DIRICHLET SERIES

The goal of discussing the topic of Dirichelet series is to apply it to the Riemann Zeta function which is the key function for the application of complex analysis to number theory such as the prime number theorem and the Riemann hypothesis. A Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

as a function of the complex variable s. One way to relate a Dirichlet series to a power series is to apply the change of variables $w = e^{-s}$ to transform a power series

$$\sum_m \gamma_m w^m$$

to get

$$\sum_m \gamma_m e^{-ms} = \sum_m \frac{\gamma_m}{(e^m)^s} = \sum_n \frac{a_n}{n^s},$$

where $a_n = \gamma_m$ with the relabeling of the index m to $n = e^m$ and a corresponding change of the set of indices where the summation is to take place. The convergence condition of $|w| < R$ involving a lower bound of the absolute value $|w|$ of w for the convergence of the power series in w corresponds to the convergence condition of $\operatorname{Re} s > -\log R$ involving an upper bound of the abscissa $\operatorname{Re} s$ of s for the convergence of the Dirichlet series in s. For a power series we talk about its radius of convergence. For a Dirichlet series we talk about its abscissa of convergence.

There are only two subtopics about Dirichlet series which we will treat.

(i) The abscissa of convergence and the abscissa of absolute convergence and the maximum possible gap between them.

(ii) The growth rate of a Dirichlet series along a vertical line.

We will conclude by applying the results of our discussion to the Riemann Zeta function. The reason for discussing the abscissa of convergence is that for our applications we need to justify the terms-by-terms integration involving Dirichlet series. The growth rate of a Dirichlet series along a vertical line is important in one crucial step of our proof of the Prime Number Theorem when the residue at 0 has to be computed along a rectangular contour around 0 which goes to a vertical strip as limit, with estimates under control in the limiting process.

Radius of Convergence for Power Series and Abscissa of Convergence for Dirichlet Series. The abscissa of convergence (or absolute convergence) for a Dirichlet series is the analogue of the radius convergence for a power series. One important difference is the gap between the abscissa of convergence and the abscissa of absolute convergence (while the radius of convergence is always equal to the radius of absolute convergence for a poer series). The gap can achieve the maximum value of 1. The interesting and important case for us is

$$\left(1-\frac{1}{2^{s-1}}\right)\zeta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

whose abscissa of convergence is $\operatorname{Re} s = 0$ and whose abscissa of absolute convergence is $\operatorname{Re} s = 1$. We now derive these properties of the abscissa of convergence and the abscissa of absolute convergence.

Definitions of Abscissa of Convergence and Abscissa of Absolute Convergence. The infimum σ_0 of all $\operatorname{Re} s$ such that the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

is convergent is the *abscissa of convergence* of the Dirichlet series. The infimum $\bar{\sigma}$ of all $\operatorname{Re} s$ such that the series

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{n^s}\right|$$

is convergent is the *abscissa of absolute convergence* of the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}.$$

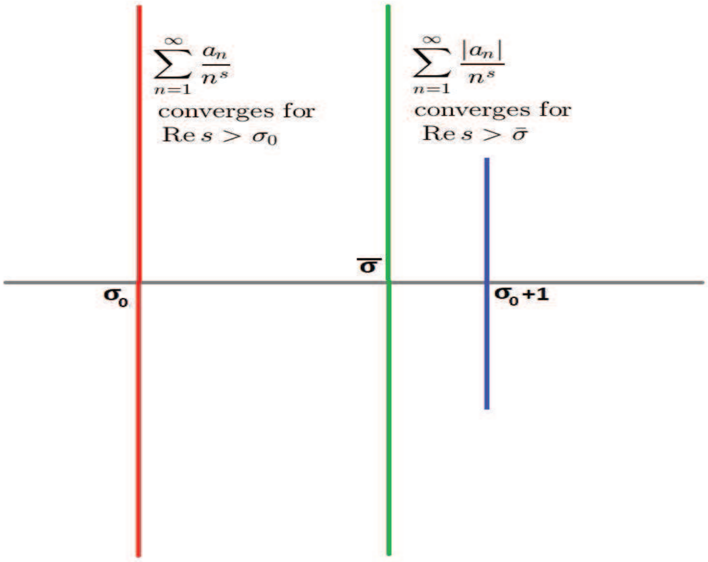

Figure 1: Abscissa of Convergence and Abscissa of Absolute Convergence

Gap between Abscissa of Convergence and Abscissa of Absolute Convergence. The relation between σ_0 and $\bar\sigma$ is $\sigma_0 \leq \bar\sigma \leq \sigma_0 + 1$, because the convergence of

