ELLIPTIC FUNCTIONS (Approach of Weierstrass)

We now discuss the approach of Weierstrass to elliptic functions. In contrast to Jacobi's approach of inverting an indefinite integral whose denominator is the root of a quartic polynomial (which is motivated by deriving the exact solution of the motion of the simple pendulum), Weierstrass starts with an infinite series of modified principal parts to define a doubly periodic global meromorphic function (called the Weierstrass $\wp$ function) on $\mathbb{C}$ with the only principal part $\frac{1}{z^2}$ in a fundamental paralellogram). From the differentiation formula for the Weierstrass $\wp$ function one can conclude that the the Weierstrass $\wp$ function is the inverse of an indefinite integral (with ∞ as the initial point of integration) whose denominator is the square root of a cubic polynomial. In other words, one can transformthe Jacobian elliptic sine function to the Weierstrass $\wp$ function by applying a Möbius transformation to the integrand of the indefinite integral to send one of the roots of the quartic polynomial to ∞ and by changing the initial point of integration from the origin to ∞.

One advantage of the approach of Weierstrass is a geometric interpretation (and as a consequence a simple natural proof) of the addition formula for the Weierstrass $\wp$ function by using the three points of intersection between a cubic curve and a line in the complex projective plane $\mathbb{P}_2$. For this interpretation and proof of the additional formula the third of the three basic properties of a general elliptic function plays a key role. As the starting point of our discussion, we introduce the notion of a general elliptic function and its three basic properties.

Three Basic Properties of General Elliptic Functions. Before we discuss the approach of Weierstrass to elliptic functions, we first look at some basic properties of doubly periodic meromorphic functions. These functions are called *elliptic functions*. The Jacobian elliptic functions we have seen and the Weierstrass elliptic functions we are introducing are special cases of these *general* elliptic functions. The three basic properties of of general elliptic functions are as follows. We denote two primitive periods of an elliptic function f by ω_1 and ω_2.

(i) The sum of the residues of the function inside a fundamental parallelogram is zero.

(ii) The number of zeroes of the function equals the number of poles inside a fundamental parallelogram.

(iii) Inside a fundamental parallelogram the sum of the coordinates of the zeroes equals the sum of the coordinates of the poles modulo a period.

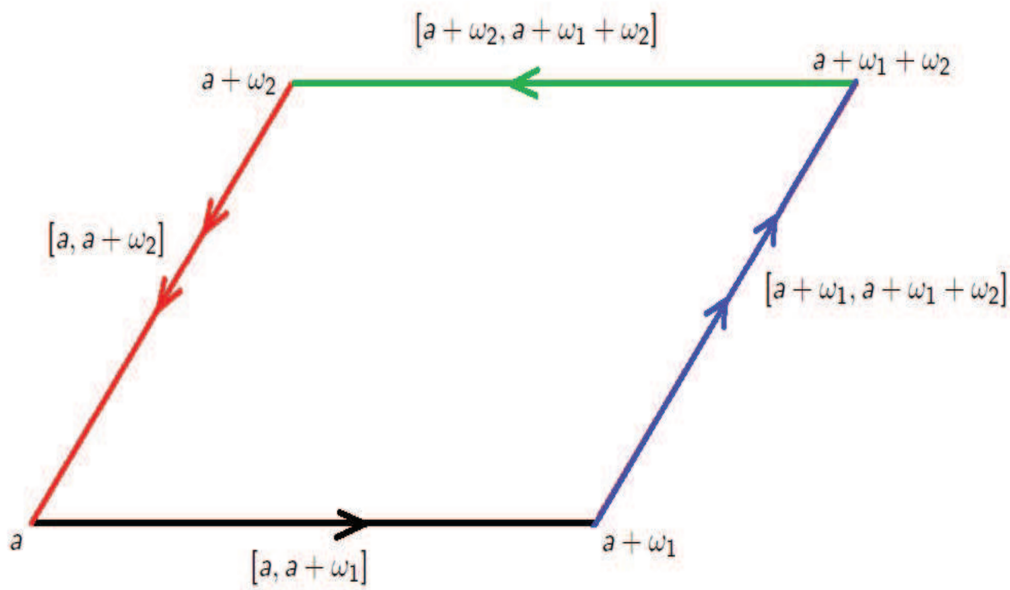

Figure 1: Fundamental Parallelogram

To prove (i) we integrate $f(z)\,dz$ along the boundary of the fundamental parallelogram. By the residue theorem the integral is simply $2\pi i$ times the sum of the residues of f inside the parallelogram. On the other hand the integral is zero, because, by the periodicity of f, the integral over $[a, a+\omega_1]$ equals the integral over $[a+\omega_2, a+\omega_1+\omega_2]$ and the integral over $[a, a+\omega_2]$ equals the integral over $[a+\omega_1, a+\omega_1+\omega_2]$.

Property (ii) follows from integrating

$$\frac{1}{2\pi i}\frac{f'(z)}{f(z)}\,dz$$

over the boundary of the fundamental parallelogram and from the argument principle. Again the integral along the boundary of the fundamental parallelogram because of cancellation on account of the periodicity of f.

