\eta^3 - g_2\eta - g_3}}{\xi - \eta} \right) \right)^2 \right).
$$
Relation Between Weierstrass $\wp$ Function and Jacobian Elliptic Sine Function

- Let $4z^3 - g_2z - g_3 = 4(z - e_1)(z - e_2)(z - e_3)$ be the factorization so that
  
  $$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

- Let $\lambda$ be a complex number to be determined later so that
  
  $$w = e_3 + \frac{e_1 - e_3}{\text{sn}^2(\lambda z, k)}$$

satisfies the differential equation for the Weierstrass $\wp$ function, where the notation $\text{sn}(\lambda z, k)$ means that the modulus of the Jacobian elliptic sine function is $k$. 

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• We have

\[
\left( \frac{dw}{dz} \right)^2 = \left( \frac{2(e_1 - e_3)\lambda cn(\lambda z)dn(\lambda z)}{sn^3(\lambda z)} \right)^2
\]

\[
= \frac{4(e_1 - e_3)^2\lambda^2 cn^2(\lambda z)dn^2(\lambda z)}{sn^6(\lambda z)}
\]

\[
= \frac{4(e_1 - e_3)^2\lambda^2 (1 - sn^2(\lambda z)) (1 - k^2 sn^2(\lambda z))}{sn^6(\lambda z)}
\]

\[
= 4(e_1 - e_3)^2\lambda^2 \left( \frac{1}{sn^2(\lambda z)} - 1 \right) \left( \frac{1}{sn^2(\lambda z)} - k^2 \right) \frac{1}{sn^2(\lambda z)}
\]

\[
= 4 \frac{\lambda^2}{e_1 - e_3} \left( \frac{e_1 - e_2}{sn^2(\lambda z)} - (e_1 - e_3) \right) \left( \frac{e_1 - e_3}{sn^2(\lambda z)} - k^2(e_1 - e_3) \right) \frac{e_1 - e_2}{sn^2(\lambda z)}
\]

\[
= 4 \frac{\lambda^2}{e_1 - e_3} \left( (w - e_3) - (e_1 - e_3) \right) \left( (w - e_3) - k^2(e_1 - e_3) \right) (w - e_3).
\]

• Set \( \lambda^2 = e_1 - e_3 \) and \( k^2 = \frac{e_2 - e_3}{e_1 - e_3} \).
Then
\[
\left( \frac{dw}{dz} \right)^2 = 4(w - e_1)(w - e_2)(w - e_3)
\]
and
\[
\wp(z + C) = e_3 + \frac{e_1 - e_3}{\text{sn}^2(\lambda z, k)}
\]
for some constant \( C \).

Since
\[
\frac{e_1 - e_3}{\text{sn}^2(\lambda z, k)} \to \infty \quad \text{as} \quad z \to 0,
\]
it follows that \( \wp(C) = \infty \) and \( C \) is a period of \( \wp \) so that
\[
\wp(z + C) = \wp(z).
\]

The relation between the Weierstrass \( \wp \) function and the Jacobi elliptic sine function is given by
\[
\wp(z) = e_3 + \frac{e_1 - e_3}{\text{sn}^2(\sqrt{e_1 - e_3}z)}
\]
with the modulus \( k \) of the elliptic sine function equal to \( \sqrt{\frac{e_2 - e_3}{e_1 - e_3}} \).
Period Lattice of Weierstrass $\wp$ Function

- The Riemann surface $X$ of $\sqrt{4z^3 - g_2z - g_3}$ is a two-sheeted branched cover of the Riemann sphere $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$ with four branch points above $\infty$ and the three roots $e_1, e_2, e_3$ of $4z^3 - g_2z - g_3$.

- By using the loop of going around once the line-segment $[e_1, \infty]$, we know that
  \[
  \omega_1 := 2 \int_{\zeta=\infty}^{e_1} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}
  \]
  is a primitive period of $\wp(z)$.

- By replacing $e_1$ by $e_2$ or $e_3$, we can use any one of
  \[
  \omega_2 := 2 \int_{\zeta=\infty}^{e_2} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},
  \]
  \[
  \omega_3 := 2 \int_{\zeta=\infty}^{e_3} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}
  \]
  as a primitive period.
• The two integrals, around \([e_1, e_2]\) and \([e_3, \infty]\), of the nowhere zero holomorphic 1-form

\[
d\frac{z}{\sqrt{4z^3 - g_2 z - g_3}}
\]

on the Riemann surface \(X\) of \(\sqrt{4z^3 - g_2 z - g_3}\) generate the fundamental group of \(X\).

