Solution of Take-Home Midterm of Math 113
(Taken in any 4-Hour Period between 12:01 a.m. March 18, 2024 and 11:59 p.m. March 20, 2024)
(Total Number of Problems = 6)

Problem 1 *(Evaluation of Definite Integral by Residue Theory).* Evaluate

$$\int_{x=0}^{\infty} \frac{\sin^2 x}{x^2}\,dx$$

and express the answer explicitly as a rational function of π with coefficients in $\mathbb{Q}$.

Hint: Consider the integral of

$$f(z) = \frac{1 - e^{2iz}}{2z^2}$$

over an appropriate sequence of contours and use the double angle formula

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

in trigonometry.

Solution. The only pole of

$$f(z) = \frac{1 - e^{2iz}}{2z^2} = \frac{-1}{2z^2}\left(2iz + \frac{(2iz)^2}{2!} + \frac{(2iz)^3}{3!} + \frac{(2iz)^4}{4!} + \cdots\right)$$

is a simple at $z = 0$ whose residue is computed as follows.

$$\mathrm{Res}_{z=0} f(z) = \mathrm{Res}_{z=0} \frac{-1}{2z^2}\left(2iz + \frac{(2iz)^2}{2!} + \frac{(2iz)^3}{3!} + \frac{(2iz)^4}{4!} + \cdots\right) = -i.$$

Let C_ρ be the upper half circle of radius ρ centered at 0 in the anticlockwise sense. The theorem of Cauchy-Goursat yields

$$\int_{x=-R}^{-\varepsilon} f(z)dz - \int_{C_\varepsilon} f(z)dz + \int_{x=\varepsilon}^{R} f(z)dz + \int_{C_R} f(z)dz = 0$$

for $0 < \varepsilon < R$. Over C_R the integral of $f(z)\,dz$ vanishes as $R \to \infty$, because the supremum of $|f(z)|$ on C_R is no greater than 2 times R^{-2} and the length of C_R is πR. Since 0 is a simple pole of $f(z)$, the limit of

$$\frac{1}{2\pi i}\int_{C_\varepsilon} f(z)dz$$

as $\varepsilon \to 0$ is equal to the "half-residue" of $f(z)$ at 0, which is $\frac{-i}{2}$. Thus,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{x=-\infty}^{-\varepsilon} f(x)dx + \lim_{\varepsilon\to 0}\int_{x=\varepsilon}^{\infty} f(x)dx \\ &= \lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)dz = \frac{1}{2}\,2\pi i\,\mathrm{Res}_{z=0} f(z) = \pi.\end{aligned}$$

Taking the real part of both sides, we obtain

$$\int_{x=-\infty}^{\infty}\frac{1-\cos(2x)}{2x^2} = \pi$$

and

$$\int_{x=-\infty}^{\infty}\frac{\sin^2 x}{x^2} = \pi$$

so that

$$\int_{x=0}^{\infty}\frac{\sin^2}{x^2} = \frac{\pi}{2}.$$

Half residue needs to be used. If the full residue or the wrong sign is used for the computation of the residue, **twenty** points will be deducted.

Problem 2 *(Evaluation of Definite Integral by Using Product of Branches of Fractional Powers).* Let $0 < \alpha < 1$. Evaluate the integral

$$\int_{x=0}^{1}\frac{dx}{(x+1)\,x^\alpha\,(1-x)^{1-\alpha}}$$

by using the theory of residues. Express your answer as a rational function of π, $2^{1-\alpha}$, and $\sin\pi\alpha$ with coefficients in $\mathbb{Q}$.

Hint: Use the holomorphic function

$$f(z)=\frac{1}{(z+1)\,z^{\alpha}\,(1-z)^{1-\alpha}}$$

on $\mathbb{C}-[0,1]$, where z^{α} and $(1-z)^{1-\alpha}$ are defined respectively as follows.

(i) For $z\in\mathbb{C}-[0,\infty)$ the function z^{α} is defined as $r^{\alpha}e^{i\alpha\theta}$ with $z=re^{i\theta}$ and $0<\theta<2\pi$.

