ELLIPTIC FUNCTIONS (Approach of Jacobi)

There are two approaches to the theory of elliptic functions, one due to Jacobi (1804 - 1851) and the other due to Weierstrass (1815 - 1897). The approach of Jacobi is from the viewpoint of real-world problems such as the exact solution of the motion of the simple pendulum and the function for the arc-length of an ellipse, emphasizing the analogy between the Jacobian elliptic functions and the trigonometric functions. The approach of Weierstrass is from the viewpoint of the geometry of a complex curve of degree 3 in the complex projective plane $\mathbb{P}_2$.

We will first discuss the approach of Jacobi and then the approach of Weierstrass, according to the historically chronological order.

Exact Solution of Motion of Simple Pendulum. Let m be the mass of the bob at the end of the pendulum, a be the length of the pendulum, θ be the angle of inclination which the pendulum makes with a vertical line, α be the initial angle of inclination when the pendulum is released from rest position at time zero, t be the time variable, and g be the constant of the gravity of the earth. The equation of the conservation of energy is

$$\frac{1}{2}ma^2\left(\frac{d\theta}{dt}\right)^2 - mga\cos\theta = -mga\cos\alpha.$$

An approximating equation replaces $\cos\theta$ by $1-\frac{\theta^2}{2}$ to give

$$\left(\frac{d\theta}{dt}\right)^2 + A\theta^2 = B$$

for some constants A and B, whose solution can be expressed by

$$t = \rho\sin(\sigma\theta + \tau)$$

for some constants ρ, σ and τ. We are interested in an exact solution of the motion of the simple pendulum and not in its approximating solution. We rewrite the exact equation

$$\frac{1}{2}ma^2\left(\frac{d\theta}{dt}\right)^2 - mga\cos\theta = -mga\cos\alpha.$$

by using the double angle formula

$$\cos 2\xi = \cos^2 \xi - \sin^2 \xi = 1 - 2\sin^2 \xi$$

for the cosine function (with $\xi = \frac{\theta}{2}$ and $\xi = \frac{\alpha}{2}$) to get

$$\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{a}\left(\cos\theta - \cos\alpha\right) = 4\frac{g}{a}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right).$$

A motivation to use the double angle formula for the cosine function is to imitate the approximate replacement of $\cos\theta$ by $1 - \frac{\theta^2}{2}$ for the approximating solution, by using the replacement of

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2},$$

which is now exact. A better motivation for using the double angle formula for the cosine function is that when one tries to simplify

$$\int \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}$$

by using the substitution $x = \sin\varphi$ to get rid of $\sqrt{1-x^2}$, one gets the integral

$$\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

and in that integral there is the expression $\sin^2\varphi$. In order to get an integral equivalent to the form

$$\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

we use the double angle formula for the cosine function and the substitution

$$\sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}.$$

One gets the differential equation

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{g}{a}\left(1 - \sin^2\frac{\alpha}{2}\sin^2\varphi\right).$$

So

$$t=\sqrt{\frac{a}{g}}\int_{\psi=0}^{\varphi}\frac{d\psi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\psi}}.$$

The final answer is

$$\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\,\mathrm{sn}\left(\sqrt{\frac{g}{a}}\,t\right)$$

with the (elliptic) modulus k equal to $\sin\frac{\alpha}{2}$.

The period of the pendulum is 4 times the time needed for the bob to go from the highest point to the lowest point, which means for θ to go between $\theta=\alpha$ and $\theta=0$, or equivalently for φ to go between $\varphi=\frac{\pi}{2}$ and $\varphi=0$, because

$$\sin\varphi=\frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}.$$

Thus, the period of the simple pendulum is given by

$$\begin{aligned}&4\sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\\&=4\sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}}\left(1+\frac{1}{2}k^2\sin^2\varphi+\frac{1\cdot 3}{2\cdot 4}k^4\sin^4\varphi+\cdots\right)d\varphi\\&=2\pi\sqrt{\frac{a}{g}}\left(1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\cdots\right),\end{aligned}$$

because

$$\int_0^{\frac{\pi}{2}}\sin^{2n}\varphi d\varphi=\frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)},$$

from

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\sin^{2n}\varphi d\varphi=\frac{1}{4}\int_0^{2\pi}\sin^{2n}\varphi d\varphi=\frac{1}{4}\int_0^{2\pi}\left(\frac{1}{2i}\left(e^{i\varphi}-e^{-i\varphi}\right)\right)^{2n}d\varphi\\&=\frac{1}{4}\int_0^{2\pi}\left(\frac{1}{2i}\right)^{2n}(-1)^n\binom{2n}{n}d\varphi=\frac{1}{4}\,2\pi\,\left(\frac{1}{2i}\right)^{2n}(-1)^n\binom{2n}{n}.\end{aligned}$$

The period for the approximating equation is $4\sqrt{\frac{a}{g}}$. The formula about the period of the exact motion of the simple pendulum shows the error involved (in terms of k) when the trigonometric sine function is used instead of the Jacobian elliptic sine function.