• This shows that \(\omega_2 - \omega_1\) and \(\omega_3\) are two primitive periods of \(\wp(z)\).

• Since \(\omega_1\) is a period, it follows that \(\omega_2, \omega_3\) are two primitive periods of \(\wp(z)\). As a matter of fact, any two of \(\omega_1, \omega_2, \omega_3\) are two primitive periods of \(\wp(z)\).
Zeros and Poles of $\wp'(z)$

- Since the only pole of $\wp(z)$ in a fundamental parallelogram is at double pole at 0, the only pole of $\wp'(z)$ in a fundamental parallelogram is a triple pole at 0.
- From
  \[
  \frac{\omega_j}{2} = \int_{\zeta=\infty}^{e_j} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}
  \]
  and the definition of $\wp$ as the inverse function of the indefinite integral
  \[
  w \mapsto \int_{\zeta=\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}
  \]
  it follows that $\wp \left( \frac{\omega_j}{2} \right) = e_j$ for $j = 1, 2, 3$. 
• Because of the differential equation

\[ \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \]

one knows that \( \frac{\omega_1}{2}, \frac{\omega_3}{2}, \frac{\omega_3}{2} \) are the roots of \( \wp' \).

• Note that by Property (iii) for a general elliptic function, the sum of the three roots of \( \wp' \) is equal to the sum of the three poles of \( \wp' \) modulo the period lattice of \( \wp' \). Thus \( \omega_1 + \omega_2 + \omega_3 = 0 \).

• Surprisingly unlike the situation with \( \wp'(z) \), there is no simple expression for the zeroes of \( \wp(z) \).
No Simple Expression for Zeros of $\wp(z)$

- Since $0$ is the only pole of $\wp(z)$ in the fundamental parallelogram and is a double pole, by Property (iii) for a general elliptic function the sum of the two roots of $\wp(z)$ in the fundamental parallelogram is a period. In particular, the two roots are $c$ and $-c$ for some $c$ in the fundamental parallelogram.

- From the definition of $w = \wp(z)$ as the inverse function of

$$w \mapsto \wp^{-1}(w) = \int_{\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},$$

we know that, modulo the period lattice $L$,

$$c = \int_{0}^{\infty} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}.$$
In general, the value of this integral cannot be expressed easily in terms of \( L, g_2, \) and \( g_3. \) For the special case of the lattice of periods \( L \) being equal to the set of all Gaussian integers \( \mathbb{Z} + \mathbb{Z}i, \) the value of the integral is simply \( \frac{1+i}{2} \) for the following reason.

From

\[
g\phi(z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{(z - m - ni)^2} - \frac{1}{(m + in)^2} \right)
\]

it follows that

\[
g\phi(iz) = \frac{1}{(iz)^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{(iz - m - ni)^2} - \frac{1}{(m + in)^2} \right)
\]

\[
= -\frac{1}{z^2} - \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{(z + mi - n)^2} - \frac{1}{(-mi + n)^2} \right)
\]

\[
= -g\phi(z).
\]
• Thus, if \( c \) is a zero of \( \wp(z) \), then \( ic \) is also a zero of \( \wp(z) \). By Property (iii) for a general elliptic function the sum \( c + ic \) of the coordinates of the two zeros \( c, ic \) is equal to the sum of the coordinates of the poles (with multiplicities counted) so that \( c + ic \) is period \( m + in \).