The proof of Property (iii) is slightly more complicated. One integrates

$$\frac{1}{2\pi i}\frac{zf'(z)}{f(z)}\,dz$$

over the boundary of the fundamental parallelogram, but in this case the integral may not be zero, because

$$\begin{aligned}&\frac{1}{2\pi i}\int_{[a+\omega_1,a+\omega_1+\omega_2]}\frac{zf'(z)}{f(z)}dz-\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{zf'(z)}{f(z)}dz\\&=\omega_1\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{f'(z)}{f(z)}dz.\end{aligned}$$

However,

$$\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{f'(z)}{f(z)}dz$$

equals $\frac{1}{2\pi i}$ times the difference of the value of $\log f(z)$ at $a+\omega_2$ and at a when z runs along $[a, a+\omega_2]$. Since $f(z)$ has the same value at a as at $a+\omega_2$, the difference of the value of $\log f(z)$ at $a+\omega_2$ and at a when z runs along $[a, a+\omega_2]$ must be $2\pi i$ times an integer. Therefore

$$\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{f'(z)}{f(z)}dz$$

is an integer and

$$\frac{1}{2\pi i}\int_{[a+\omega_1,a+\omega_1+\omega_2]}\frac{zf'(z)}{f(z)}dz-\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{zf'(z)}{f(z)}dz$$

is a period of f. Likewise

$$\frac{1}{2\pi i}\int_{[a,a+\omega_1]}\frac{zf'(z)}{f(z)}dz-\frac{1}{2\pi i}\int_{[a+\omega_2,a+\omega_1+\omega_2]}\frac{zf'(z)}{f(z)}dz$$

is also a period of f. This concludes the proof of (iii).

Remarks on Relation Between 3 Fundamental Properties and Theorems of Riemann-Roch and Abel for Compact Riemann Surfaces of Genus 1. As we see later, Property (i) above (of the vanishing of the sum of all residues) is actually a necessary and sufficient condition for a collection of principal parts on a compact Riemann surface of genus 1 to be achieved by a meromorphic function. For a general compact Riemann surface, it evolves into the theorem of Riemann-Roch which, at the level of counting the dimensions of vector spaces, gives a necessary and sufficient condition for a collection of principal parts to be achieved by a meromorphic function. As we see later, Property (ii) and Property (iii) (of the sum of the coordinates of zeroes being equal to the sum of the coordinates of poles modulo the period lattice) is a actually a necessary and sufficient condition for given zero-set and pole-set to be achieved by a meromorphic function. For a general compact Riemann surface, it evolves into the theorem of Abel which gives a necessary and sufficient condition for given zero-set and pole-set to be achieved by a meromorphic function. In the course we will not discuss the theorems of Riemann-Roch and Abel for general compact Riemann surfaces.

Reduction of Polynomial in Elliptic Integral from Quartic to Cubic. As mentioned before, in the approach of Jacobi using Abel's inversion, the indefinite integral

$$\int\frac{dz}{\sqrt{P_4(z)}},$$

where $P_4(z)$ is a quartic polynomial with four distinct finite roots, especially of the form $(1-z^2)(1-k^2z^2)$, can be transformed to the indefinite integral

$$\int\frac{dz}{\sqrt{P_3(z)}},$$

where $P_3(z)$ is a cubic polynomial with three distinct finite roots, by using a Möbius transformation as follows.

Let the four distinct finite roots of the quartic polynomial $P_4(x)$ be γ_ν (for $1 \leq \nu \leq 4$). Apply the Möbius transformation

$$z = \frac{a\zeta + b}{c\zeta + d}$$

(with $ad - bc \neq 0$) to z. Then

$$dz = \frac{ad - bc}{(c\zeta + d)^2}$$

and

$$\int \frac{d\zeta}{\sqrt{P_4(\zeta)}} = (ad - bc) \int \frac{d\zeta}{\sqrt{Q(\zeta)}}\, d\zeta$$

with

$$Q(t) = \prod_{\nu=1}^{4} ((a\zeta + b) - \gamma_\nu(c\zeta + d)).$$

We can choose a, b, c, d so that $a - \gamma_1 c = 0$. Then $b - \gamma_1 d \neq 0$ from $ad - bc \neq 0$. The degree of $Q(\zeta)$ is 3 in ζ. Geometrically this means that the Möbius transformation

$$z = \frac{a\zeta + b}{c\zeta + d}$$

maps the point $\zeta = \infty$ to the point $z = \gamma_1$ and with respect to the ζ coordinate our integral comes from a polynomial of degree 3. An affine variable change can reduce the polynomial to $4\zeta^3 - g_2\zeta - g_3$, where g_2, g_3 are complex constants.

Let the three roots of the cubic polynomial $4\zeta^3 - g_2\zeta - g_3$ be e_1, e_2, e_3. The discriminant of the cubic polynomial $4\zeta^3 - g_2\zeta - g_3$ is

$$\prod_{1 \leq j < k \leq 3} (e_j - e_k)^2 = -16(g_2^3 - 27g_3^2).$$

The condition for the the three roots e_1, e_2, e_3 of the cubic polynomial $4\zeta^3 - g_2\zeta - g_3$ to be distinct is the same as $g_2^3 - 27g_3^2 \neq 0$.