Inversion by Addition Formula by Abel. We consider the integration of the 1-form

$$\omega := \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

with $z \in \mathbb{C}$. The function

$$\sqrt{(1-z^2)(1-k^2z^2)}$$

of the complex variable z is double-valued except at the four roots ± 1, $\frac{\pm 1}{k}$ of the quartic polynomial. To make the double-valued function

$$\sqrt{(1-z^2)(1-k^2z^2)}$$

single-valued, we construct a new structure which replaces a single point z by two points (except when $z = \pm 1, \frac{\pm 1}{k}$) so that

$$\sqrt{(1-z^2)(1-k^2z^2)}$$

becomes single-valued on this new structure. To construct this new structure, which is the Riemann surface X we seek, we consider the two branches of

$$\sqrt{(1-z^2)(1-k^2z^2)}$$

on the domain obtained by removing the two branch-cuts $[\frac{-1}{k}, -1]$ and $\left[1, \frac{1}{k}\right]$ from the Riemann sphere $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$. The two branches are the negative of each other.

Any of the two branches take values of opposite sign on both edges of each of the two branch-cuts. To get the Riemann surface X we can take two copies of $\mathbb{P}_1 - \left(\left[\frac{-1}{k}, -1\right] \cup \left[1, \frac{1}{k}\right]\right)$ and join them by identifying the upper edge of $\left[\frac{-1}{k}, -1\right]$ in one copy identified with the lower edge of $\left[\frac{-1}{k}, -1\right]$ in the other copy and at the same time identifying the upper edge of $\left[1, \frac{1}{k}\right]$ in one copy identified with the lower edge of $\left[1, \frac{1}{k}\right]$ in the other copy.

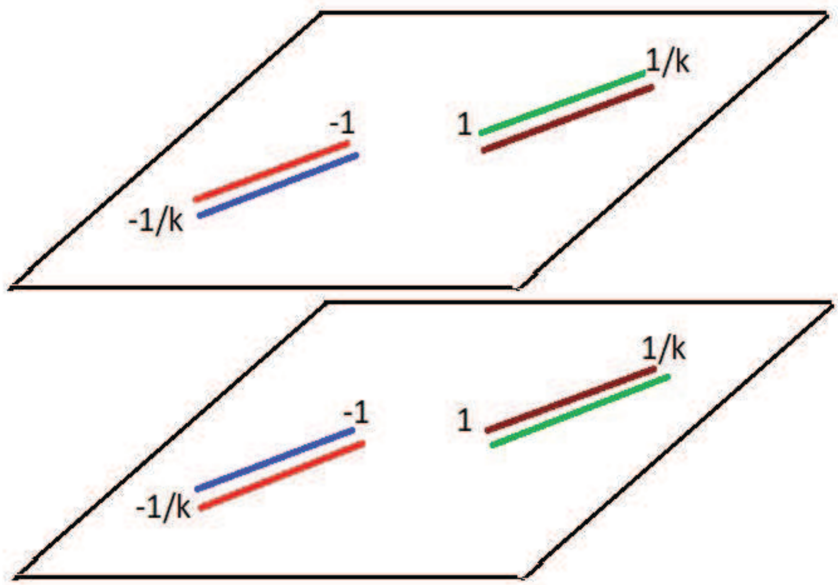


Figure 1: Two red-colored edges identified. Two blue-colored edges identified. Two green-colored edges identified. Two brown-colored edges identified.