• Hence modulo the Gaussian integers \( \mathbb{Z} + \mathbb{Z}i \),

\[
c = \frac{m + in}{1 + i} = \frac{m - n}{1 + i} + n \equiv \frac{m - n}{1 + i} = \frac{(m - n)(1 - i)}{2} \equiv \frac{1 + i}{2}.
\]

• For a general Weierstrass \( \wp \) function, the only result is in the following 1982 paper of Eichler and Zagier.


which still involves the evaluation of a definite integral and does not give any simple expression of its two zeros (with one always the negative of the other).
The Weierstrass $\wp$ function

$$\wp(z) = \frac{1}{z^2} + \sum_{\ell \in L - \{0\}} \left( \frac{1}{(z - \ell)^2} - \frac{1}{\ell^2} \right)$$

for the lattice $L$

satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3$$

with

$$g_2 = 60 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4}$$

and

$$g_3 = 140 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^6}.$$
This means that \( w = \wp(z) \) is the inverse function of the indefinite integral

\[
w \mapsto z = \wp^{-1}(w) = \int_{\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}.
\]

This conclusion starts out with a lattice \( L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) which gives rise to the two complex coefficients \( g_2 \) and \( g_3 \).

Conversely, when one starts out with two complex coefficients \( g_2, g_3 \) with \( g_2^3 - 27g_3^2 \neq 0 \), one can use the argument of the Riemann surface for \( \sqrt{4z^3 - g_2z - g_3} \) and its fundamental group to recover the \( \wp(z) \) as the inverse function of the indefinite integral with the primitive periods

\[
2 \int_{e_1}^{e_2} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},
2 \int_{e_3}^{\infty} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},
\]

where \( 4z^3 - g_2z - g_3 = 4(z - e_1)(z - e_2)(z - e_3) \).
• Instead of using the method of the Riemann surface and its fundamental group, we now introduce another method of using the $j$-invariant to recover the lattice $L$ when $g_2, g_3$ are given with $g_2^3 - 27g_3^2 \neq 0$.

• The reason of introducing the $j$-invariant to solve the inversion problem of the indefinite integral is that the $j$-invariant itself opens up an important new area of modular functions and forms.

• We need to introduce first some notions and terminology.

• The modulus $k$ parametrizes the Jacobian elliptic function $sn(z, k)$, where we put the $k$ in to emphasize the dependence of $sn z$ on $k$ in its definition

$$sn^{-1}(w) = \int_{\zeta}^{w} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - k^2\zeta^2)}}.$$
• An alternative way to parametrize \( \text{sn}(z, k) \) is to use its period lattice \( L := \mathbb{Z}(4K) + \mathbb{Z}(2iK') \) with

\[
K = \int_{z=0}^{1} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}},
\]

\[
iK' = \int_{z=1}^{\frac{1}{k}} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.
\]

• Concerning the two functions \( k^2 \mapsto K \) and \( k^2 \mapsto K' \), we would like to make a side remark, which is already indicated in our homework assignment of September 17, 2020. When we expand the above two integrands as two power series of \( k^2 \), the coefficients which are definite integrals which can be easily evaluated explicitly and the two power series satisfy the hypergeometric differential equation

\[
z(1 - z) \frac{d^2 w}{dz^2} + (c - (a + b + 1)z) \frac{dw}{dz} - abw = 0
\]

in the special case \( a = b = \frac{1}{2} \) and \( c = 1 \).
• So $K$ and $K'$ are linked to $k^2$ in a differential relation as well as in an integral relation. In the book of Stein and Shakarchi the power series and the integral representation of the hypergeometric function $F(a, b, c; z)$ are mentioned respectively in Exercise 16(e) of Chapter 1 on p.28 and Exercise 9 of Chapter 6 on p.176.

• Now we come back to the period lattice $L$. Any pair of generators $(\omega_1, \omega_2)$ of $L$ over $\mathbb{Z}$ is mapped to a pair of generators by by any element of $SL(2, \mathbb{Z})$ which means the set of all $2 \times 2$ matrix with integer entries whose determinant is 1.

• The group $SL(2, \mathbb{Z})$, because of this link to the modulus (or parameter) of the elliptic sine function and the associated complex elliptic curve, is known as the modular group.

• The term modular group also is used to mean the quotient group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/ \pm I$ of $SL(2, \mathbb{Z})$ by its center $\{I, -I\}$, because elements of $SL(2, \mathbb{Z})$ acts on the quotient $\tau := \frac{\omega_2}{\omega_1}$ of the pair of generators $(\omega_1, \omega_2)$ as linear fractional transformations.
• Consider the action of the modular group $PSL(2, \mathbb{Z})$ as linear fractional transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + b}$$

on the upper half-plane $\mathbb{H}$.