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

implies

$$\sup_{1\leq n<\infty}\frac{|a_n|}{n^\sigma} < \infty,$$

which in turn implies that

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+1+\delta}} < \infty$$

for any $\delta > 0$. The relation σ_0 and $\bar\sigma$ is sharp, as is shown by the example

$$\left(1-\frac{1}{2^{s-1}}\right)\zeta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}.$$

Note that this example is obtained by taking the difference between the Dirichlet series of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and two times the sum over only the even-indexed terms

$$2\sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \frac{2}{2^s}\sum_{n=1}^{\infty} \frac{1}{n^s}$$

to result in the series of all odd-indexed terms minus all even-indexed terms.

Uniform Convergence on Sector (by Argument of Summation by Parts). Let

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series. If the series converges at some s_0, then for any $0 < \delta < \frac{\pi}{2}$ it converges uniformly on

$$-\frac{\pi}{2} + \delta \leq \arg(s - s_0) \leq \frac{\pi}{2} - \delta.$$

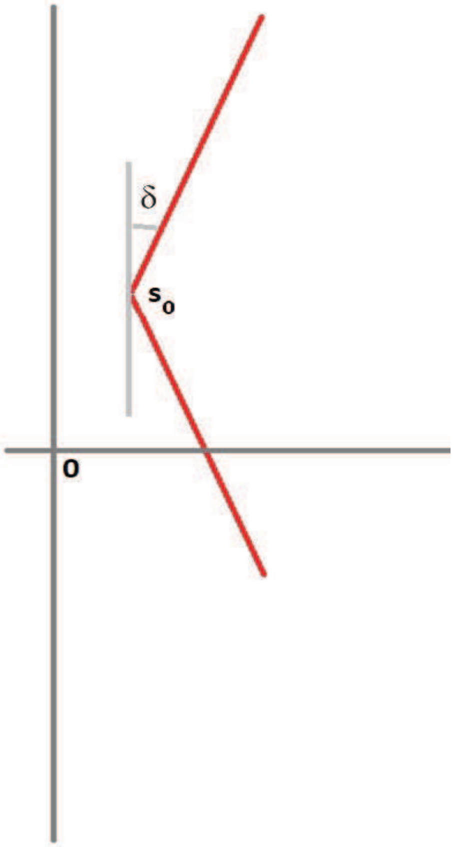

Figure 2: Uniform Convergence of Dirichlet Series on Sector $-\frac{\pi}{2}+\delta \leq \arg(s-s_0) \leq \frac{\pi}{2}-\delta$

For the verification it suffices to consider the case $s_0 = 0$. Let $r_n = \sum_{k=n+1}^{\infty} a_k$. By summation by parts

$$\sum_{n=M}^{N} \frac{a_n}{n^s} = \sum_{n=M}^{N} \frac{r_{n-1}-r_n}{n^s} = \sum_{n=M}^{N} r_n \left(\frac{1}{(n+1)^s} - \frac{1}{n^s}\right) + \frac{r_{M-1}}{M^s} - \frac{r_N}{(N+1)^s}.$$