Advantages of Weierstrass Approach of Cubic Polynomial over Jacobi Approach of Quartic Polynomial. Jacobi's approach of using a quartic polynomial is its parallelism with the theory of trigonometric functions so that in using Jacobi's elliptic functions one has a rough guide from the well-known properties of the trigonometric functions. Weierstrass's approach of using a cubic polynomial has a number of advantages from the viewpoint of pure mathematics.

The first one is that the Riemann surface constructed from Weierstrass of using a cubic polynomial can be algebro-geometrically identified with a smooth complex curve of degree 3 in the complex projective plane $\mathbb{P}_2$.

The second one is that in Weierstrass's approach the addition theorem can be proved by a simple and intuitive algebro-geometrical argument without complicated arguments with formulas obtained by differentiation. We now explain the first advantage of identifying the Riemann surface with a smooth cubic complex curve in $\mathbb{P}_2$ by presenting Hurwitz's method of using Euler numbers to compute the genus of a smooth complex curve in the complex projective plane. Then we use a "light-source projection" in $\mathbb{P}_2$ to identity explicitly the Riemann surface with a nonsingular cubic curve in $\mathbb{P}_2$.

Hurwitz's Method of Determining Genus of Nonsingular Complex Curve in the Complex Projective Plane. Let $F(\zeta_0, \zeta_1, \zeta_2) = 0$ a homogeneous polynomial of degree m defines a nonsingular complex curve C in $\mathbb{P}_2$. It means that the gradient

$$\left(\frac{\partial F}{\partial \zeta_0}, \frac{\partial F}{\partial \zeta_1}, \frac{\partial F}{\partial \zeta_2}\right)$$

and F vanish at the same time only at the origin $(\zeta_0, \zeta_1, \zeta_2)$ of $\mathbb{C}^3$. After a linear change of the homogeneous coordinates, we can assume that the infinite line $\{\zeta_0 = 0\}$ intersects C transversely precisely at m points and the point $[0, 0, 1]$ is not on C. In the affine coordinates

$$(z_1, z_2) = \left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}\right)$$

of $\mathbb{C}^2$, the affine part C_0 of C is defined by the vanishing of $f(z_1, z_2) = F(1, z_1, z_2)$. In the light-source projection $\pi : C \to \mathbb{P}_1$ with light-source $[0, 0, 1]$ and target $\mathbb{P}_1 = \{\zeta_1 = 0\}$, the number of branching points with multiplicities counted is the product $m(m-1)$ of the degrees of f and $\frac{\partial f}{\partial z_2}$. Use the triangulations of C and $\mathbb{P}_1$ compatible with the π such that all the branching points in C are vertices. Let g be the genus of C. The Euler number of C is $2-2g$, which is equal to the number of vertices minus the number of sides plus the number of faces of the triangulation of C. By comparing the numbers of vertices, sides, and faces of the triangulations of C and $\mathbb{P}_1$ and using the fact that the Euler number of $\mathbb{P}_1$ is 2, we get $2m = 2-2g+m(m-1)$ or

$$2g = 2-2m+m(m-1) = m^2-3m+2 = (m-1)(m-2)$$

so that

$$g = \frac{(m-1)(m-2)}{2}.$$

The genus g is 1 for $m=3$. The genus g is 3 for $m=4$.

Riemann Surface Defined by Graph of Equation. Consider the case of the Weierstrass formulation of using the polynomial $z_2^2 = f(z_1)$, where

$$f(z_1) = \alpha(z_1-a_1)(z_1-a_2)(z_1-a_3)$$

(with $\alpha \neq$) is a cubic polynomial with three finite distinct roots a_1, a_2, a_3, for example,

$$f(z_1) = 4z_1^3 - g_2 z_1 - g_3,$$

where g_2, g_3 are complex numbers with $g_2^3 - 27g_3^2 \neq 0$. In $\mathbb{P}_2$ with homogeneous coordinates $[\zeta_0, \zeta_1, \zeta_2]$ we use the homogeneous polynomial

$$F(\zeta_0,\zeta_1,\zeta_2) = \zeta_0\zeta_2^2 - \alpha(\zeta_1-a_1\zeta_0)(\zeta_1-a_2\zeta_0)(\zeta_1-a_3\zeta_0)$$

of degree 3 to define the complex curve, which one straightforwardly checks that the origin of $\mathbb{C}^3$ is the only common zero of $F(\zeta_0,\zeta_1,\zeta_2)$ and its gradient in ζ_0,ζ_1,ζ_2. The point $[\zeta_0,\zeta_1,\zeta_2] = [0,0,1]$ is on C. Use the point $[\zeta_0,\zeta_1,\zeta_2] = [0,0,1]$ of C as the light-source and the complex line $\{\zeta_2 = 0\}$ as the target $\mathbb{P}_1 =$ for the projection $\pi : C \to \mathbb{P}_1$. In the affine part $\mathbb{C}^2$ of $\mathbb{P}_2$ with affine coordinates $(z_1,z_2) = \left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_1}{\zeta_0}\right)$, the projection π is the same as the projection map $(z_1,z_2) \mapsto z_1$ from $C \cap \{\zeta_0 \neq 0\}$ to $\mathbb{C}$. The only difference between π and its restriction to the affine part is that π maps the light-source point to ∞ in the target $\mathbb{P}_1$. The intersection of