• (When we use the quotient of $\tau := \frac{\omega_2}{\omega_1}$ of the pair of generators $(\omega_1, \omega_2)$, the imaginary part of $\tau$ must be zero due to the $\mathbb{R}$-linear independence of $\omega_1, \omega_2$ and we can choose to use $-\omega_2$ instead of $\omega_2$ to make sure that $\tau$ is in $\mathbb{H}$.)

• A holomorphic function $f$ on $\mathbb{H}$ is called a modular function if it is invariant under the modular group $PSL(2, \mathbb{Z})$, i.e.,

$$f \left( \frac{a\tau + b}{c\tau + d} \right) = f(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \quad \text{and} \quad \tau \in \mathbb{H}.$$
• In general, a holomorphic function \( f \) on \( \mathbb{H} \) is called a modular form of weight \( k \) if
\[
f \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^k f(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \quad \text{and} \quad \tau \in \mathbb{H}.
\]
and \( f \) is holomorphic as \( \tau \to i \infty \).

• (At this point we will use only the notion of a modular function and later we will explain more about the condition of a modular form \( f \) being holomorphic as \( \tau \to i \infty \).)

• The introduction of new notions and terminology is finished. We are returning now to the question of recovering a pair of generators \( \omega_1 \) and \( \omega_2 \) of a period lattice \( L \) from \( g_2 \) and \( g_3 \) given by
\[
g_2 = 60 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4} \quad \text{and} \quad g_3 = 140 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^6},
\]
by using a method other than using a biholomorphic map between the Riemann surface to a torus as done earlier.
• For given complex numbers \( a \) and \( b \) with \( a^3 - 27b^2 \neq 0 \), to solve the system of two equations

\[
g_2(\omega_1, \omega_2) = a \quad \text{and} \quad g_3(\omega_1, \omega_2) = b
\]

is equivalent to solving the system of two equations

\[
\frac{g_2(\omega_1, \omega_2)^3}{g_2(\omega_1, \omega_2)^3 - 27g_3(\omega_1, \omega_2)^2} = \frac{a^3}{a^3 - 27b^2} \quad \text{and} \quad \frac{g_2(\omega_1, \omega_2)}{g_3(\omega_1, \omega_2)} = \frac{a}{b}
\]

for the following reason.

• Clearly the first system implies the second system. Conversely, by taking the reciprocal of both sides of the first of the second system we obtain

\[
1 - 27 \frac{g_3(\omega_1, \omega_2)^2}{g_2(\omega_1, \omega_2)^3} = 1 - 27 \frac{b^2}{a^3}
\]

and

\[
\frac{g_3(\omega_1, \omega_2)^2}{g_2(\omega_1, \omega_2)^3} = \frac{b^2}{a^3}.
\]

• Moreover, from the second equation of the second system we have
Moreover, squaring both sides of the second equation of the second system we have

\[ \frac{g_3(\omega_1, \omega_2)^2}{g_2(\omega_1, \omega_2)^2} = \frac{a^2}{b^2} \]

which when multiplied by the preceding equation yields \( \frac{1}{g_2(\omega_1, \omega_2)} = \frac{1}{a} \) and \( g_2(\omega_1, \omega_2) = a \) so that from the second equation of the second system we obtain \( g_3(\omega_1, \omega_2) = b \).

With \( \tau = \frac{\omega_2}{\omega_1} \) we introduce the \( J \) function on \( \mathbb{H} \) defined by

\[ J(\tau) = \frac{g_2^3}{g_2^3 - 27g_3^2}, \]

where

\[ g_2 = 60 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4} \quad \text{and} \quad g_3 = 140 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^6} \]

with \( L = \mathbb{Z} + \mathbb{Z}\tau \).
• It turns out that amazingly all the Fourier coefficients of $1728J(\tau)$ are integers and the coefficient of $e^{2\pi in\tau}$ for positive $n$ is the dimension of the part of the “moonshine module” of grade $n$.

• It motivates the introduction of the $j$-invariant which is defined as $j(\tau) = 1728J(\tau)$.

• The problem of recovering $\omega_1, \omega_2$ from $g_2(\omega_1, \omega_2), g_3(\omega_1, \omega_2)$ is now reduced to solving the equation $J(\tau) = c$ for the unknown $\tau \in \mathbb{H}$ when $c \in \mathbb{C}$ is given. The reason is as follows.

