TAKE-HOME FINAL EXAMINATION OF MATH 113

Closed-Book Exam to Be Taken in Any Two 4-Hour Periods
(Separated by a 2-Hour Intermission) of Your Choice
Between 12:01 a.m. April 29, 2024 and 11:59 p.m. May 5, 2024
(Total Number of Problems = 12)

Please submit the PDF file of your solution
to the CANVAS website for Math 113

Instructions. This is a closed-book exam. That is, the same as you are sitting in an examination hall. You are free to choose your two 4-hour periods to work on the exam problems. It is an honor system. You simply put down your starting time and ending time of each of the two periods in your submitted PDF file. The time you use to type up your file in TeX does not count toward the two periods of four hours each you are allowed to work on the problems. Do not read the problems unless you are ready to start working on them in your chosen two periods of four hours each.

N.B. Solutions with rigorous details are expected. For problems which can be solved by techniques in some homework problems, complete self-contained solutions are required and homework problems cannot be quoted simply as known facts in the solutions.

Problem 1 (*Evaluation of Integrals by Residue Theory*).

(a) Evaluate
$$\int_{\theta=0}^{2\pi} \frac{1}{5+4\sin\theta}\,d\theta.$$

(b) Evaluate
$$\int_{x=-\infty}^{\infty} \frac{\sin x}{x\,(1+x^2)}\,dx.$$

(c) Evaluate

$$\int_{x=0}^{\infty} \frac{x^{\alpha}}{(1+x^2)^2}\,dx$$

for $-1 < \alpha < 3$.

Problem 2. (*Use of Half-Residues and Gamma and Beta Functions in Evaluation of Integrals by Residue Theorem*).

(a) Evaluate the integral

$$\int_{x=0}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx$$

by using the theory of residues. Give your answer as a rational function of π with coefficients in $\mathbb{Z}$.

Hint: Use

$$\sin^3 x = \frac{3\sin x - \sin(3x)}{4} = \operatorname{Im}\left(\frac{3e^{ix} - e^{3ix} - 2}{4}\right)$$

and integrate

$$\frac{3e^{iz} - e^{3iz} - 2}{z^3}\,dz$$

over the boundary of the set which is equal to the upper half-disk of radius $R > 0$ minus the upper half-disk of radius r with $0 < r < R$.

(b) Let $b > a > 1$. Evaluate

$$\int_{\theta=0}^{\frac{\pi}{2}} \cos^a\theta\cos(b\theta)\,d\theta = C\,\frac{\Gamma(a+1)}{2^a\,\Gamma\left(\frac{a}{2}+\frac{b}{2}+1\right)\Gamma\left(\frac{a}{2}-\frac{b}{2}+1\right)}$$

to determine the constant C explicitly in the form of a rational function of π with coefficients in $\mathbb{Z}$.

Hint: Consider the integral of $\left(z+\frac{1}{z}\right)^a z^{b-1}dz$ over the right half of the unit circle and use the symmetric version of the integral formula for the beta function. Recall the following material about the Gamma and Beta functions.

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \quad \textit{(Relation between Gamma and Beta functions)},$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \textit{(Euler's reflection formula)},$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

and

$$B(x,y) = \int_0^\infty \frac{v^{y-1} dv}{(1+v)^{x+y}} = \int_0^1 \lambda^{x-1} (1-\lambda)^{y-1} d\lambda.$$

Problem 3 (*Summation of Series by Residue Theory*).

(a) Sum the following two series

$$\sum_{n=-\infty}^{\infty} \frac{n^2+n+1}{n^4+1} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} (-1)^n \frac{n^2+n+1}{n^4+1}$$

by integrating respectively $R(z) \cot \pi z$ and $R(z) \operatorname{cosec} \pi z$ for some appropriate rational function $R(z)$ over the boundary of the square whose four vertices are $\left(n+\frac{1}{2}\right)(\pm 1 \pm i)$.

(b) Verify the following series expansion

$$\frac{\pi}{\tan \pi z} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{k+z}$$

for $z \in \mathbb{C} - \mathbb{Z}$ by using the theorem of Mittag-Melffler stated below. Note that the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+z}$$

does not converge absolutely.