Since

$$\begin{aligned} &\left|\frac{1}{(n+1)^s} - \frac{1}{n^s}\right| = \left| s \int_{u=n}^{n+1} \frac{du}{u^{s+1}} \right| \\ &\leq |s| \int_{u=n}^{n+1} \frac{du}{u^{\sigma+1}} = \frac{|s|}{\sigma}\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right), \end{aligned}$$

it follows that, when $|r_n| < \varepsilon$ for $M \geq n_0$,

$$\begin{aligned}\left|\sum_{n=M}^{N} \frac{a_n}{n^s}\right| &< \frac{\varepsilon|s|}{\sigma}\sum_{n=M}^{N}\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right) + \frac{\varepsilon}{M^\sigma} + \frac{\varepsilon}{(N+1)^\sigma} \\ &= \frac{\varepsilon|s|}{\sigma}\left(\frac{1}{M^\sigma} - \frac{1}{(N+1)^\sigma}\right) + \frac{\varepsilon}{M^\sigma} + \frac{\varepsilon}{(N+1)^\sigma} \\ &< \frac{2\varepsilon|s|}{\sigma} + 2\varepsilon \leq 2\varepsilon\left(\frac{1}{\cos\delta} + 1\right).\end{aligned}$$

We would like to make some remarks about this proof of the uniformity of the convergence in the sector

$$-\frac{\pi}{2} + \delta \leq \arg(s - s_0) \leq \frac{\pi}{2} - \delta.$$

There are two basic techniques. One is summation by parts (which is the discrete analogue of integration by parts). The convergence of a Dirichlet series is like the convergence of a p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where the key role is played by the exponent p. Summation by parts enables us to use the assumption of the convergence of the Dirichlet series at a given point to increase the exponent in the denominator. The second technique is to use the assumption of the real part of s in the estimate of the

$$\frac{1}{(n+1)^s} - \frac{1}{n^s}$$

by expressing it as an integral and then replacing the integrand by its absolute value. The difference of consecutive terms has the effect of increasing the exponent of the denominator by 1. These two techniques are also used below in estimating the growth rate of a Dirichlet series along a vertical line.

Estimate of Riemann Zeta Function on a Vertical Line from the Comparison with the Integral of $\frac{1}{x^s}$ *over* $[1, \infty)$. We estimate the growth rate of $\zeta(s)$ on a vertical line by comparison between $\zeta(s)$ and the integral of $\frac{1}{x^s}$ over $[1, \infty)$. By comparing the Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

with the integral

$$\int_{x=1}^{\infty}\frac{1}{x^s}\,dx=\left[\frac{-1}{1-s}\,x^{1-s}\right]_{s=1}^{\infty}=\frac{1}{1-s}$$

for $\operatorname{Re}s>1$ and writing

$$\frac{1}{n^s}=\int_{x=n}^{n+1}\frac{1}{n^s}\,dx$$

and

$$\int_{x=1}^{\infty}\frac{1}{x^s}\,dx=\sum_{n=1}^{\infty}\int_{x=n}^{n+1}\frac{1}{x^s}\,dx,$$

we get

$$\zeta(s)-\frac{1}{1-s}=\sum_{n=1}^{\infty}\int_{x=n}^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)dx.$$

We now apply, to the integrand

$$\frac{1}{n^s}-\frac{1}{x^s}$$

for $n\leqslant x\leqslant n+1$, the estimate

$$|f(x_2)-f(x_1)|\leq(x_2-x_1)\sup_{x_1\leqslant t\leqslant x_2}|f'(t)|$$

from the fundamental theorem of calculus for a complex-valued function of a real variable

$$f(x_2)-f(x_1)=\int_{t=x_1}^{x_2}f'(t)dt.$$

Note that the mean-value theorem is only for a real-valued function of a real variable and cannot be applied directly to a complex-valued function of a real variable, unless it is applied separately to the real part and the imaginary part of the function, in which case the point chosen for the derivative for the real part may not different from that for the imaginary part. We get

$$\left|\frac{1}{n^s}-\frac{1}{x^s}\right|\leqslant(x-n)\sup_{n\leqslant t\leqslant x}\left|-\frac{s}{t^{s+1}}\right|\leqslant\frac{|s|}{n^{\sigma+1}}$$

for $n\leqslant x\leqslant n+1$. For any fixed $0<\varepsilon<1$, we can use a multiplicative interpolation of the two estimates

