ELLiptic Functions (Approach of Weierstrass)

Three Basic Properties of General Elliptic Functions. Before we discuss the approach of Weierstrass to elliptic functions, we first look at some basic properties of doubly periodic meromorphic functions. These functions are called elliptic functions. The Jacobian elliptic functions we have seen and the Weierstrass elliptic functions we are introducing are special cases of these general elliptic functions. The three basic properties of general elliptic functions are as follows. We denote two primitive periods of an elliptic function $f$ by $\omega_1$ and $\omega_2$.

(i) The sum of the residues of the function inside a fundamental parallelogram is zero.

(ii) The number of zeroes of the function equals the number of poles inside a fundamental parallelogram.

(iii) Inside a fundamental parallelogram the sum of the coordinates of the zeroes equals the sum of the coordinates of the poles modulo a period.
To prove (i) we integrate $f(z)\, dz$ along the boundary of the fundamental parallelogram. By the residue theorem the integral is simply $2\pi i$ times the sum of the residues of $f$ inside the parallelogram. On the other hand the integral is zero, because, by the periodicity of $f$, the integral over $[a, a + \omega_1]$ equals the integral over $[a + \omega_2, a + \omega_1 + \omega_2]$ and the integral over $[a, a + \omega_2]$ equals the integral over $[a + \omega_1, a + \omega_1 + \omega_2]$.

Property (ii) follows from integrating
\[
\frac{1}{2\pi i} \frac{f'(z)}{f(z)} \, dz
\]
over the boundary of the fundamental parallelogram and from the argument principle. Again the integral along the boundary of the fundamental parallelogram because of cancellation on account of the periodicity of $f$.

The proof of Property (iii) is slightly more complicated. One integrates
\[
\frac{1}{2\pi i} \frac{zf'(z)}{f(z)} \, dz
\]
over the boundary of the fundamental parallelogram, but in this case the integral may not be zero, because
\[
\frac{1}{2\pi i} \int_{[a,a+\omega_1]} \frac{zf'(z)}{f(z)} \, dz - \frac{1}{2\pi i} \int_{[a+a+\omega_2]} \frac{zf'(z)}{f(z)} \, dz = \omega_1 \frac{1}{2\pi i} \int_{[a,a+\omega_2]} f'(z) \, dz.
\]
However,
\[
\frac{1}{2\pi i} \int_{[a,a+\omega_2]} f'(z) \, dz
\]
equals $\frac{1}{2\pi i}$ times the difference of the value of $\log f(z)$ at $a + \omega_2$ and at $a$ when $z$ runs along $[a,a+\omega_2]$. Since $f(z)$ has the same value at $a$ as at $a+\omega_2$, the difference of the value of $\log f(z)$ at $a + \omega_2$ and at $a$ when $z$ runs along $[a,a+\omega_2]$ must be $2\pi i$ times an integer. Therefore
\[
\frac{1}{2\pi i} \int_{[a,a+\omega_2]} \frac{f'(z)}{f(z)} \, dz
\]
is an integer and
\[
\frac{1}{2\pi i} \int_{[a+\omega_1,a+\omega_1+\omega_2]} \frac{zf'(z)}{f(z)} \, dz - \frac{1}{2\pi i} \int_{[a,a+\omega_2]} \frac{zf'(z)}{f(z)} \, dz
\]
is a period of $f$. Likewise
\[
\frac{1}{2\pi i} \int_{[a,a+\omega_1]} \frac{zf'(z)}{f(z)} \, dz - \frac{1}{2\pi i} \int_{[a+\omega_2,a+\omega_1+\omega_2]} \frac{zf'(z)}{f(z)} \, dz
\]
is also a period of $f$. This concludes the proof of (iii).