Theorem of Mittag-Leffler. Let $f(z)$ be a meromorphic function on $\mathbb{C}$ whose poles $\{a_n\}_{1\leq n<\infty}$ are simple with $0 < |a_1| \leq |a_2| \leq \cdots$ so that the residue of $f(z)$ at a_n is b_n. Suppose that there is a sequence of closed contours C_n such that the enclosure of C_n includes $a_1, \cdots, a_n$ but no other poles. Assume that the distance R_n from C_n to the origin goes to infinity as $n \to \infty$ and the length C_n is $\leq K R_n$ for some constant K independent of n. Assume that for some nonnegative integer p one has $\frac{1}{R_n^{p+1}} \sup_{C_n} |f(z)| \to 0$ as $n \to \infty$. Then

$$f(z) = \sum_{\nu=0}^{p} \frac{z^\nu}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z-a_n} + \frac{1}{a_n} + \frac{z}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}} \right).$$

Problem 4 (a) (*Generalization of Hurwitz's Theorem on Limits of Univalent Functions*). Let Ω be a connected open subset of $\mathbb{C}$ and $f_n : \Omega \to \mathbb{C} \cup \{\infty\}$ be a holomorphic map from Ω to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ such that f_n is injective in the sense that $f(z_1) \neq f(z_2)$ as elements of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ if z_1 and z_2 are distinct points of Ω. Let f be a nonconstant meromorphic function on Ω with pole-set Z. Assume that f_n converges to f uniformly on compact subsets of $\Omega - Z$ Prove that f is injective as a map from Ω to the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

(b) (*Examples of Application of Rouchè's Theorem*). Let $\alpha > 1$ be a real number and n be a positive integer.

(i) Use Rouché's theorem to find out the number of roots, *with multiplicities counted*, of the equation $z^n e^{\alpha - z} = 1$ in $|z| < 1$.

(ii) Find out the number of *distinct* roots of the same equation $z^n e^{\alpha - z} = 1$ in $|z| < 1$.

Problem 5 (*Elliptic Functions Constructed from Schwarz-Christoffel Transformations and Schwarz Reflection Principle with Respect to Sides of Triangles*). Let a, b, c be distinct complex numbers not on a straight line. Let Ω be the disk whose boundary contains the three points a, b, and c. Let m, n, p be positive integers such that

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = 1$$

and $m \leq n \leq p$. Consider the holomorphic function

$$z = g(w) = \int_{\zeta=0}^{w} (\zeta - a)^{\frac{1}{m} - 1} (\zeta - b)^{\frac{1}{n} - 1} (\zeta - c)^{\frac{1}{p} - 1} d\zeta$$

defined on Ω, where the integrand is any chosen branch defined on Ω. This is a Schwarz-Christoffel transformation whose domain is a disk instead of the upper half-plane.

(a) Let $A = g(a)$, $B = g(b)$, and $C = g(c)$. Denote by ΔABC the triangle with vertices A, B, C. Verify that the interior angles of the triangle ΔABC at the vertices A, B, C are respectively $\frac{\pi}{m}$, $\frac{\pi}{n}$, and $\frac{\pi}{p}$.

(b) Verify that $z = g(w)$ maps the disk Ω biholomorphically onto the triangle ΔABC.

(c) For positive integers $m \leq n \leq p$ satisfying

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = 1,$$

verify that the only possibilities are $(m, n, p) = (2, 3, 6)$, $(2, 4, 4)$, and $(3, 3, 3)$. *Hint:* m can only be 2 or 3. When m is 2, $(n-2)(p-2) = 4$.

(d) By going over the three possible cases listed in Part**(c)**, prove that the inverse function $w = F(z)$ of $z = g(w)$ can be holomorphically continued to a meromorphic function f on all of $\mathbb{C}$ by repeatedly applying the Schwarz reflection principle. Moreover, show that the meromorphic function f on $\mathbb{C}$ is an elliptic function with two primitive periods. Write down, with justification, for each of the three cases, a pair of primitive periods in terms of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$. Note that though the lattice of periods of an elliptic function is unique, yet two primitive periods as two generators of the period lattice are not unique.

Hint: For z in the interior of the triangle ΔABC, let $\hat{w}$ be the reflection of $w = F(z)$ with respect to the boundary $\partial\Omega$ of Ω. Keep track of the iterated reflections of z with respect to the sides of the triangle ΔABC and with respect to the sides of those triangles obtained by such reflections, to determine the first time such an iterated reflection $\tilde{z}$ of z has image w (rather than $\hat{w}$) under F and the map $z \mapsto \tilde{z}$ is a translation by a vector which is independent of small perturbation of z in ΔABC.