(ii) For $z\in\mathbb{C}-[1,\infty)$ (which means that $1-z$ is a point of $\mathbb{C}-(-\infty,0]$) the function $(1-z)^{1-\alpha}$ is defined as $\rho^{1-\alpha}e^{i(1-\alpha)\varphi}$ with $1-z=\rho e^{i\varphi}$ and $-\pi<\varphi<\pi$.

Let C_R be the circle of radius R centered at the origin and let Γ_r be composed of the following four pieces: the right-half of the circle $|z-1|=r$, the line-segment joining $1+ri$ to ri, the left-half of the circle $|z|=r$, and the line-segment joining $-ri$ to $1-ri$. Use as contour of integration the boundary of the domain enclosed by C_R and Γ_r for $R>0$ sufficiently large and for $r>0$ sufficiently small.

Solution. Over C_R the integral of $f(z)\,dz$ vanishes as $R\to\infty$, because the supremum of $|f(z)|$ on C_R is no greater than a positive constant times R^{-2} and the length of C_R is πR. Over the right-half of the circle $|z-1|=r$ the integral of $f(z)\,dz$ vanishes as $r\to 0$, because the supremum of $|f(z)|$ on that half-circle is no greater than a positive constant times $r^{-(1-\alpha)}$ and the length of that half-circle is πr. Likewise, over the left-half of the circle $|z|=r$ the integral of $f(z)\,dz$ vanishes as $r\to 0$, because the supremum of $|f(z)|$ on that half-circle is no greater than a positive constant times $r^{-\alpha}$ and the length of that half-circle is πr.

For z approaching $(0,1)$ from the upper half-plane, the value of θ approaches 0 and, since $1-z$ approaches $(0,1)$, the value of φ approaches also 0. When we integrate $f(z)\,dz$ along $[ir,1+ir]$ from left to right and take limit as $r\to 0$, we get

$$\int_{x=0}^{1}\frac{dx}{(x+1)\,x^{\alpha}\,(1-x)^{1-\alpha}}.$$

For z approaching $(0,1)$ from the lower half-plane, the value of θ approaches 2π and, since $1-z$ approaches $(0,1)$, the value of φ approaches 0. When we integrate $f(z)\,dz$ along $[-ir, 1-ir]$ from left to right and take limit as $r \to 0$, we get

$$\int_{x=0}^{1} \frac{dx}{(x+1)\,x^{\alpha}e^{2\pi i\alpha}\,(1-x)^{1-\alpha}}.$$

When $z=-1$ the value of θ is π and the value of $r=|z|$ is 1, and from $1-z=2$ it follows that the value of φ is 0 and the value of $\rho=|1-z|$ is 2. Hence the value of $(z-(-1))\,f(z)$ as $z\to -1$ is equal to

$$\frac{1}{r^{\alpha}e^{i\alpha\theta}\rho^{1-\alpha}e^{i(1-\alpha)\varphi}} = \frac{1}{e^{\pi i\alpha}2^{1-\alpha}}.$$

Thus we have

$$\mathrm{Res}_{z=-1} f(z) = \frac{1}{e^{\pi i\alpha}2^{1-\alpha}}.$$

From the theory of residues we have

$$\left(1-e^{-2\pi i\alpha}\right)\int_{x=0}^{1} \frac{dx}{(x+1)\,x^{\alpha}\,(1-x)^{1-\alpha}} = 2\pi i\,\frac{1}{e^{\pi i\alpha}2^{1-\alpha}}$$

and

$$\begin{aligned}\int_{x=0}^{1} \frac{dx}{(x+1)\,x^{\alpha}\,(1-x)^{1-\alpha}} &= \frac{2\pi i}{e^{\pi i\alpha}2^{1-\alpha}\left(1-e^{-2\pi i\alpha}\right)} \\ &= \frac{2\pi i}{2^{1-\alpha}\left(e^{\pi i\alpha}-e^{-\pi i\alpha}\right)} = \frac{\pi}{2^{1-\alpha}\sin\pi\alpha}.\end{aligned}$$

Problem 3 *(Computation of Infinite Sum and Derivation of Product Expansion with Use of Logarithmic Derivative).* **(a)** Use the theorem below (labelled as Mittag-Leffler's theorem) to prove that

$$\tan z = 2z\left(\frac{1}{\left(\frac{\pi}{2}\right)^2-z^2}+\frac{1}{\left(\frac{3\pi}{2}\right)^2-z^2}+\frac{1}{\left(\frac{5\pi}{2}\right)^2-z^2}+\cdots\right)$$

(b) Use the result of Part **(a)** and the logarithmic derivative of $\cos z$ to derive an infinite product expansion of $\cos z$.