$$F(\zeta_0,\zeta_1,\zeta_2) = \zeta_0\zeta_2^2 - \alpha(\zeta_1-a_1\zeta_0)(\zeta_1-a_2\zeta_0)(\zeta_1-a_3\zeta_0) = 0$$

and $\zeta_0 =$ is given by $\zeta_1^3 = 0$. From the fact that the light-source point $[\zeta_0, \zeta_1, \zeta_2] = [0, 0, 1]$ itself is already on C we know that every complex line in $\mathbb{P}_2$ whose intersection with the target $\mathbb{P}_1 = \{\zeta_1 = 0\}$ intersects C at the light-source point. The triple intersection of the infinity line with C at the light-source point means that the light-source point is a branch point of multiplicity 2 for the map $\pi : C \to \mathbb{P}_1$ whose image is the infinity point of $\mathbb{P}_1$. In the affine part, from the implicit function theorem to solve for z_2 in terms of z_1 in $z_2^2 = f(z_1)$ the map $\pi : C \to \mathbb{P}_1$ has precisely three branch points in C, each of which is of multiplicity 2 and is mapped to a root of the cubic polynomial $f(z_1)$. This description tells us that the Riemann surface in the Weierstrass formulation is actually the nonsingular complex curve in $\mathbb{P}_2$ whose affine part is defined by $z_2^2 = 4z_1^3 - g_2 z_1 - g_3$. On the other hand, for the Jacobian formulation, Hurwitz's method of computing the genus of a nonsingular complex curve in $\mathbb{P}_2$ tells us that we cannot describe the Riemann surface in the Jacobian formulation by using a nonsingular complex curve in $\mathbb{P}_2$ of degree 4.

We now describe the elliptic function in the Weierstrass approach which is obtained in a way analogous to the construction of the Jacobian elliptic sine function by Abel's inversion.

Weierstrass $\wp$ Function as Inverse Function of Elliptic Integral Defined by Cubic Polynomial. As in the case of the Jacobian elliptic sine function $w = \operatorname{sn} z$ which is defined by

$$\operatorname{sn}^{-1} w = \int_{\zeta=0}^{w} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$

with the point $\zeta = 0$ and $\sqrt{(1-\zeta^2)(1-k^2\zeta^2)} = 1$ as the initial point of integration, we defined the Weierstrass $\wp$ function $w = \wp(z)$ as the inverse of indefinite elliptic integral

$$\wp^{-1}(w) = \int_{\zeta=\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},$$

where g_2, g_3 are complex numbers with $g_2^3 - 27g_3^2 \neq 0$ and the point $\zeta = \infty$ as the initial point of integration. From the argument of Abel's inversion, we know that the Weierstrass $\wp$ function is an elliptic function (*i.e.,* a doubly periodic meromorphic function on $\mathbb{C}$). The choice of ∞ as the initial point of integration means that the inverse function of the indefinite elliptic integral

(i) has a double pole at the origin and no other pole in the fundamental parallelogram, and

(ii) is an even function.

Just like the computation of K and K' by definite integrals

$$K = \int_{\zeta=0}^{1} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}},$$
$$K' = \int_{\zeta=1}^{\frac{1}{k}} \frac{d\zeta}{\sqrt{(\zeta^2-1)(1-k^2\zeta^2)}}$$

involving k for the Jacobian elliptic sine function, the two primitive periods ω_1, ω_2 of the elliptic function $w = \wp(z)$ can be computed by definite integrals involving the three distinct finite roots e_1, e_2, e_3 of the cubic polynomial $4z^3 - g_2 z - g_3$. For example,

$$\omega_1 = 2\int_{\zeta=e_1}^{e_2} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},$$
$$\omega_2 = 2\int_{\zeta=e_3}^{\infty} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}.$$

Principal Part of Weierstrass $\wp$ Function at the Origin. We would like to see what the principal part at the origin should be chosen for the elliptic function so that its translates by the periods can be used in an expansion of the Weierstrass $\wp$ function as an infinite sum of partial fractions.

By the fundamental theorem of calculus, the elliptic function $\wp(z)$ defined as the inverse function of

$$z = \wp^{-1}(w) = \int_{\zeta=\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}$$

satisfies

$$\frac{dz}{dw} = \frac{1}{\sqrt{4w^3 - g_2 w - g_3}}$$

which is the same as

$$\left(\frac{d\wp(z)}{dz}\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

If the principal part of $\wp(z)$ at 0 is

$$\sum_{\nu=-\ell}^{-1} \frac{c_{-\nu}}{z^{\nu}},$$

equating the terms of most negative power on both sides yields

$$\frac{\ell^2 c_{-\ell}^2}{z^{2(\ell+1)}} = 4\frac{c_{-\ell}^3}{z^{3\ell}},$$

which implies $2(\ell+1) = 3\ell$ and $\ell^2 c_{-\ell}^2 = 4c_{-\ell}^3$, *i.e.*, $\ell = 2$, and $c_{-2} = 1$. By the evenness of $\wp(z)$ (from the initial point of integration being a branch-point of the Riemann surface), the principal part of $\wp(z)$ at 0 must be $\frac{1}{z^2}$.