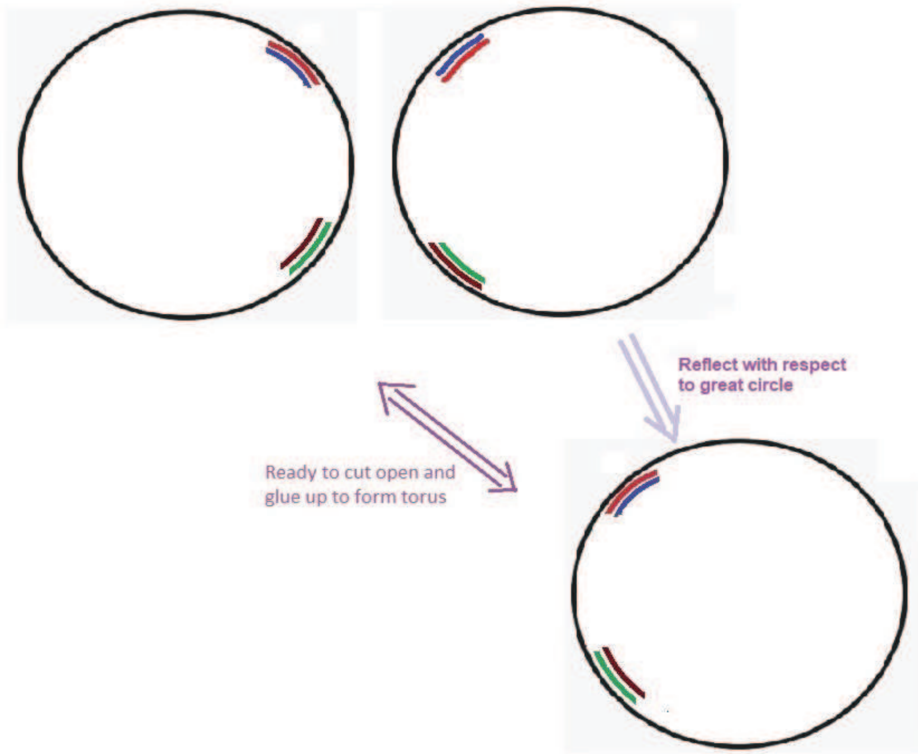


Figure 2: Reflect one sphere with respect to its great circle to prepare for cutting up and glueing to form torus

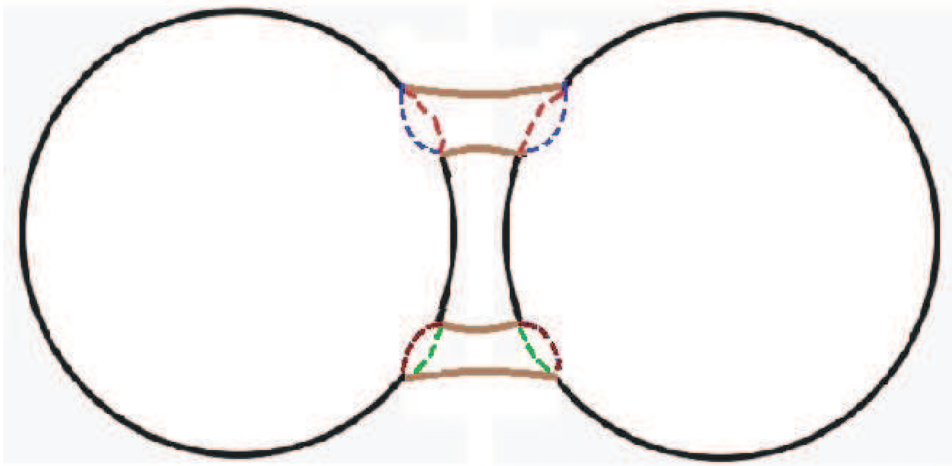
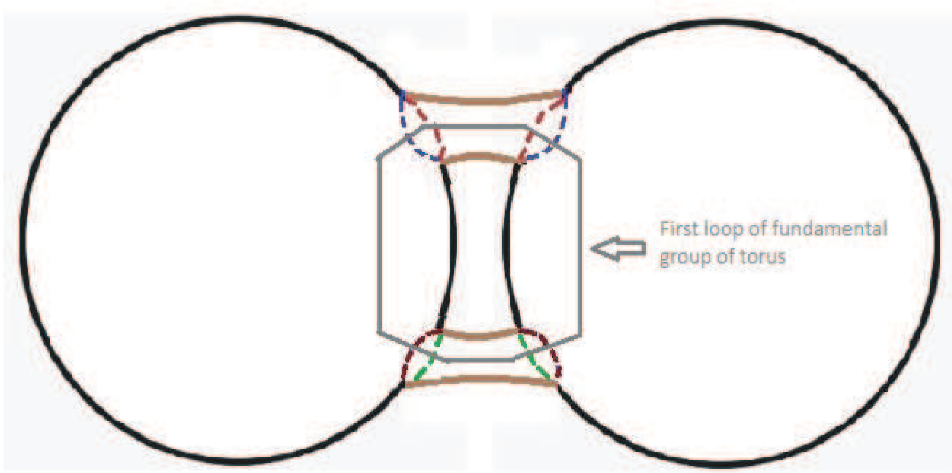
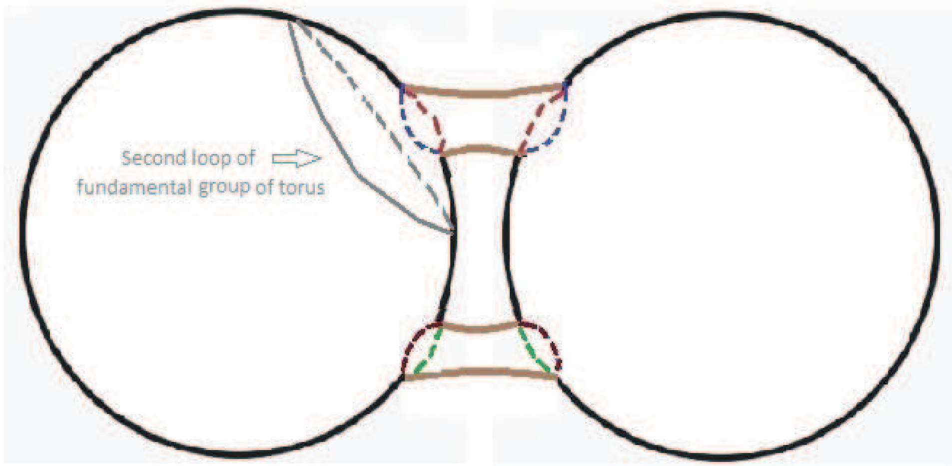