Problem 6 (*Riemann Mapping from Riemann Mapping Theorem as Logarithmic Derivative of Bergman Kernel with Respect to Conjugate Complex Variable*). Let Ω be a simply connected bounded domain in $\mathbb{C}$. Let $f_\nu(z)$ $(1 \leq \nu < \infty)$ be an orthonormal basis of the Hilbert space of all square integrable holomorphic functions on Ω.

(a) Show that $\sum_{\nu=1}^{\infty} |f_\nu(z)|^2$ converges uniformly on compact subsets of Ω.

Hint: For any relatively compact disk D in Ω, use the fact that for any holomorphic function g on Ω the value $|g|^2$ at the center c of D is no more than the average of $|g|^2$ on D and apply this to the case $g = \sum_{\nu=1}^{n} a_\nu f_\nu(z)$ with $a_\nu = \overline{f_\nu(c)}$.

(b) Define $K_\Omega(z,\bar w)=\sum_{\nu=1}^\infty f_\nu(z)\overline{f_\nu(w)}$ and use Part**(a)** to prove that the series converges uniformly on compact subsets of $\Omega\times\Omega$ and $K_\Omega(z,\bar w)$ is holomorphic in z and anti-holomorphic in w. "Anti-holomorphic in the variable w" means "holomorphic in the variable $\bar w$". The function $K_\Omega(z,\bar w)$ is known as the *Bergman kernel function* of Ω. For $w_0\in\Omega$, introduce the map

$$\Phi_{\Omega,w_0}:\Omega\to\mathbb{C}\cup\{\infty\}$$

from Ω to the Riemann sphere $\mathbb{C}\cup\{\infty\}$ by defining it as the logarithmic derivative

$$\Phi_{\Omega,w_0}=\left(\frac{\partial}{\partial\bar w}\log K_\Omega(z,\bar w)\right)_{w=w_0}$$

of the Bergman kernel with respect to the conjugate complex variable $\bar w$ at the point $w=w_0$. Verify that $\Phi_{\Omega,w_0}:\Omega\to\mathbb{C}\cup\{\infty\}$ is holomorphic as a map from from Ω to the Riemann sphere $\mathbb{C}\cup\{\infty\}$.

(c) For the special case where Ω is the open unit disk, verify from its definition that the Bergman kernel $K_{\mathbb{D}}(z,\bar w)$ is of the form

$$K_{\mathbb{D}}(z,\bar w)=\frac{a}{(b-z\bar w)^2}$$

for some complex constants a and b and explicitly determine a and b. Hence show that $\Phi_{\mathbb{D},w_0}(z)$ (for any $w_0\in\mathbb{D}$) is a linear fractional transformation (also known as a Möbius transformation).

(d) Let $F_\Omega:\Omega\to\mathbb{D}$ be a biholomorphic map from Ω to $\mathbb{D}$ whose existence is guaranteed by the Riemann mapping theorem. From the definition of Bergman kernel, verify that

$$K_\Omega(z,\bar w)\left(\frac{dz}{d\zeta}\right)\overline{\left(\frac{dw}{d\tau}\right)}=K_{\mathbb{D}}(\zeta,\bar\tau)$$

where $\zeta=F_\Omega(z)$ and $\tau=F_\Omega(w)$. Hence, show that (for any $w_0\in\Omega$) the Riemann mapping F_Ω must be the composite of the map Φ_{Ω,w_0} (defined by the Bergman kernel of Ω) and a linear fractional transformation (which acts on the target space of Φ_{Ω,w_0}).

Problem 7 (*Three-Lines Lemma Derived from Maximum Modulus Principle – from Problem #3 on p.133 of Stein's Complex Analysis Book*). In this problem, we investigate the behavior of certain bounded holomorphic functions in an infinite strip. The particular result described here is sometimes called the "three-lines lemma".

(a) Suppose $F(z)$ is holomorphic and bounded in the strip $0 < \mathrm{Im}(z) < 1$ and continuous on its closure. If $|F(z)| \leq 1$ on the boundary lines, then $|F(z)| \leq 1$ through out the strip.