Partial Fraction Expansion of Weierstrass $\wp$ Function. We can attempt to get a partial fraction expansion of the Weierstrass $\wp$ function by taking the sum of all the principal parts

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} \frac{1}{\left(z - (n_1\omega_1 + n_2\omega_2)\right)^2}.$$

Unfortunately, such a sum does not converge uniformly on compact subsets (after a removal of a finite number of terms).

In our review of basic complex analysis, we discussed the problem of partial fraction expansion, in the case of a meromorphic function on $\mathbb{C}$ with simple poles and a polynomial growth order on a sequence of contours going out to infinity. By increasing the growth order of the Cauchy kernel and using the technique of modifying each principal part by its Taylor polynomial of an appropriate degree at the origin, we can express such a global meromorphic function on $\mathbb{C}$ in terms of its modified principal parts. Such a method works also for double poles instead of simple poles.

As a global meromorphic function with double poles on $\mathbb{C}$ with two $\mathbb{R}$-linearly independent primitive periods, the Weierstrass $\wp$ function has growth order $o(R_n^{p+1})$ with $p = 0$ on a sequence of rectangles C_n with distance $R_n \to \infty$. To get an infinite-sum expansion of the Weierstrass $\wp$ function, we can use the technique of modifying each principal part by its Taylor polynomial of an appropriate degree at the origin, in order to guarantee convergence of the infinite sum of its principal parts. Instead of specifying the degree of the Taylor polynomial at the origin we should use, we can start with degree zero and then test convergence to increase the degree if necessary. The case of Taylor polynomial of degree zero means that we should try

$$\frac{1}{z^2}+\sum_{(n_1,n_2)\in\mathbb{Z}^2-(0,0)}\left(\frac{1}{\left(z-(n_1\omega_1+n_2\omega_2)\right)^2}-\frac{1}{(n_1\omega_1+n_2\omega_2)^2}\right),$$

where we treat the principal part at 0 differently because the Taylor polynomial at 0 is meaningless for a function with a pole at 0 and, to test the convergence of an infinite series, we can always ignore a finite number of terms in the infinite series.

It turns out that it suffices to use Taylor polynomials of degree 0. To simplify notations, let L can be the period lattice $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$. The series whose convergence is being tested is now written in the simpler form

$$\frac{1}{z^2}+\sum_{\ell\in L-\{0\}}\left(\frac{1}{(z-\ell)^2}-\frac{1}{\ell^2}\right).$$

Its convergence can be argued by considering the infinite sum as the limit of the sequence of partial sums

$$\sum_{\ell\in L-\{0\},|\ell|\leq p}\left(\frac{1}{(z-\ell)^2}-\frac{1}{\ell^2}\right)$$

as $p\to\infty$ and using the Cauchy criterion for the convergence of a sequence as follows. The difference between the q-th and the p-th partial sums

$$\sum_{\substack{\ell\in L,\\ p\leq|\ell|\leq q}}\left(\frac{1}{(z-\ell)^2}-\frac{1}{\ell^2}\right)$$

is comparable to

$$\sum_{k=p}^{q}\left(\sum_{k-1\leq|\ell|\leq k+1}\frac{1}{|\ell|^3}\right)\sim\sum_{k=p}^{q}k\frac{1}{k^3}=\sum_{k=p}^{q}\frac{1}{k^2},$$

which goes to 0 as $p,q\to\infty$.

We have to argue that the meromorphic function

$$F(z) = \frac{1}{z^2} + \sum_{\ell \in L - \{0\}} \left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2} \right)$$

on $\mathbb{C}$ is in fact equal to the Weierstrass $\wp$ function $\wp(z)$. First, we have to check that $F(z)$ is periodic with period lattice L so that the two functions $F(z)$ and $\wp(z)$ having the same principal parts at points of L must be constant. Then we have to check that the constant $F(z) - \wp(z)$ is in fact 0.

Because the term $\frac{1}{z^2}$ corresponding to $\ell = 0$ is different from the term

$$\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2}$$

for $\ell \in L - \{0\}$, it is by no means clear that L is a period lattice for $F(z)$. One way to handle the problem is to get rid of $-\frac{1}{\ell^2}$ in each term for $\ell \in L - \{0\}$ by differentiating and then integrating back. The equation

$$F'(z) = -2 \sum_{\ell \in L} \frac{1}{(z-\ell)^3}$$

from term-by-term differentiation is justified from the convergence of the right-hand side absolutely and uniformly on compact subsets of $\mathbb{C}$. Now clearly

$$F'(z+\ell) = F'(z)$$

for $\ell \in L$. Integrating with respect to z yields

$$F(z+\ell) = F(z) + C_\ell$$

for some constant C_ℓ. Since $F(z)$ defined by the infinite series is clearly an even function, evaluating at $z = -\frac{\ell}{2}$ yields

$$F\left(\frac{\ell}{2}\right) = F\left(-\frac{\ell}{2}\right) + C_\ell$$

which from the evenness of $F(z)$ implies that $C_\ell = 0$. Thus, we know that $F(z)$ is periodic. Because $F(z)$ and $\wp(z)$ have the same principal parts and the same period lattice, we conclude that $F(z)$ and $\wp(z)$ differ at most by a constant. Instead of verifying the constant to be zero, we going to show by another method that $F(z)$ and $\wp(z)$ agree.