Theorem of Mittag-Leffler. Let $f(z)$ be a meromorphic function on $\mathbb{C}$ whose poles $\{a_n\}_{1\le n<\infty}$ are simple with $0<|a_1|\le|a_2|\le\cdots$ so that the residue of $f(z)$ at a_n is b_n. Suppose that there is a sequence of closed contours C_n such that the enclosure of C_n includes $a_1,\cdots,a_n$ but no other poles. Assume that the distance R_n from C_n to the origin goes to infinity as $n \to \infty$ and the length C_n is of the order $\mathrm{O}(R_n)$, where $\mathrm{O}(\cdot)$ is the symbol to mean at most of the same order. Assume that for some nonnegative integer p one has $f(z) = \mathrm{o}(R_n^{p+1})$ on C_n as $n \to \infty$, where $\mathrm{o}(\cdot)$ is the symbol to mean higher order. Then

$$f(z) = \sum_{\nu=0}^{p} \frac{z^\nu}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} + \frac{z}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}} \right).$$

Solution of Problem 3(a). Consider $f(z) = \tan z$. The poles of $f(z)$ are the zeroes of $\cos z$, which are $\frac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$. The residue of $f(z)$ at $\frac{(2n+1)\pi}{2}$ is equal to

$$\left(z - \frac{(2n+1)\pi}{2} \right) \left(\frac{\sin z}{\cos z} \right) \bigg|_{z=\frac{(2n+1)\pi}{2}} = \frac{\sin z}{\frac{d}{dz} \cos z} \Bigg|_{z=\frac{(2n+1)\pi}{2}} = -1.$$

We now pair n with $-(n+1)$, because $-(2n+1) = 2(-n-1)+1$, so that $\cup_{n=0}^{\infty} \{n, -(n+1)\} = \mathbb{Z}$.

To apply the theorem of Mittag-Leffler, for the definition of the sequence of contours we label the simple poles of $\tan z$ as

$$\frac{\pi}{2}, \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{-3\pi}{2}, \cdots, \frac{(2n-1)\pi}{2}, \frac{-(2n-1)\pi}{2}, \cdots$$

and label the contours as

$$C_1, C_1', C_2, C_2', \cdots, C_n, C_n', \cdots,$$

where the four vertices of C_n are $n\pi, \pm n\pi i, -(n-1)\pi$ and the four vertices of C_n' are $n\pi, \pm n\pi i, -n\pi$. From the definition

$$\tan z = -i \, \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

we have

$$|\tan z| \leq \frac{e^{|y|} + e^{-|y|}}{e^{|y|} - e^{-|y|}} = 1 + \frac{2e^{-|y|}}{e^{|y|} - e^{-|y|}} \leq 1 + \frac{2}{e^\pi - 1}$$

for $|y| \geq \pi$. This estimate, together with $\tan(z + \pi) = \tan z$ from the period π of $\tan z$, implies that $\tan z$ is bounded uniformly on C_n and C_n' independent of $n \in \mathbb{N}$. We can use $p = 0$ in the application of the theorem of Mittag-Leffler.

Let $a_n = \frac{(2n+1)\pi}{2}$, and $b_n = -1$, when a_n is paired with $a_{-(n+1)}$, we have

$$\begin{aligned}
&b_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right)+b_{-n-1}\left(\frac{1}{z-a_{-n-1}}+\frac{1}{a_{-n-1}}\right)\\
&=-\left(\frac{1}{z-\frac{(2n+1)\pi}{2}}+\frac{1}{\frac{(2n+1)\pi}{2}}\right)-\left(\frac{1}{z+\frac{(2n+1)\pi}{2}}-\frac{1}{\frac{(2n+1)\pi}{2}}\right)\\
&=\frac{2z}{\left(\frac{(2n+1)\pi}{2}\right)^2-z^2}.
\end{aligned}$$

Thus,

$$\tan z = 2z\sum_{n=0}^{\infty}\frac{1}{\left(\frac{(2n+1)\pi}{2}\right)^2-z^2}.$$

Though the sequence of closed contours should be constructed and the uniform bound of $\tan z$ should be checked, **no** point is deducted for not going through the complete process of careful checking.