Hint: Apply the maximum modulus principle to $F_\varepsilon(z) = F(z)e^{-\varepsilon z^2}$ for $\varepsilon > 0$ and then let $\varepsilon \to 0$.

(b) For the more general F, let $\sup_{x\in\mathbb{R}} |F(x)| = M_0$ and $\sup_{x\in\mathbb{R}} |F(x+i)| = M_1$. Then

$$\sup_{x\in\mathbb{R}} |F(x+iy)| \leq M_0^{1-y} M_1^y \quad \text{if} \quad 0 \leq y \leq 1.$$

Hint: Apply **(a)** to $M_0^{-iz-1} M_1^{iz} F(z)$.

(c) As a consequence, prove that $\log \sup_{x\in\mathbb{R}} |F(z+iy)|$ is a convex function of y when $0 \leq y \leq 1$.

Problem 8 (*Gauss's Identity on Elliptic Integrals and Arithmetic and Geometric Means – from Exercise #24(b) on p.254 and Problem #9 on p.259 of Stein's Complex Analysis Book*).

(a) (*Complementary Elliptic Modulus*) For $0 < k < 1$, consider the elliptic integral

$$K(k) = \int_{x=0}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

and the complementary modulus $\tilde{k}$ of the modulus k given by

$$\tilde{k}^2 = 1 - k^2,$$

with $0 < \tilde{k} < 1$. Verify

$$K(k) = \frac{2}{1+\tilde{k}} K\left(\frac{1-\tilde{k}}{1+\tilde{k}}\right)$$

by using the change of variables

$$x = \frac{2t}{1 + \tilde{k} + (1-\tilde{k})t^2}.$$

Note that this is Problem 5(b) of Homework #9 and redoing it here is required. You cannot just say that it was already done as a homework problem.

(b) (*Common Limit of Iterated Arithmetic and Geometric Means*) Let $a \geq b > 0$. Define the iterated arithmetic and geometric means a_n, b_n by $a_0 = a$ and $b_0 = b$ and

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n\, b_n}$$

inductively. Prove that the two sequences $\{a_n\}$ and $\{b_n\}$ have a common limit $M(a,b)$ by showing that

$$a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq b_n \geq \cdots \geq b_2 \geq b_1 \geq b_0$$

and

$$a_n - b_n \leq \frac{a-b}{2^n}.$$

(c) (*Gauss's Identity*) Let

$$I(a,b) = \frac{2}{\pi} \int_{\theta=0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}}.$$

Prove the identity of Gauss

$$\frac{1}{M(a,b)} = I(a,b)$$

by observing that

$$I(a,b) = \frac{2}{\pi a} K(k) = \frac{2}{\pi a} \int_{x=0}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

with $k^2 = 1 - \frac{b^2}{a^2}$ and using Part **(a)** to show that

$$I(a,b) = I\left(\frac{a+b}{2}, \sqrt{a\,b}\right).$$

Problem 9 (*Determination of Number of Zeros of Theta Function by Argument Principle*). Let $\omega_1,\, \omega_2 \in \mathbb{C}$ be $\mathbb{R}$-linearly independent. Let $\eta_1, \eta_2, \xi_1, \xi_2$ be complex numbers. Suppose $f(w)$ is an entire function on $\mathbb{C}$ such that

$$f(w + \omega_\nu) = e^{\eta_\nu w + \xi_\nu} f(w) \quad \text{for} \quad \nu = 1, 2 \quad \text{and} \quad w \in \mathbb{C}.$$

Let $c \in \mathbb{C}$ such that f is nowhere zero on the four sides of the parallelogram Ω with vertices $c, c+\omega_1, c+\omega_2, c+\omega_1+\omega_2$.

(a) Use the argument principle (applied to the logarithmic derivative of $f(w)$) to show that the case $\eta_1 = 0$ and $\eta_2 = i\,\overline{\omega_1}$ can never occur.

(b) Use the argument principle to compute the number of zeroes of f (with multiplicities counted) in the interior of the parallelogram Ω if $\omega_1 = \pi$ and $\omega_2 = \pi\tau$ and $\eta_1 = 0$ and $\eta_2 = -8i$.