For Abel's inversion, we compare the fundamental groups of a covering space and its base. When Weierstrass developed his theory, he actually avoided the use of Abel's inversion and relied only on purely analytic arguments. We now use Weierstrass purely analytic approach to show that $F(z)$ agrees with $\wp(z)$ by showing $F(z)$ satisfies the same differential equation

$$\left(\frac{dF(z)}{dz}\right)^2 = 4F(z)^3 - g_2F(z) - g_3.$$

Then $w = F(z)$ must be equal to the inverse of

$$z = \int_{\zeta=w_0}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}$$

for some initial point of integration w_0 which must be a value of $F(0)$. Since we know that 0 is a pole of $w = F(z)$, the initial point of integration w_0 must be infinity, which means that $F(z)$ agrees with $\wp(z)$.

Derivation of Differential Equation by Infinite Series of Principal Parts by Eliminating the Principal Part at the Only Pole in Fundamental Parallelogram. We now derive the first-order differential equation for

$$F(z) = \frac{1}{z^2} + \sum_{\ell \in L - \{0\}} \left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2}\right)$$

by using the technique of eliminating the principal part of the only pole in a fundamental parallelogram for a polynomial of $F(z)$ and $F'(z)$. The derivative $F'(z)$ of the even function $F(z)$ is odd. In order to work only with even elliptic functions (to minimize the number of terms in the principal part at the origin), we use $F'(z)^2$ whose principal part at 0 is

$$\frac{4}{z^6} + \frac{a}{w^4} + \frac{b}{w^2}$$

for some $a, b \in \mathbb{C}$ due to its evenness. To cancel the term

$$\frac{4}{z^6}$$

in the principal part at 0 with the most negative power, we should use $4F(z)^3$, but the process still leaves a principal part

$$\frac{c}{z^4} + \frac{d}{z^2}$$

for some $c, d \in \mathbb{C}$. Thus, we know that

$$F'(z)^2 = 4F^3 + \alpha_2 F^2 + \alpha_1 F(x) + \alpha_0$$

for some constants $\alpha_0, \alpha_1, \alpha_2$. To determine the constants $\alpha_0, \alpha_1, \alpha_2$, we have to explicitly write down a few terms in the Laurent series expansion of $F(z)$ at 0.

Equating Terms of Nonpositive Powers in Laurent Series Expansion at 0. For $n \geq 3$ Let

$$s_n = \sum_{\ell \in L - \{0\}} \frac{1}{\ell^n}.$$

Since

$$\frac{1}{(z-\ell)^2} = \frac{1}{\ell^2 \left(1 - \frac{z}{\ell}\right)^2} = \frac{1}{\ell^2} + 2\frac{z}{\ell^3} + 3\frac{z^2}{\ell^4} + \cdots \text{ for } |z| < |\ell|,$$

it follows from straightforward computation that

$$\begin{aligned} F(z) &= \frac{1}{z^2} + 3s_4 z^2 + 5s_6 z^4 + \cdots, \\ F'(z) &= -\frac{2}{z^3} + 6s_4 z + 20 s_6 z^3 + \cdots, \\ F'(z)^2 &= \frac{4}{z^6} - \frac{24 s_4}{z^2} - 80 s_6 + \cdots, \\ F(z)^3 &= \frac{1}{z^6} + \frac{9s_4}{z^2} + 15 s_6 + \cdots, \end{aligned}$$

and

$$F'(z)^2 - 4F(z)^3 + 60 s_4 F(z) = -140 s_6 + \cdots.$$

Thus we have the differential equation

$$F'(z)^2 = 4F(z)^3 - g_2 F(z) - g_3,$$

where

$$g_2 = 60 s_4 = 60 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4} \quad \text{and} \quad g_3 = 140 s_6 = 140 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^6}.$$

We now can conclude that $w = F(z)$ agrees with $w = \wp(z)$, which is defined as the inverse function of the indefinite integral

$$w \mapsto z = \int_{\zeta=\infty}^{w} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}.$$

An important bonus of this argument is the two relations between the period lattice L and the coefficients g_2 and g_3 of the cubic polynomial $4z^3 - g_2 z - g_3$.

$$g_2 = 60 s_4 = 60 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4} \quad \text{and} \quad g_3 = 140 s_6 = 140 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^6}.$$

It is interesting to note that at the beginning of our discussion on elliptic functions, when we compute the actual period of a simple pendulum (with the constants a and g normalized to be 1), the period $4K$ is given by

$$4K = 2\pi \left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots \right),$$

which is a relation between one primitive period $4K$ and the coefficient k in the quartic polynomial $(1 - z^2)(1 - k^2 z^2)$. This relation between K and k is actually a hypergeometric series which satisfies a second-order differential equation called the Gauss hypergeometric differential equation. We will have a homework problem about it. Later we will discuss more about the relation between g_2, g_3 and the lattice L.