**Weierstrass Elliptic Functions to be Introduced as Infinite Sum of Partial Fractions.** Though a Weierstrass elliptic function is to be introduced as an infinite sum of partial fractions which is doubly periodic as a meromorphic function on $\mathbb{C}$, we will first discuss how the partial fractions in the infinite sum is motivated from the approach of Abel and Jacobi of introducing an elliptic function as the inverse function of an elliptic integral whose integrand is the reciprocal of the square root of a quartic polynomial. Our discussion of the motivation starts with the replacement of the quartic polynomial by a cubic polynomial.
**Reduction of Polynomial in Elliptic Integral from Quartic to Cubic.** The approach of Abel and Jacobi inverts the indefinite integral

\[ \int \frac{dz}{\sqrt{F(z)}}, \]

where \( F(z) \) is a quartic polynomial, especially of the form \((1 - z^2)(1 - k^2 z^2)\). Actually the case of \( \deg F(x) = 3 \) and the case of \( \deg F(x) = 4 \) are the same. Suppose \( F(x) \) is of degree 4, which without loss of generality can be written as \( \prod_{\nu=1}^{4} (x - \gamma_{\nu}) \) with all four distinct points \( \gamma_{\nu} \) of \( \mathbb{C} \). Apply the Möbius transformation

\[ z = \frac{a\zeta + b}{c\zeta + d} \]

with \( ad - bc \neq 0 \) to \( z \). Then

\[ dz = \frac{ad - bc}{(c\zeta + d)^2} \]

and

\[ \int \frac{d\zeta}{\sqrt{F(\zeta)}} = (ad - bc) \int \frac{d\zeta}{\sqrt{Q(\zeta)}} \]

with

\[ Q(t) = \prod_{\nu=1}^{4} ((a\zeta + b) - \gamma_{\nu}(c\zeta + d)). \]

We can choose \( a, b, c, d \) so that \( a - \gamma_1 c = 0 \). Then \( b - \gamma_1 d \neq 0 \) from \( ad - bc \neq 0 \). Then the degree of \( Q(\zeta) \) is 3 in \( \zeta \). Geometrically this means that the Möbius transformation

\[ z = \frac{a\zeta + b}{c\zeta + d} \]

maps the point \( \zeta = \infty \) to the point \( z = \gamma_1 \) and with respect to the \( \zeta \) coordinate our integral comes from a polynomial of degree 3. An affine variable change can reduce the polynomial to \( 4\zeta^3 - g_2\zeta - g_3 \).

**Elliptic Integral in Weierstrass Form and Principal Part of Weierstrass \( \wp \) Function at 0.** An indefinite elliptic integral of Weierstrass is of the form

\[ \int_{\zeta=\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}, \]
where $g_2, g_3$ are complex numbers and the three roots $e_1, e_2, e_3$ of the cubic polynomial $4\zeta^3 - g_2\zeta - g_3$ are distinct, which means that its discriminant

$$\prod_{1 \leq j < k \leq 3} (e_j - e_k)^2 = -16(g_2^3 - 27g_3^2)$$

is nonzero. The initial point of integration is chosen to be $\infty$, because it is one of the roots of the quartic polynomial and is a branching point of the Riemann surface.

The choice of $\infty$ as the initial point of integration means that the inverse function of the indefinite elliptic integral

(i) has a double pole at the origin and

(ii) is an even function.

We would like to see what the principal part at the origin should be chosen for the elliptic function so that its translates by the periods can be used in the infinite sum of partial fractions to define the Weierstrass elliptic function.

By the fundamental theorem of calculus, the elliptic function $\wp(z)$ defined as the inverse function of

$$z = \wp^{-1}(w) = \int_{\zeta=\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}$$

satisfies

$$\frac{dz}{dw} = \frac{1}{\sqrt{4w^3 - g_2w - g_3}}$$

which is the same as

$$\left(\frac{d\wp(z)}{dz}\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$ 

If the principal part of $\wp(z)$ at 0 is

$$\sum_{\nu=-\ell}^{-1} \frac{c_{-\nu}}{z^\nu},$$
equating the terms of most negative power on both sides yields
\[ \frac{\ell c_+^2}{z^{-2(\ell+1)}} = 4 \frac{c_{-\ell}^3}{z^{3\ell}}, \]
which implies \(2(\ell+1) = 3\ell\) and \(\ell^2 c_{-\ell}^2 = 4c_{-\ell}^3\), i.e., \(\ell = 2\), and \(c_{-2} = 1\). By the evenness of \(\wp(z)\) (from the initial point of integration being a branch-point of the Riemann surface), the principal part of \(\wp(z)\) at 0 must be \(\frac{1}{z^2}\).