Solution of Problem 3(b). Since

$$\frac{\frac{d}{dz}\cos z}{\cos z} = -\tan z = -2z\sum_{n=0}^{\infty}\frac{1}{\left(\frac{(2n+1)\pi}{2}\right)^2-z^2}$$

and

$$\frac{d}{dz}\log\left(1-\frac{z^2}{\left(\frac{(2n+1)\pi}{2}\right)^2}\right)=\frac{\frac{-2z}{\left(\frac{(2n+1)\pi}{2}\right)^2}}{1-\frac{z^2}{\left(\frac{(2n+1)\pi}{2}\right)^2}}=-2z\left(\frac{1}{\left(\frac{(2n+1)\pi}{2}\right)^2-z^2}\right),$$

it follows that

$$\frac{d}{dz}\sum_{n=0}^{\infty}\log\left(1-\frac{z^2}{\left(\frac{(2n+1)\pi}{2}\right)^2}\right)=-2z\sum_{n=0}^{\infty}\frac{1}{\left(\frac{(2n+1)\pi}{2}\right)^2-z^2}=\frac{\frac{d}{dz}\cos z}{\cos z}.$$

We can now integrate from $z=0$ and then exponentiate to get

$$\cos z = \prod_{n=0}^{\infty}\left(1-\frac{z^2}{\left(\frac{(2n+1)\pi}{2}\right)^2}\right).$$

It is important that any infinite product written down has to **converge**. **Ten** points will be deducted if $\cos z$ is given as a constant C times the **nonconvergent** infinite product

$$\prod_{n=0}^{\infty}\left(\left(\frac{(2n+1)\pi}{2}\right)^2 - z^2\right)$$

and then the constant C is determined to be the reciprocal of the **nonconvergent** infinite product

$$\prod_{n=0}^{\infty}\left(\frac{(2n+1)\pi}{2}\right)^2 .$$

The appropriate constant of integration should be included in each term (for infinite sum) or each factor (for infinite product) to make the argument mathematically meaningful and rigorous.

Problem 4 *(Location of Zeroes by Argument Principle and Rouché's Theorem).* **(a)** By using the argument principle, show that if α, β are positive numbers, the equation

$$z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$$

has $n-1$ roots with positive real parts if n is odd, and n roots with positive real parts if n is even (when the roots are counted with multiplicities).

Hint: Apply the argument principle after computing the net change of argument of

$$z^{2n} + \alpha^2 z^{2n-1} + \beta^2$$

while z goes around the boundary of the right half circle of radius $R > 0$ centered at 0 in the counterclockwise sense as $R \to \infty$.

(b) Find the number of roots of the equation (counting multiplicities)

$$z^8 - 4z^5 + z^2 - 1 = 0$$

in the disk $|z| < 1$.

Hint: Apply Rouché's theorem by writing the polynomial as the perturbation of a polynomial of 2 terms by another polynomial of 2 terms.

Solution of Problem 4(a). Along the right half circle of radius R as $R \to \infty$, the change of argument of

$$\frac{1}{R^{2n}}\left(z^{2n}+\alpha^2 z^{2n-1}+\beta^2\right)=e^{i2n\theta}+\frac{\alpha^2}{R}e^{i(2n-1)\theta}+\frac{\beta^2}{R^{2n}}$$

can be computed by using just the first term $e^{i2n\theta}$ which yields $2n\pi$. Going down along the imaginary axis with $z = iy$ from iR to $-iR$ the net change of the argument of

$$z^{2n}+\alpha^2 z^{2n-1}+\beta^2=(-1)^n y^{2n}+i(-1)^{n-1}\alpha^2 y^{2n-1}+\beta^2$$

is given by 2π times the number of times the point P with abscissa $(-1)^n y^{2n}+\beta^2$ and ordinate $(-1)^{n-1}\alpha^2 y^{2n-1}$ goes around the origin in $\mathbb{R}^2$. We now distinguish between the case of odd n and the case of even n.