As the second advantage of Weierstrass's approach over that of Jacobi's, we mentioned that in Weierstrass's approach the addition theorem can be proved by a simple and intuitive algebro-geometrical argument without complicated arguments with formulas obtained by differentiation. We now present the simple and intuitive algebro-geometrical proof of the addition theorem for the Weierstrass $\wp$ function.

Addition Theorem for Weierstrass $\wp$ Function. In the case of the trigonometric functions, the unit circle $x^2 + y^2 = 1$ is parametrized by

$$\theta \mapsto (x, y) = (\cos\theta, \sin\theta),$$

because of the identity

$$(\sin x)^2 + (\sin' x)^2 = 1$$

from

$$\sin^{-1} w = \int_{\zeta=0}^{w} \frac{d\zeta}{\sqrt{1-\zeta^2}}.$$

In the case of the Weierstrass $\wp$ function, the identity

$$(\sin x)^2 + (\sin' x)^2 = 1$$

is replaced by

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

which is from

$$z = \wp^{-1}(w) = \int_{\zeta=\infty} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}.$$

Because of the identity

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

we can use

$$z \mapsto (x, y) = (\wp(z), \wp'(z))$$

as a parametrization of

$$y^2 = 4x^3 - g_2 x - g_3.$$

The addition formula is to be considered as the interpretation of the addition $(z_1, z_2) \mapsto z_1 + z_2$ in the parameter space (with coordinate z) in terms of a relation among the three points

$$(\wp(z_1), \wp'(z_1)),\ (\wp(z_2), \wp'(z_2)),\ (\wp(z_1 + z_2), \wp'(z_1 + z_2))$$

on the smooth cubic complex curve in $\mathbb{P}_2$ whose affine part in $\mathbb{C}^2$ is defined by the equation $y^2 = 4x^3 - g_2 x - g_3$ with affine coordinates $(x, y) \in \mathbb{C}^2$.

This relation (which is the addition theorem for the Weierstrass $\wp$ function) will be obtained by using Property (iii) of doubly periodic meromorphic functions, which states that on a fundamental parallelogram of a doubly periodic meromorphic function the sum of the coordinates of the zeroes equals the sum of the coordinates of the poles of the doubly periodic meromorphic function modulo the period lattice.

Let $x = \wp(w)$ and $y = \wp'(w)$ and, for some complex numbers $a \neq 0$ and b to be determined later, we would like to apply Property (iii) of a general elliptic function to the doubly periodic meromorphic function

$$y + ax + b = \wp'(z) + a\wp(z) + b.$$

This doubly periodic meromorphic function has a pole of order 3 at the origin and no other poles inside a fundamental parallelogram. So the sum of its three zeroes must be zero modulo a period. The reason of using $\wp'(z) + a\wp(z) + b$ instead of $\wp'(z)$ is that we need an elliptic function with a pole of order 3 at the origin and *with two parameters (a and b) to be determined.* The addition of $a\wp(z) + b$ to $\wp'(z)$ does not preserve the property of having a pole of order 3 at the origin and adds the two parameters a and b to be determined.

We are free to choose values for the two parameters a and b. We can choose a and b so that the doubly periodic function

$$y + ax + b = \wp'(z) + a\wp(z) + b$$

vanishes at w_1 and w_2. Then

$$y + ax + b = \wp'(z) + a\wp(z) + b$$

must also vanish at $-(w_1 + w_2)$.

On the other hand we have the equation $y^2 = 4x^3 - g_2x - g_3$, which is satisfied by the three points

$$\begin{aligned}
&(x, y) = (\wp(w_1), \wp'(w_1)),\\
&(x, y) = (\wp(w_2), \wp'(w_2)),\\
&(x, y) = (\wp\left(-(w_1 + w_2)\right), \wp'\left(-(w_1 + w_2)r\right)),
\end{aligned}$$

because all three points lie on the Riemann surface which is the nonsingular cubic complex curve in $\mathbb{P}_2$ defined by the equation $y^2 = 4x^3 - g_2x - g_3$,. So by solving the two equation

$$\begin{cases} y + ax + b = 0, \\ \\ y^2 = 4x^3 - g_2x - g_3, \end{cases}$$

we would get the values of x and y at $-(w_1 + w_2)$. Since one equation is a linear equation and the second one is a cubic equation, we expect to get 3 solutions for (x, y).

The other two solutions are the values of (x, y) at w_1 and w_2. Knowing these two solutions makes getting the third solution very easy, because one can use the fact that for a cubic equation with unit leading coefficient the sum of the three roots is the negative of the second coefficient.