$$\left|\frac{1}{n^s}-\frac{1}{x^s}\right|\leqslant\frac{|s|}{n^{\sigma+1}}$$

and

$$\left|\frac{1}{n^s}-\frac{1}{x^s}\right| \leqslant \frac{2}{n^\sigma}$$

for $n \leqslant x \leqslant n+1$ to get

$$\left|\frac{1}{n^s}-\frac{1}{x^s}\right| \leqslant \left(\frac{|s|}{n^{\sigma+1}}\right)^{\varepsilon}\left(\frac{2}{n^\sigma}\right)^{1-\varepsilon} = \frac{2^{1-\varepsilon}|s|^\varepsilon}{n^{\sigma+\varepsilon}}$$

for $n \leqslant x \leqslant n+1$.

Then, for $s=\sigma+it$ with $|t| \geqslant 1$ and $1 \leqslant \sigma \leqslant 2$, from

$$\zeta(s)-\frac{1}{1-s}=\sum_{n=1}^{\infty}\int_{x=n}^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)dx$$

we get

$$|\zeta(s)| \leqslant C_1 + C_2\sum_{n=1}^{\infty}\frac{2^{1-\varepsilon}|s|^\varepsilon}{n^{\sigma+\varepsilon}} \leqslant C^*_\varepsilon|t|^\varepsilon.$$

Since $\zeta(s)$ is uniformly bounded for $\sigma \geqslant 2$, we obtain the estimate

$$|\zeta(s)| \leqslant C_\varepsilon|t|^\varepsilon$$

for $s=\sigma+it$ with $|t| \geqslant 1$ and $\sigma \geqslant 1$, with $C_\varepsilon > 0$ depending only on $\varepsilon > 0$.

This argument can be refined a little bit by allowing $\sigma+\varepsilon>1$ instead of just $\sigma>1$, because

$$\sum_{n=1}^{\infty}\frac{2^{1-\varepsilon}|s|^\varepsilon}{n^{\sigma+\varepsilon}} < \infty$$

implies that

$$\sum_{n=1}^{\infty}\int_{x=n}^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)dx.$$

is holomorphic on $\sigma+\varepsilon>1$ and the equation

$$\zeta(s)-\frac{1}{1-s}=\sum_{n=1}^{\infty}\int_{x=n}^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)dx$$

by analytic continuation holds on $\sigma+\varepsilon>1$ so that

$$|\zeta(s)| \leqslant C^\dagger_\varepsilon|t|^\varepsilon$$

for $s=\sigma+it$ with $|t|\geqslant 1$ and $\sigma\geqslant 1-\frac{\varepsilon}{2}$, with $C^{\dagger}_{\varepsilon}>0$ depending only on $\varepsilon>0$. This estimate implies the following estimate for $\zeta'(s)$ on vertical lines with abscissa ≥ 1.

$$|\zeta'(s)|\leqslant C'_{\varepsilon}|t|^{\varepsilon}$$

for $s=\sigma+it$ with $|t|\geqslant 1$ and $\sigma\geqslant 1$, with C'_{ε} depending only on $\varepsilon>0$. To verify the estimate, it suffices to apply the Cauchy integral formula for derivatives to the function $\zeta(s)$ on the disk $|s-1|=\frac{\varepsilon}{2}$ and use the refined estimate given above.

Growth Rate of Dirichlet Series Along Vertical Lines in Terms of Abscissa of Convergence. Let $f(s)$ be the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

with abscissa of convergence σ_0. Let $\varepsilon>0$. Then

$$f(s)=O\left(|t|^{1-(\sigma-\sigma_0)+\varepsilon}\right)$$

uniformity on $\sigma_0+\delta\leq\sigma\leq\sigma_0+1$ for any $\delta>0$, where $s=\sigma+i\,t$.