**Rigorous Definition of Weierstrass \(\wp\) Function.** The above discussion involving the Riemann surface for the square root of a cubic polynomial with leading coefficient 4 and without second-highest degree term is just for the sake of giving the background motivation for choosing the principal part at the origin to be \(\frac{1}{z^2}\). The discussion helps to understand why the infinite sum of partial fractions to define the Weierstrass \(\wp\) function is chosen to be of that particular form, but it is completely unnecessary for the rigorous definition of \(\wp(z)\) which we now give. We can actually start our treatment of the Weierstrass elliptic function \(\wp(z)\) from this point and forget about all the preceding material concerning elliptic functions. Let \(\omega_1, \omega_2\) be two complex numbers which are \(\mathbb{R}\)-linearly independent. Denote by \(L\) the lattice \(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\) generated by \(\omega_1, \omega_2\).

The Weierstrass \(\wp(z)\) for the lattice \(L\) is defined as the infinite series
\[ \wp(z) = \frac{1}{z^2} + \sum_{n \in L - \{0\}} \left( \frac{1}{(z-\ell)^2} - \frac{1}{\ell^2} \right). \]

The reason for the infinite sum over \(\ell \in L\) is the need to end up with a function for which every element of \(L\) is a period. The term
\[ \frac{1}{(z-\ell)^2} - \frac{1}{\ell^2} \]
is used instead of just
\[ \frac{1}{(z-\ell)^2}, \]
because without subtracting \(\frac{1}{\ell^2}\) the infinite sum fails to converge for the following reason. One is not able to conclude that for \(z\) in a compact subset of \(\mathbb{C}\) the Cauchy partial sum
\[ \sum_{\ell \in L, \ p \leq |\ell| \leq q} \frac{1}{(z-\ell)^2} \]
to go to 0 as \( p, q \to \infty \), because it is comparable to
\[
\sum_{k=p}^{q} \left( \sum_{k-1 \leq |\ell| \leq k+1} \frac{1}{|\ell|^2} \right) \sim \sum_{k=p}^{q} \frac{1}{k^2} = \sum_{k=p}^{q} \frac{1}{k^2}.
\]

On the other hand,
\[
\sum_{\ell \in L, \ p \leq |\ell| \leq q} \left( \frac{1}{(z - \ell)^2} - \frac{1}{\ell^2} \right)
\]
is comparable to
\[
\sum_{k=p}^{q} \left( \sum_{k-1 \leq |\ell| \leq k+1} \frac{1}{|\ell|^3} \right) \sim \sum_{k=p}^{q} \frac{1}{k^3} = \sum_{k=p}^{q} \frac{1}{k^3},
\]
which goes to 0 as \( p, q \to \infty \).

Note that to determine the convergence of an infinite series of partial fractions uniformly on a compact subset \( K \) of \( \mathbb{C} \), one removes first the finite number of terms with poles in \( K \) so that the convergence question is only for an infinite series of holomorphic functions instead of meromorphic functions. An unexpected difficulty in the verification of the periodicity property of \( \wp(z) \) arises with the use of the term
\[
\frac{1}{(z - \ell)^2} - \frac{1}{\ell^2}
\]
for \( \ell \in L - \{0\} \), because the term for the case of \( \ell = 0 \) is different due to the difficulty of dividing 1 by 0. In fact, \( \wp(z) \) so defined still has period \( \ell \) for each \( \ell \in L \). One can verify this in one of the following ways. The first way differentiates the infinite series of \( \wp(z) \) term-by-term to get rid of \( -\frac{1}{2} \ell \) in each term for \( \ell \in L - \{0\} \) and then integrate back. The vanishing of the constant of integration is guaranteed by the evenness of \( \wp(z) \).

More precisely
\[
\wp'(z) = -2 \sum_{\ell \in L} \frac{1}{(z - \ell)^3}
\]
which is justified from the convergence of the right-hand side absolutely and uniformly on compact subsets of \( \mathbb{C} \). Now clearly
\[
\wp'(z + \ell) = \wp'(z)
\]
for $\ell \in L$. Integrating with respect to $z$ yields

$$\varphi(z + \ell) = \varphi(z) + C_\ell$$

for some constant $C_\ell$. Evaluating at $z = -\frac{\ell}{2}$ yields

$$\varphi\left(-\frac{\ell}{2}\right) = \varphi\left(-\frac{\ell}{2}\right) + C_\ell,$$

which from the evenness of $\varphi(z)$ implies that $C_\ell = 0$.