We now carry out the details to get our addition theorem. From

$$\begin{aligned}\wp'(w_1) + a\wp(w_1) + b = 0 \\ \wp'(w_2) + a\wp(w_2) + b = 0\end{aligned}$$

we get

$$a = -\frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)}.$$

(As we see later we do not need to solve for b.) From the equation

$$(ax + b)^2 = 4x^3 - g_2 x - g_3$$

we obtain (by the evenness of $\wp(w)$)

$$\wp(w_1) + \wp(w_2) + \wp(w_1 + w_2) = \frac{a^2}{4} = \frac{1}{4}\left(\frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)}\right)^2.$$

Thus we have the addition formula

$$\wp(w_1 + w_2) = -\wp(w_1) - \wp(w_2) + \frac{1}{4}\left(\frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)}\right)^2.$$

Remark. This simple algebro-geometric argument for the derivation of the addition formula works for the Weierstrass $\wp$ function, because the Riemann surface can be identified with a smooth cubic complex curve of $\mathbb{P}_2$. For this reason, the argument does not work for the Jacobian elliptic sine function. The two advantages of the Weierstrass approach over the Jacobi approach are actually not unrelated.

Indefinite Integral Formulation of Addition Theorem for $\wp$. Historically, the addition formula was first derived for the arc-length of the lemniscate by Giulio Carlo Fagnano in 1750. It is a formula relating the radial coordinates of two points on a lemniscate (defined by $r^2 = \cos(2\theta)$ in polar coordinates) such that the value of the arc-length function $s(r_2)$ for one point $r = r_2$ is twice that for the arc-length function $s(r_1)$ at another point $r = r_1$ if

$$r_2 = \frac{2r_1\sqrt{1-r_1^4}}{1+r_1^4}.$$

From this, Euler in 1761 obtained the following addition theorem for the arc-length of the lemniscate.

$$w = \frac{u\sqrt{1-v^4}+v\sqrt{1-u^4}}{1+u^2v^2} \quad \text{when} \quad s(w) = s(u)+s(v).$$

This helps to pave the way to the addition theorem of the Jacobian elliptic sine function.

A full discussion of the proof of Fagnano's additional formula for arc-length of the the lemniscate is given in the Appendix A.

Since the arc-length function is a definite integral from some initial point to a variable point (which is the upper limit of the integral), historically before Abel's inversion the addition formula was written as w_3 in terms of w_1 and w_2 when the sum of two definite integrals with upper limits w_1 and w_2 is equal to the definite integral with upper limit w_3. Whenever we have an addition formula for a function defined as the inverse of an indefinite integral, there is a corresponding formulation in terms of the sum of two definite integrals being equal to one definite integral with relation among the three upper limits.

Suppose we have an addition theorem $f(z_1+z_2) = \Xi(f(z_1), f(z_2))$ for a function $w = f(z)$ whose inverse function is defined by an indefinite integral $f^{-1}(w) = \int_{\zeta=a}^{w}\Phi(\zeta)d\zeta$. We can rewrite it in terms of the defining integrals as

$$\int_{\zeta=a}^{w_1}\Phi(\zeta)d\zeta + \int_{\zeta=a}^{w_2}\Phi(\zeta)d\zeta = \int_{\zeta=a}^{\Xi(w_1,w_2)}\Phi(\zeta)d\zeta$$

when $w_1 = f(z_1)$ and $w_2 = f(z_2)$, because the above equation simply states that

$$\begin{aligned} f^{-1}(w_1) + f^{-1}(w_2) &= z_1 + z_2 = f^{-1}\left(f(z_1+z_2)\right) \\ &= f^{-1}\left(\Xi(f(z_1), f(z_2)\right) = f^{-1}\left(\Xi(w_1, w_2)\right). \end{aligned}$$

Such a derivation was already mentioned in the lectures notes on elliptic functions in Jacobi's approach. In the case of the addition theorem for the Weierstrass $\wp$ function, the formulation in terms of indefinite integrals is

$$\int_{\zeta=\infty}^{\xi} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}} + \int_{\zeta=\infty}^{\eta} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}$$
$$= \int_{\zeta=\infty}^{-\xi-\eta+\frac{1}{4}\left(\frac{\sqrt{4\xi^3-g_2\xi-g_3}-\sqrt{4\eta^3-g_2\eta-g_3}}{\xi-\eta}\right)^2} \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}},$$

because when we let $\xi = \wp(w_1)$ and $\eta = \wp(w_2)$, we have

$$\wp^{-1}(\xi) + \wp^{-1}(\eta) = w_1 + w_2 = \wp^{-1}\left(\wp(w_1 + w_2)\right)$$
$$= \wp^{-1}\left(-\wp(w_1) - \wp(w_2) + \frac{1}{4}\left(\frac{\wp'(w_1) - \wp'(w_2)}{\wp(w_1) - \wp(w_2)}\right)^2\right)$$
$$= \wp^{-1}\left(-\xi - \eta + \frac{1}{4}\left(\frac{\sqrt{4\xi^3 - g_2\xi - g_3} - \sqrt{4\eta^3 - g_2\eta - g_3}}{\xi - \eta}\right)^2\right).$$

TO BE CONTINUED ...