Note that the exponent $1-(\sigma-\sigma_0)+\varepsilon$ means a little bit more than the difference η between 1 plus the abscissa of convergence σ_0 minus the abscissa σ of the variable s, as shown in the figure below. The abscissa σ of the variable s must be uniformly strictly bigger than the abscissa of convergence σ_0 and no more than 1 plus abscissa of convergence σ_0 to guarantee uniformity of the estimate in σ.

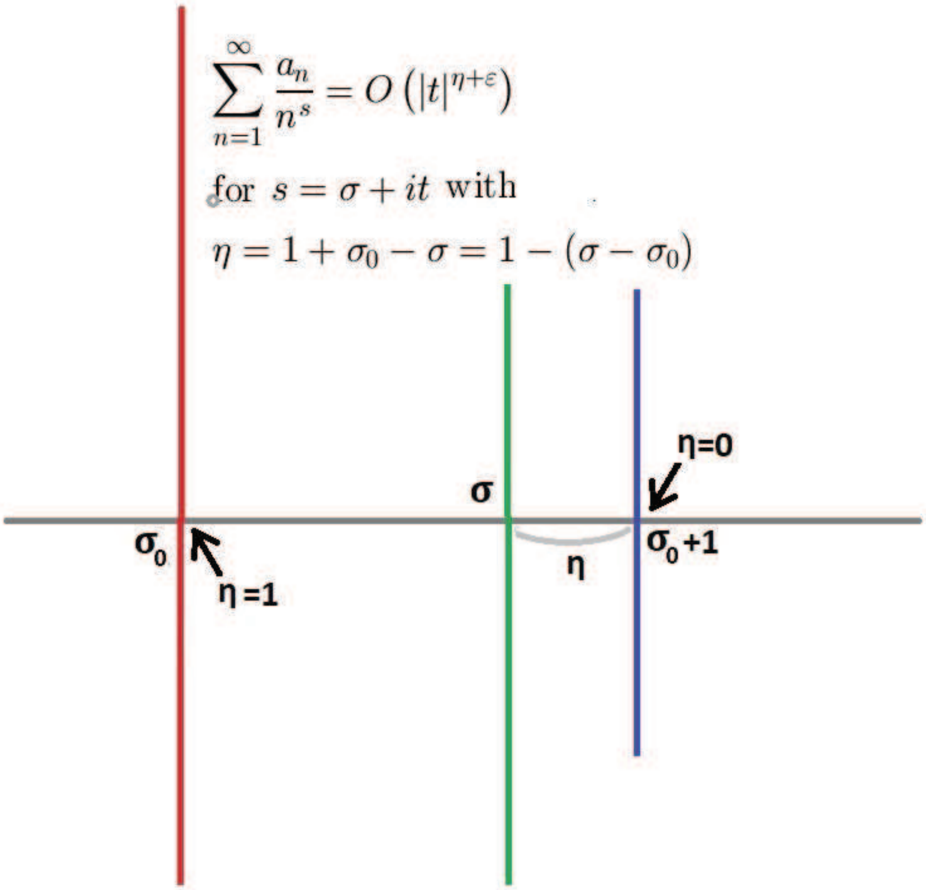

Figure 3: Growth Rate of Dirichlet Series Along Vertical Lines

The proof is by summation by parts and by using the partial sum

$$\sum_{n=1}^{M} \frac{a_n}{n^s}$$

with the number N of terms comparable to the imaginary part t of the variable $s = \sigma + it$. By a translation along the real part of the variable we can assume without loss of generality that σ_0 is negative and very close to 0 (relative to ε) so that $\sum_{n=1}^{\infty} a_n$ converges and we need only verify

$$f(s) = O\left(|t|^{1-\sigma+\varepsilon}\right)$$

uniformly for $\delta \leq \sigma \leq 1$.