Figure 3: Torus constructed after cutting up branch-cuts of both spheres and glueing corresponding edges.

Topologically there are two loops to generate the fundamental group of the torus. They are described as follows.





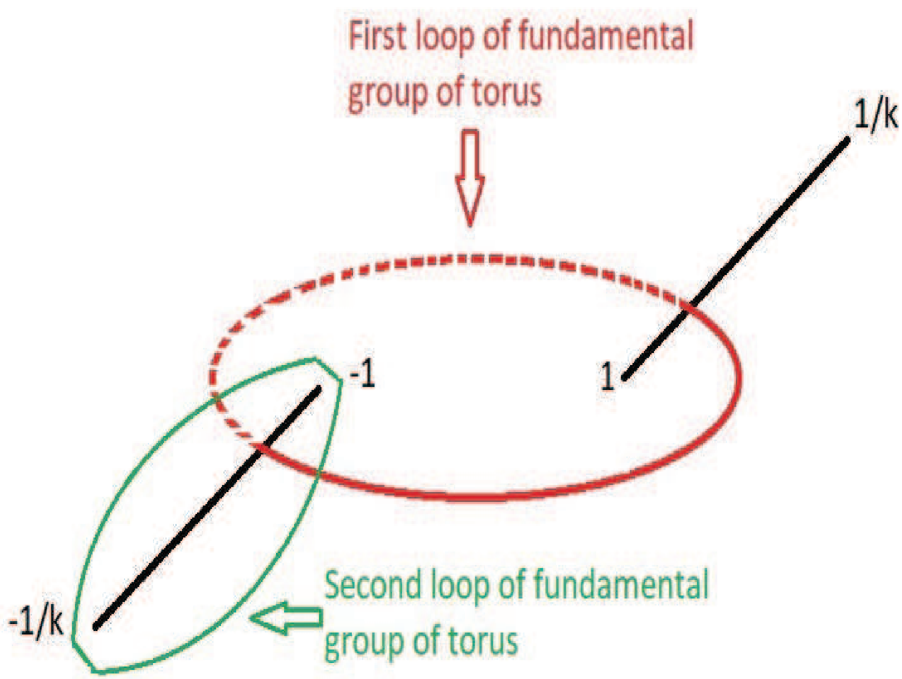


Figure 4: Both loops of the fundamental group of the torus. First goes to another sheet through one branch-cut (represented by dotted curve) and then returns to the original sheet through another branch-cut.