Case of n Odd. When n is odd, to count the number of times the point P with abscissa $-y^{2n}+\beta^2$ and ordinate $\alpha^2 y^{2n-1}$ goes around the origin in $\mathbb{R}^2$, we focus on when P crosses the two coordinate axes and the sense of rotation at each crossing and the initial and final position of P. The initial point of P is when $y = R$ with $R > 0$ very large, which, with abscissa $-y^{2n}+\beta^2$ very negative and the ordinate $\alpha^2 y^{2n-1}$ very positive, means that P is in the second quadrant and is very close to the real axis, because

$$\lim_{y\to\infty}\frac{\alpha^2 y^{2n-1}}{-y^{2n}+\beta^2}=0.$$

The crossing of either axis by P occurs when y is a real zero of either $-y^{2n}+\beta^2=0$ or $\alpha^2 y^{2n-1}=0$, which means $y=\beta^{\frac{1}{n}}$, $y=0$, and $y=-\beta^{\frac{1}{n}}$. At $y=\beta^{\frac{1}{n}}$ as the point P crosses the imaginary axis when y goes past $\beta^{\frac{1}{n}}$ from right to left, the ordinate $\alpha^2 y^{2n-1}$ of P stays positive while the abscissa $-y^{2n}+\beta^2$ of P goes from negative to positive, which means that P goes from the second quadrant to the first quadrant. At $y=0$ as the point P crosses the real axis when y goes past $y=0$ from the right, the abscissa $-y^{2n}+\beta^2$ of P stays positive and the ordinate $\alpha^2 y^{2n-1}$ of P goes from positive to negative, which means that P goes from the first quadrant to the fourth quadrant. At $y=-\beta^{\frac{1}{n}}$ as the point P crosses the imaginary axis when y goes past $-\beta^{\frac{1}{n}}$ from right to left, the ordinate $\alpha^2 y^{2n-1}$ of P stays negative while the abscissa $-y^{2n}+\beta^2$ of P goes from positive to negative, which means that P goes from the fourth quadrant to the third quadrant. The final point of P is when $y=-R$ with $R>0$ very large, which, with abscissa $-y^{2n}+\beta^2$ very negative and the ordinate $\alpha^2 y^{2n-1}$ very negative, means that P is in the third quadrant and is very close to the real axis, because

$$\lim_{y\to-\infty}\frac{\alpha^2 y^{2n-1}}{-y^{2n}+\beta^2}=0.$$

In summary, as y goes from $y=-R$ to $y=R$ for R very large, the point goes in the clockwise sense from a position very near the negative real axis, crossing first the positive imaginary axis at $y=\beta^{\frac{1}{n}}$, then crossing the positive real axis at $y=0$, crossing first the negative imaginary axis at $y=-\beta^{\frac{1}{n}}$, and finally back to a position very near the negative real axis. Since the net change of the argument of $z^{2n}+\alpha^2 z^{2n-1}+\beta^2$ along the boundary of the right half circle of radius $R>0$ centered at 0 approaches $2n\pi-2\pi=2(n-1)\pi$, there are $n-1$ roots of of $z^{2n}+\alpha^2 z^{2n-1}+\beta^2=0$ with positive real parts if n is odd.

Case of n Even. When n is even, as y goes from R to $-R$ for $R>0$ very large, the point P in $\mathbb{R}^2$ with abscissa $y^{2n}+\beta^2$ and ordinate $-\alpha^2 y^{2n-1}$ stays always on the right half plane, because $y^{2n}+\beta^2\geq\beta^2>0$ for all $y\in\mathbb{R}$. It means that the number of times P goes around the origin as y goes from $y=R$ to $y=-R$ is 0 (as $R\to\infty$). Since the net change of the argument of $z^{2n}+\alpha^2 z^{2n-1}+\beta^2$ along the boundary of the right half circle of radius $R>0$ centered at 0 approaches $2n\pi$, there are n roots of of $z^{2n}+\alpha^2 z^{2n-1}+\beta^2=0$ with positive real parts if n is even.