(c) Let τ be an element in the open upper half-plane $\mathbb{H}$. Denote by $\Theta(w|\tau)$ the function (with w as the complex variable for fixed τ) defined by

$$\Theta(w|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n w}.$$

Verify that the entire function $\Theta(w|\tau)$ is a theta function for the period lattice $\mathbb{Z} + \mathbb{Z}\tau$ in the sense that the translation by a period results in the function being multiplied by the exponential of a polynomial of degree ≤ 1 (with polynomial depending on the period). Use the argument principle to determine the number of zeros of $\Theta(w|\tau)$ in a fundamental parallelogram of the period lattice $\mathbb{Z} + \mathbb{Z}\tau$.

Problem 10 (*Use of Möbius Transformation to Solve Dirichlet Problem with Locally Constant Boundary Value on Domain Between Two Disjoint Nonconcentric Circles*). Recall that the cross-ratio of four distinct points z_1, z_2, z_3, z_4 in $\mathbb{C}$ is defined as

$$\frac{\frac{z_1 - z_3}{z_1 - z_4}}{\frac{z_2 - z_3}{z_2 - z_4}}.$$

(a) Let C_1 be the circle $|z - 2| = 2$ and C_2 be the circle $|z| = 5$ in the z-plane. Compute the cross-ratio of the four points (in some order of your choice) where the circles C_1 and C_2 intersect the line joining the centers of C_1 and C_2.

(b) For any $R > 1$ let A_R be the open annulus $1 < |w| < R$ in the w-plane. Find the cross-ratio of the four points (in some order of choice) where the boundary of A_R intersects the real axis in the w-plane and express it in terms of R.

(c) Let Ω be the domain between the two non-concentric circles C_1 and C_2. Use the two cross-ratios obtained in Part **(a)** and Part **(b)** to determine $R > 1$ such that there exists a Möbius transformation

$$w = T(z) = \frac{az + b}{cz + d}$$

which maps Ω onto the annulus A_R.

(d) Find the coefficients a, b, c, d of the Möbius transformation $w = T(z)$ in Part**(c)**.

(e) Use the Möbius transformation to find a harmonic function u on the domain Ω which is continuous up to the boundary of Ω such that the boundary value of u on C_1 is identically 1 and the value of u on C_2 is identically 2.

Problem 11 (*Derivation of Addition Theorem for Weierstrass $\wp$ Function from its Representation as Quotient of Products of Translates of Weierstrass σ Function*). Let $\omega_1,\ \omega_2 \in \mathbb{C}$ be $\mathbb{R}$-linearly independent and let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $L^* = L - \{0\}$. Recall that the Weierstrass σ function is defined by

$$\sigma(w) = w \prod_{\ell \in L^*} \left(1 - \frac{w}{\ell}\right) e^{\frac{w}{\ell} + \frac{1}{2}\left(\frac{w}{\ell}\right)^2}$$

so that

$$\wp(w) = -\frac{d\zeta(w)}{dw} = \frac{1}{w^2} + \sum_{\ell \in L^*} \left(\frac{1}{(w-\ell)^2} - \frac{1}{\ell^2}\right)$$

with

$$\zeta(w) = \frac{\sigma'(w)}{\sigma(w)} = \frac{1}{w} + \sum_{\ell \in L^*} \left(\frac{1}{w-\ell} + \frac{1}{\ell} + \frac{w}{\ell^2}\right).$$

(a) Prove the formula

$$\wp(w) - \wp(\tau) = -\frac{\sigma(w-\tau)\sigma(w+\tau)}{\sigma^2(w)\sigma^2(\tau)}$$

by comparing the zeroes and the poles in w of both sides for fixed τ and considering the principal parts of both sides at $w = 0$. Note that this is Problem 3(a) of Homework #11 and redoing it here is required. You cannot just say that it was already done as a homework problem.

(b) Prove the formula

$$\frac{1}{2}\frac{\wp'(w) - \wp'(\tau)}{\wp(w) - \wp(\tau)} = \zeta(w+\tau) - \zeta(w) - \zeta(\tau)$$

by taking the logarithmic derivatives of both sides of the equation in Part**(a)**.

(c) Prove the addition theorem

$$\wp(w+\tau) = -\wp(w) - \wp(\tau) + \frac{1}{4}\left(\frac{\wp'(w)-\wp'(\tau)}{\wp(w)-\wp(\tau)}\right)^2.$$

by differentiating both sides of the equation in Part**(b)** and using a representation of $\wp''(w)$ as a polynomial of degree 2 in $\wp(w)$.