Denote by $\pi : X \to \mathbb{P}_1$ the projection which is 2-to-1 except over the four points ± 1, $\frac{\pm 1}{k}$. On X we have the well-defined 1-form

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

which we will integrate with some initial point to get a map and then use the inverse of the map. The initial point we use is the point in X above 0 where the value of

$$\sqrt{(1-z^2)(1-k^2z^2)}$$

is $+1$.

Nowhere Zero Holomorphic Form on Riemann Surface and Inverse Map of its Indefinite Integral. After the construction of the Riemann surface X to make

$$\sqrt{(1-z^2)(1-k^2z^2)}$$

single-valued, we have a well-defined 1-form

$$\omega := \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

The complex-analytic coordinate charts on Riemann surface X can naturally be defined from the complex-analytic coordinate charts on the Riemann sphere $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$, except at the four points of X which lie above ± 1 and $\pm\frac{1}{k}$. At these four points of X above $c = \pm 1, \pm\frac{1}{k}$, we let $z - c = \zeta^2$ so that ζ can define a complex-analytic coordinate chart on X centered at the point above c. With respect to these complex-analytic coordinate charts on X the 1-form ω is holomorphic and nowhere zero. This is clear outside the four points of X above ± 1 and $\pm\frac{1}{k}$.

At one of the four points $c = \pm 1, \pm\frac{1}{k}$, with respect to the complex-analytic coordinate ζ, we have $dz = 2\zeta d\zeta$ and the denominator yields a factor of ζ from the factor $z - c$ of the quartic polynomial $(1-z^2)(1-k^2z^2)$. The map Φ obtained from integrating ω on X (with the initial point above 0 where the value of

$$\sqrt{(1-z^2)(1-k^2z^2)}$$

is 1) is multi-valued holomorphic with nowhere zero derivative. Denote by τ_1 and τ_2 the two periods of ω from the two loops in the fundamental group of X.

Then the map Ψ from X to $\mathbb{C}\,/(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$ induced by Φ is biholomorphic, because it maps the fundamental group of X one-one onto the fundamental group of $\mathbb{C}\,/(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$. The inverse map of the indefinite integral now is the composite map $\pi\circ\Psi^{-1}\circ\Pi$ from $\mathbb{C}$ to $\mathbb{P}_1=\mathbb{C}\cup\{\infty\}$, where $\Pi:\mathbb{C}\to\mathbb{C}\,/(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$ is the natural quotient map. It is a meromorphic function on $\mathbb{C}$ with two primitive periods τ_1,τ_2, which is the *Jacobian elliptic sine function* $z=\operatorname{sn}w$, where z is the affine coordinate of $\mathbb{P}_1$ and w is the coordinate of $\mathbb{C}$.

$$\begin{array}{ccc} X & \stackrel{\Psi}{\longrightarrow} & \mathbb{C}\,/(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2) \\ \downarrow\pi & & \uparrow\Pi \\ \mathbb{P}_1 & \stackrel{\operatorname{sn}}{\longleftarrow} & \mathbb{C} \end{array}$$

It has two distinct simple zeros (respectively poles) in a fundamental domain which correspond to the two points of X above 0 (respectively above ∞).

Two Periods from Two Loops of Fundamental Group of Torus. We now express, as explicit definite integrals, the two primitive periods from the integration of the nowhere-zero holomorphic 1-form

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

on X over the two loops. The period from the first loop which goes around the line segment $[-1,1]$ with one part in one sheet and another part in the other sheet is $4K$ with

$$K=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

The period from the second loop which goes around the line segment $\left[1,\frac{1}{k}\right]$ (or equivalently around the line-segment $\left[-\frac{1}{k},-1\right]$), all on one single sheet, is $2iK'$ with

$$iK'=\int_1^{1/k}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Note that for the exact solution of the motion of the simple pendulum (with the constants a and g both normalized to 1) the period of the motion of the simple pendulum is equal to

$$
\begin{aligned}
&4\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \\
&\qquad \text{(after using the substitution } x=\sin\varphi) \\
&= 4\int_{x=0}^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = 4K,
\end{aligned}
$$

which is the period from the first loop.