The second way of verifying the periodicity of $\varphi(z)$ is simply to rewrite the infinite sum as

$$\varphi(z) = \sum_{n_1 \in \mathbb{Z}} \frac{1}{(z - n_1 \omega_1)^2} - \sum_{n_1 \in \mathbb{Z} - \{0\}} \frac{1}{n_1^2 \omega_1^2}$$

$$+ \sum_{n_2 \in \mathbb{Z} - \{0\}} \sum_{n_1 \in \mathbb{Z}} \left( \frac{1}{(z - n_1 \omega_1 - n_2 \omega_2)^2} - \frac{1}{(n_1 \omega_1 - n_2 \omega_2)^2} \right)$$

which clearly satisfies $\varphi(z + \omega_1) = \varphi(z)$. For the same reason $\varphi(z + \omega_2) = \varphi(z)$ holds. The key point of this second way of verification is that the two series of single summation

$$\sum_{n_1 \in \mathbb{Z}} \frac{1}{(z - n_1 \omega_1)^2} \text{ and } \sum_{n_1 \in \mathbb{Z} - \{0\}} \frac{1}{n_1^2 \omega_1^2}$$

converge.

**Derivation of Differential Equation for $\varphi(z)$ by Eliminating Principal Part of Only Pole in Fundamental Parallelogram for Elliptic Function.**

We now derive the first-order differential equation for $\varphi(z)$ by eliminating the principal part of the only pole in a fundamental parallelogram for an elliptic function constructed as a polynomial of $\varphi(z)$ and $\varphi'(z)$. Since $\varphi'(z)$ as the derivative of an even function is odd, to work with even elliptic functions to minimize the number of terms in the principal part at the origin, we use $\varphi'(z)^2$ whose principal part at 0 is

$$\frac{4}{z^6} + \frac{a}{w^4} + \frac{b}{w^2}$$
for some \( a, b \in \mathbb{C} \) due to its evenness. To cancel the term
\[
\frac{4}{z^6}
\]
in the principal part at \( 0 \) with the most negative power, we should use \( 4\varphi(z)^3 \), but the process still leaves a principal part
\[
\frac{c}{z^4} + \frac{d}{z^2}
\]
for some \( c, d \in \mathbb{C} \). Thus, we know that
\[
\varphi'(z)^2 = 4\varphi(z)^3 + \alpha_2\varphi(x)^2 + \alpha_1\varphi(x) + \alpha_0
\]
for some constants \( \alpha_0, \alpha_1, \alpha_2 \). To determine the constants \( \alpha_0, \alpha_1, \alpha_2 \), we have to explicitly write down a few terms in the Laurent series expansion of \( \varphi(z) \) at \( 0 \).

**First Few Terms in Laurent Series Expansion of \( \varphi(z) \) at 0.** For \( n \geq 3 \)
Let
\[
s_n = \sum_{\ell \in L-\{0\}} \frac{1}{\ell^n}.
\]
Since
\[
\frac{1}{(z-\ell)^2} = \frac{1}{\ell^2 (1 - \frac{z}{\ell})^2} = \frac{1}{\ell^2} + \frac{2z}{\ell^3} + \frac{3z^2}{\ell^4} + \cdots \text{ for } |z| < |\ell|,
\]
it follows that
\[
\varphi(z) = z^{-2} + 3s_4z^{-2} + 5s_6z^4 + \cdots,
\]
\[
\varphi'(z) = -2z^{-3} + 6s_4z + 20s_6z^3 + \cdots,
\]
\[
\varphi'(z)^2 = 4z^{-6} - 24s_4z^{-2} - 80s_6 + \cdots,
\]
\[
\varphi(z)^3 = z^{-6} + 9s_4z^{-2} + 15s_6 + \cdots,
\]
and
\[
\varphi'(z)^2 - 4\varphi(z)^3 + 60s_4\varphi(z) = -140s_6 + \cdots.
\]
Thus we have the differential equation
\[
\varphi'(z)^2 = 4\varphi(z)^3 - g_2\varphi(z) - g_3.
\]
where
\[ g_2 = 60s_4 = 60 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4} \quad \text{and} \quad g_3 = 140s_6 = 140 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^6}. \]

From the differential equation it follows that \( w = \wp(z) \) is the inverse function of the indefinite integral
\[ w \mapsto z = \int_{\zeta = \infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}. \]

The inversion step of using the fundamental group of the Riemann surface of the denominator of the integrand has been replaced by the elimination of the principal part at 0 of a polynomial in the doubly periodic functions \( \wp(z) \) and \( \wp'(z) \) when \( \wp(z) \) is defined as the infinite sum of partial fractions.