Let $s_n = \sum_{k=1}^{n} a_k$. Summation by parts from $n = M+1$ to $n = N$ yields

$$\sum_{n=1}^{N} \frac{a_n}{n^s} = \sum_{n=1}^{M} \frac{a_n}{n^s} + \sum_{n=M+1}^{N} s_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) - \frac{s_M}{(M+1)^s} + \frac{s_N}{(N+1)^s}.$$

It follows from $\sigma > 0$ that as $N \to \infty$ we have

$$f(s) = \sum_{n=1}^{M} \frac{a_n}{n^s} + \sum_{n=M+1}^{\infty} s_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) - \frac{s_M}{(M+1)^s}.$$

Let A be a bound for $|a_n|$ and $|s_n|$. Since

$$\begin{aligned}\left|\frac{1}{(n+1)^s} - \frac{1}{n^s}\right| &= \left|s \int_{u=n}^{n+1} \frac{du}{u^{s+1}}\right| \\ &\leq |s| \int_{u=n}^{n+1} \frac{du}{u^{\sigma+1}} = \frac{|s|}{\sigma}\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right),\end{aligned}$$

it follows that

$$|f(s)| \leq A \left(\sum_{n=1}^{M} \frac{1}{n^\sigma} + \sum_{n=M+1}^{\infty} \frac{|s|}{\sigma}\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right) + \frac{1}{(M+1)^\sigma}\right).$$

Since

$$\sum_{n=2}^{M} \frac{1}{n^\sigma} \leq \int_{x=1}^{M} \frac{dx}{x^\sigma} = \frac{1}{1-\sigma}\left(M^{1-\sigma} - 1\right),$$

it follows that for $\delta \leq \sigma < 1$ and $t > 0$,

$$|f(s)| \leq A \left(\frac{1}{1-\sigma} M^{1-\sigma} + \left(1 + \frac{t}{\sigma}\right)\frac{1}{M^\sigma} + \frac{1}{(M+1)^\sigma}\right).$$

We obtain

$$f(s) = O\left(t^{1-\sigma}\right)$$

by choosing M to be the integral part of t so that $M^\sigma = \lfloor t \rfloor^\sigma$ yields the order t^σ. This choice make the contributions of the first two terms of

$$\frac{1}{1-\sigma} M^{1-\sigma} + \left(1 + \frac{t}{\sigma}\right)\frac{1}{M^\sigma} + \frac{1}{(M+1)^\sigma}$$

of the same growth order $t^{1-\sigma}$ while the last term is of lower order $t^{-\sigma}$.

Growth Rate of Riemann Zeta Function Along Vertical Lines Derived from Dirichlet Series of Difference Between Odd Terms and Even Terms. Riemann's Zeta function can be expressed in another Dirichlet series

$$\zeta(s)=\frac{1}{1-\frac{1}{2^{s-1}}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

which is obtained from its odd terms minus its even terms. The advantage of the new Dirichlet-series representation is that its abscissa of convergence differs from its abscissa of absolute convergence by 1, enabling us to get a statement on the growth rate of Riemann's zeta function along vertical lines inside the critical strip $\{0\leq \operatorname{Re} s\leq 1\}$ from the above general result.

Using the new odd-even-term-difference Dirichlet-series representation of $\zeta(s)$, we have

$$\zeta(s)=O\left(|t|^{1-\sigma+\varepsilon}\right)\quad\text{for }\ 0<\sigma\leq 1\ \text{ and }\ \varepsilon>0.$$

Earlier we already have

$$|\zeta(\sigma+it)|\leqslant C_\varepsilon|t|^\varepsilon\quad\text{for }\ \sigma\geqslant 1\ \text{ and }\ |t|\geqslant 1$$

for some $C_\varepsilon>0$ depending on ε for $\varepsilon>0$. Also we already have

$$|\zeta'(\sigma+it)|\leqslant C'_\varepsilon|t|^\varepsilon\quad\text{for }\ \sigma\geqslant 1\ \text{ and }\ |t|\geqslant 1$$

for some $C'_\varepsilon>0$ depending on ε for $\varepsilon>0$. While the earlier two results for the vertical growth of $\zeta(\sigma+it)$ and $\zeta'(\sigma+it)$ apply to $\sigma\geq 1$, the new result of using the odd-even-term-difference Dirichlet-series representation of $\zeta(s)$ applies to $0<\sigma\leq 1$.