Derivative of Elliptic Sine, Cosine and Delta Amplitude Functions. By applying the fundamental theorem of calculus to the infinite integral

$$\mathrm{sn}^{-1}w = \int_{\zeta=0}^{w} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$

we obtain

$$\frac{dw}{dz} = \sqrt{(1-w^2)(1-k^2w^2)},$$

which motivates the definition of the Jacobian elliptic function

$$\mathrm{cn}\, z = \sqrt{1-\mathrm{sn}^2 z},$$

with value 1 at $z=0$, analogous to the trigonometric functions. There is another factor $\sqrt{1-k^2\mathrm{sn}^2 z}$ without analogue in the theory of trigonometric functions, for which one introduces the Jacobian elliptic function

$$\mathrm{dn}\, z = \sqrt{1-k^2\mathrm{sn}^2 z},$$

with value 1 at $z=0$, known as the *Jacobian delta amplitude function*. We will later explain geometrically the reason for the description "Jacobian delta amplitude function" for $\mathrm{dn}\, z$.

The formula for the derivative of $w=\mathrm{sn}\, z$ is now simply

$$\frac{d}{dz}\,\mathrm{sn}\, z = \mathrm{cn}\, z\, \mathrm{dn}\, z$$

and is the same as the trigonometric case except for the factor $\mathrm{dn}\, z$, which becomes 1 when the (elliptic) modulus k degenerates to 0. From the definitions

$$\mathrm{cn}\, z = \sqrt{1-\mathrm{sn}^2 z}$$

and

$$\operatorname{dn} z = \sqrt{1 - k^2 \operatorname{sn}^2 z},$$

both with value 1 at 0, it is clear that they are defined in some open neighborhood of origin (from the vanishing of $\operatorname{sn} z$ at $z = 0$) but it is far from clear that they can be defined to be global meromorphic functions on all of $\mathbb{C}$, though their squares obviously are meromorphic functions on $\mathbb{C}$. The way to handle this is to use the chain rule to derive differential equations for them so that each can be defined as the inverse function of an indefinite integral whose integrand is the reciprocal of the square root of a quartic polynomial.

By the chain rule, on some open neighborhood of the origin, from

$$\frac{d}{dz} \operatorname{sn} z = \operatorname{cn} z \operatorname{dn} z$$

we obtain the following formulas for the derivatives of $\operatorname{cn} z$ and $\operatorname{dn} z$.

$$\frac{d}{dz} \operatorname{cn} z = \frac{d}{dz} \sqrt{1 - \operatorname{sn}^2 z} = -\frac{\operatorname{sn} z \operatorname{cn} z \operatorname{dn} z}{\sqrt{1 - \operatorname{sn}^2 z}} = -\operatorname{sn} z \operatorname{dn} z$$

and

$$\frac{d}{dz} \operatorname{dn} z = \frac{d}{dz} \sqrt{1 - k^2 \operatorname{sn}^2 z} = -k^2 \frac{\operatorname{sn} z \operatorname{cn} z \operatorname{dn} z}{\sqrt{1 - k^2 \operatorname{sn}^2 z}} = -k^2 \operatorname{sn} z \operatorname{cn} z.$$

From

$$\begin{aligned} \frac{d}{dz} \operatorname{cn} z &= -\operatorname{sn} z \operatorname{dn} z = -\sqrt{1 - \operatorname{cn}^2 z} \sqrt{1 - k^2 \operatorname{sn}^2 z} \\ &= -\sqrt{1 - \operatorname{cn}^2 z} \sqrt{1 - k^2 \left(1 - \operatorname{cn}^2 z\right)} \\ &= -\sqrt{\left(1 - \operatorname{cn}^2 z\right)\left(1 - k^2 + k^2 \operatorname{cn}^2 z\right)} \end{aligned}$$

we can use the inverse function of the infinite integral

$$\operatorname{cn}^{-1} w = \int_{\zeta = w}^{1} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - k^2 + k^2 \zeta^2)}}$$

to define the meromorphic function on $\mathbb{C}$. Likewise, from

$$\begin{aligned} \frac{d}{dz} \operatorname{dn} z = -k^2 \operatorname{sn} z \operatorname{cn} z &= -k^2 \sqrt{\frac{1}{k^2}\left(1 - \operatorname{dn}^2 z\right)} \sqrt{1 - \frac{1}{k^2}\left(1 - \operatorname{dn}^2 z\right)} \\ &= -\sqrt{\left(1 - \operatorname{dn}^2 z\right)\left(k^2 - 1 + \operatorname{dn}^2 z\right)} \end{aligned}$$

we can use the inverse function of the infinite integral

$$\mathrm{dn}^{-1} w = \int_{\zeta=w}^{1} \frac{d\zeta}{\sqrt{(\zeta^2-1)(1-k^2-\zeta^2)}}$$

to define the meromorphic function on $\mathbb{C}$.

TO BE CONTINUED ...