**Addition Theorem for Weierstrass \( \wp \) Function.** The addition theorem for the Weierstrass \( \wp \) function is obtained by using Property (iii) of doubly periodic functions.

Let \( x = \wp(w) \) and \( y = \wp'(w) \) and, for some complex numbers \( a \neq 0 \) and \( b \) to be determined later, we consider the doubly periodic function \( y + ax + b \).

This doubly periodic function has a pole of order 3 at the origin and no other poles inside a fundamental parallelogram. So the sum of its three zeroes must be zero modulo a period.

We are free to choose \( a \) and \( b \). We can choose \( a \) and \( b \) so that the doubly periodic function \( y + ax + b \) vanishes at \( w_1 \) and \( w_2 \). Then \( y + ax + b \) must also vanish at \(- (w_1 + w_2)\).

On the other hand we have the equation
\[ y^2 = 4x^3 - g_2x - g_3. \]

So by solving the two equation
\[ y + ax + b = 0 \]
and
\[ y^2 = 4x^3 - g_2x - g_3, \]
we would get the values of \( x \) and \( y \) at \(- (w_1 + w_2)\). Since one equation is a linear equation and the second one is a cubic equation, we expect to get 3 solutions for \((x, y)\).
The other two solutions are the values of \((x, y)\) at \(w_1\) and \(w_2\). Knowing these two solutions makes getting the third solution very easy, because one can use the fact that for a cubic equation with unit leading coefficient the sum of the three roots is the negative of the second coefficient.

We now carry out the details to get our addition theorem. From

\[
\varphi'(w_1) + a\varphi(w_1) + b = 0
\]
\[
\varphi'(w_2) + a\varphi(w_2) + b = 0
\]

we get

\[
a = -\frac{\varphi'(w_1) - \varphi'(w_2)}{\varphi(w_1) - \varphi(w_2)}.
\]

(As we see later we do not need to solve for \(b\).) From the equation

\[
(ax + b)^2 = 4x^3 - g_2x - g_3
\]

we obtain (by the evenness of \(\varphi(w)\))

\[
\varphi(w_1) + \varphi(w_2) + \varphi(w_1 + w_2) = \frac{a^2}{4} = \frac{1}{4} \left( \frac{\varphi'(w_1) - \varphi'(w_2)}{\varphi(w_1) - \varphi(w_2)} \right)^2.
\]

Thus we have the addition formula

\[
\varphi(w_1 + w_2) = -\varphi(w_1) - \varphi(w_2) + \frac{1}{4} \left( \frac{\varphi'(w_1) - \varphi'(w_2)}{\varphi(w_1) - \varphi(w_2)} \right)^2.
\]

**Indefinite Integral Formulation of Addition Theorem for \(\varphi\).** The addition theorem for the Weierstrass \(\varphi\) function can be formulated in terms of indefinite integrals as

\[
\int_{\xi=\infty}^{\xi} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}} + \int_{\xi=\infty}^{\eta} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}
\]
\[
= \int_{\xi=\infty}^{-\xi-\eta+\frac{1}{2}} \left( \frac{\sqrt{4\eta^3 - g_2\eta - g_3} - \sqrt{4\xi^3 - g_2\xi - g_3}}{\xi - \eta} \right)^2 \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},
\]
because when we let $\xi = \varphi(w_1)$ and $\eta = \varphi(w_2)$, we have

$$
\varphi^{-1}(\xi) + \varphi^{-1}(\eta) = w_1 + w_2 = \varphi^{-1}(\varphi(w_1 + w_2))
$$

$$
= \varphi^{-1}
\left(-\varphi(w_1) - \varphi(w_2) + \frac{1}{4} \left( \frac{\varphi'(w_1) - \varphi'(w_2)}{\varphi(w_1) - \varphi(w_2)} \right)^2 \right)
$$

$$
= \varphi^{-1}
\left(-\xi - \eta + \frac{1}{4} \left( \frac{\sqrt{4\xi^3 - g_2\xi - g_3} - \sqrt{4\eta^3 - g_2\eta - g_3}}{\xi - \eta} \right)^2 \right).
$$

TO BE CONTINUED ...