• When $a, b \in \mathbb{C}$ with $a^3 - 27b^2 \neq 0$, we let $c = \frac{a}{b}$. If we can solve $J(\tau) = c = \frac{a}{b}$ for $\tau \in \mathbb{H}$, then we can form

$$g_2(1, \tau) = 60 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4} \quad \text{and} \quad g_3(1, \tau) = 140 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^6}$$

with $L = \mathbb{Z} + \mathbb{Z}\tau$ to define $\omega_1$ by

$$\frac{\omega_1^2 g_2(1, \tau)}{g_3(1, \tau)} = \frac{a}{b}$$

and define $\omega_2 = \tau \omega_1$. 

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Elliptic Functions (Approach of Weierstrass)
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• Since \( J(\tau) \) is invariant under \( PSL(2, \mathbb{Z}) \), to prove that the equation \( J(\tau) = c \) for the unknown \( \tau \in \mathbb{H} \) can be solved for any given \( c \in \mathbb{C} \), it suffices to solve for \( \tau \) in the quotient space \( \mathbb{H} / PSL(2, \mathbb{Z}) \), i.e., in a fundamental domain in \( \mathbb{H} \) (with respect to the group action of \( PSL(2, \mathbb{Z}) \) on \( \mathbb{H} \)).

• Because \( \tau \mapsto \tau + 1 \) is in \( PSL(2, \mathbb{Z}) \), to look for a fundamental domain, we can confine ourselves to the half-strip \( \{ -\frac{1}{2} \leq \Re \tau < \frac{1}{2} \} \) in \( \mathbb{H} \).

• Because \( \tau \mapsto -\frac{1}{\tau} \) is also in \( PSL(2, \mathbb{Z}) \), to look for a fundamental domain, we can confine ourselves to the exterior of the open unit disk \( \mathbb{C} \) in \( \mathbb{H} \).

• We introduce the domain

\[
\Omega = \left\{ \tau \in \mathbb{H} \left| -\frac{1}{2} \leq \Re \tau \leq 0, |\tau| \geq 1 \right. \right\} \cup \left\{ \tau \in \mathbb{H} \left| 0 < \Re \tau < \frac{1}{2}, |\tau| > 1 \right. \right\}.
\]
• The reason why in the definition of $\Omega$ we take great pains to specify which part of the boundary of $\Omega$ is included in $\Omega$ is to make sure that there is a unique solution $\tau$ for the equation $J(\tau) = c$ for any given $c \in \mathbb{C}$.

• When we define $\Omega$, we only know that $\Omega$ contains a fundamental domain in the sense that it is mapped onto the quotient $\mathbb{H} / PSL(2, \mathbb{Z})$ and we do not know yet that $\Omega$ is actually a fundamental domain in the sense that it is mapped one-one onto the quotient $\mathbb{H} / PSL(2, \mathbb{Z})$.

• To prove the solvability of $J(\tau) = c$ uniquely in $\Omega$ for any given $c \in \mathbb{C}$, we are going to apply the argument principle to the truncation $\Omega \cap \{\text{Im}\, \tau \leq s\}$ of the domain $\Omega$ for a sufficiently large $> 0$.

• with the cancellation of the most of the increase of the argument of $J(\tau)$ along the boundary of $\Omega \cap \{\text{Im}\, \tau \leq A\}$. The truncation is needed because the argument principle is only for a bounded domain.
Figure: Fundamental Domain of Modular Group
We are going to apply the argument principle to the function $J(\tau) - c$ on the domain $\Omega$. Our goal is to show that $J(\tau) - c$ has a root. Suppose the contrary.

Then we consider the change of the argument of $J(\tau) - c$ as one goes around the boundary of $\Omega$. Because of the symmetry relations $J(\tau + 1) = J(\tau)$ and $J\left(\frac{-1}{\tau}\right) = J(\tau)$ the change of the argument of $J(\tau) - c$ is zero along the boundary of $\Omega \cap \{\text{Im} \, \tau < s\}$ minus $\{|\text{Re} \, \tau| \leq \frac{1}{2}, \text{Im} \, \tau = s\}$.

So we are left with the calculation of the change of the argument of $J(\tau) - c$ along $\{|\text{Re} \, \tau| \leq \frac{1}{2}, \text{Im} \, \tau = s\}$ from right to left.