Problem 12 (*Lift Force on Circular Arc as Airfoil in Fluid Flow*). Before posing the problem, we first recall the following background material from the lectures. Consider the case of a 2-dimensional fluid flow (which is steady, irrotational and incompressible with constant density 1) outside a bounded connected closed subset E with velocity $e^{i\alpha}$ at infinity (represented by a complex number) such that the boundary of E is a streamline. The complex velocity function f is a holomorphic function on $\mathbb{C}-E$ with the property that $\overline{f'}$ is the fluid velocity (represented as a complex number). The streamlines are the level curves of the imaginary part of f. When E is considered as an airfoil (which means the constant cross section of an airplane wing of infinite length), the *Kutta condition* is assumed that at the sharp rear end of the airfoil the fluid velocity is zero. Let

$$f'(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots$$

be the power series expansion of f' at infinity and let F be the force acting on the airfoil by the fluid flow (represented as a complex number). According to Bernoulli's principle and the analysis done by Kutta and Joukowski,

$$\bar{F} = -2\pi c_0 c_1$$

and the vertical lift force is the vertical component of F.

In the case where E is the closed interval $[-2,2]$ as an airfoil with 2 as the sharp rear end and where the velocity of the fluid flow at infinity is $e^{i\alpha}$ (for some $0 < \alpha < \frac{\pi}{4}$), under the transformation $z \mapsto w = z + \frac{1}{z}$ which biholomorphically maps $\mathbb{C} - \bar{\mathbb{D}}$ to $\mathbb{C} - [-2,2]$, the setting is transferred to $E = \bar{\mathbb{D}}$ with the sharp rear end at the point 1 so that

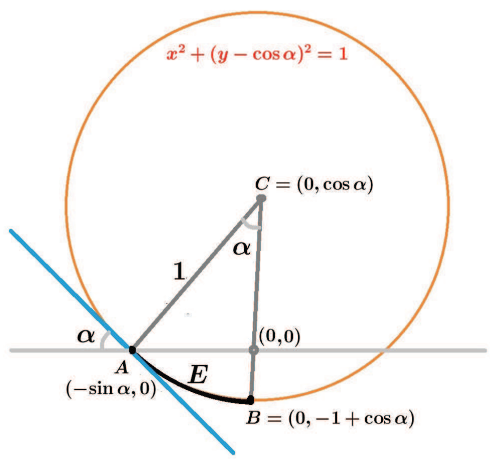
$$f'(z) = e^{-i\alpha} - \frac{1}{e^{-i\alpha}z^2} + \frac{ai}{z}$$

with $a = 2\sin\alpha$. The force F is given by

$$\bar{F} = -4\pi i e^{-i\alpha}\sin\alpha,$$

which leads to the conclusion that the vertical lift force on $[-2e^{-i\alpha}, 2e^{-i\alpha}]$ as an airfoil (obtained by rotating $[-2,2]$ clockwise by the angle α) is equal to $4\pi\sin\alpha$. We now pose the problem.

(a) Let $0 < \alpha < \frac{\pi}{4}$. Let E be the circular arc between the point $A = (-\sin\alpha, 0)$ and the point $B = (0, -1+\cos\alpha)$ on the circle $x^2+(y-\cos\alpha)^2 = 1$ with radius 1 and center $C = (0, \cos\alpha)$, as shown in the figure below. Find a biholomorphic map $z \mapsto w = g(z)$ from $\mathbb{C} - E$ to $\mathbb{C} - [z_1, z_2]$ such that asymptotically $g(z)$ is cz as $z \to \infty$ (for some nonzero constant c), where z_1, z_2 are two distinct points of $\mathbb{C}$ and $[z_1, z_2]$ is the line segment joining z_1 to z_2.



Hint: Consider the linear fractional transformation

$$z \mapsto w = \Phi(z) = \frac{z+\sin\alpha}{z-\sin\alpha}.$$

(b) Consider the circular arc E as an airfoil outside which there is a steady, irrotational and incompressible 2-dimensional fluid flow with constant density 1. Suppose the velocity of the fluid flow is 1 from left to right at infinity. Assume that the point $B = (0, -1+\cos\alpha)$ is the sharp rear end of the airfoil E. Use the background material recalled at the beginning of this problem and the biholomorphic map from Part **(a)** to find the vertical lift force on the airfoil E.