Solution of Problem 4(b). Since on $|z|=1$

$$|z^8-4z^5|=|z^3-4|\geq 4-|z^3|=3$$

and

$$|z^2-1|\leq|z^2|+1=2,$$

it follows that

$$|z^8-4z^5|>|z^2-1|$$

on $|z|=1$ so that the number of roots of $z^8-4z^5+z^2-1$ on $|z|<1$ with multiplicities counted is the same as the number of roots of $z^8-4z^5=z^5(z^3-4)$ on $|z|<1$. The nowhere vanishing of z^3-4 on $|z|<1$ implies that the number of roots of the equation (counting multiplicities)

$$z^8-4z^5+z^2-1=0$$

in the disk $|z|<1$ is 5.

Problem 5 *(Applications of Maximum Modulus Principle).* **(a)** Let k be a positive integer and let $f(z)$ be a holomorphic function on $|z|<R$ for some $R>0$ such that $f^{(\ell)}(z)$ vanishes at $z=0$ for $0\leq\ell<k$ (where $f^{(\ell)}(z)$ is the ℓ-th derivative of f at z). Let M be the supremum of $|f(z)|$ on $|z|<R$. Prove that

$$|f(z)|\leq\left(\frac{|z|}{R}\right)^k M$$

for $|z| < R$. Moreover, prove that if the equality

$$|f(z)| = \left(\frac{|z|}{R}\right)^k M$$

holds for some z with $0 < |z| < R$, or if $f^{(k)}(0) = \frac{k!M}{R^k}$, then $f(z) = \frac{e^{i\alpha}M}{R^k} z^k$ for some $\alpha \in \mathbb{R}$.

Hint: Consider the holomorphic function

$$h(z) = \begin{cases} \frac{f(z)}{z^k} & \text{for } 0 < |z| < R \\ \\ \frac{f^{(k)}(0)}{k!} & \text{for } z = 0. \end{cases}.$$

Apply the maximum modulus principle to $h(z)$ on $|z| < r$ for $0 < r < R$.

(b) Let k and $f(z)$ be as in Part**(a)**. Let A be the supremum of $\operatorname{Re} f(z)$ on $|z| < R$. Prove that

$$\sup_{|z| \leq r} |f(z)| \leq \frac{2Ar^k}{R^k - r^k}.$$

for $0 < r < R$.

Hint: Apply Part**(a)** to the function

$$g(z) = \frac{f(z)}{-f(z) + 2A}$$

on $|z| < R$.

Solution of Problem 5(a). Since $f^{(\ell)}(0) = 0$ for $0 \leq \ell < k$, from the Taylor expansion

$$f(z) = \sum_{\ell=0}^{k} \frac{f^{(\ell)}(0)}{\ell!} z^\ell + \mathrm{O}(|z|^{k+1}) = \frac{f^{(k)}(0)}{k!} z^k + \mathrm{O}(|z|^{k+1})$$

as $z \to 0$, it follows that because of

$$\lim_{z \to 0} \left(\frac{f(z)}{z^k} - \frac{f^{(k)}(0)}{k!} \right) = 0,$$

the isolated singularity of $h(z)$ at $z = 0$ can be removed to that $h(z)$ is holomorphic on $|z| < R$. For any $0 < r < R$, the maximum modulus principle applied to the holomorphic function $h(z)$ yields

$$\sup_{|z| \leq r} |h(z)| = \sup_{|z| = r} |h(z)| \leq \frac{M}{r^k}.$$

Since this holds for any $0 < r < R$, it follows by letting $r \to R$ for any fixed z with $|z| < R$ that

$$\sup_{|z|<R} |h(z)| \leq \frac{M}{R^k},$$

which implies that

$$|f(z)| = \left(\frac{|z|}{R}\right)^k M$$

for $|z| < R$. If the equality

$$|f(z)| = \left(\frac{|z|}{R}\right)^k M$$

holds for some z with $0 < |z| < R$, or if $f^{(k)}(0) = \frac{k!M}{R^k}$, then the maximum modulus of $h(z)$ on $|z| < R$ is achieved at some point z with $|z| < R$, which implies that $h(z)$ is equal to a constant whose absolute value is M. Thus $h(z) \equiv \frac{e^{i\alpha}M}{R^k}$ for some $\alpha \in \mathbb{R}$ and $f(z) \equiv \frac{e^{i\alpha}M}{R^k} z^k$.

Solution of Problem 5(b). For $|z| < R$, $-2A + u \leq u \leq 2A - u$ so that $(u - 2A)^2 \geq u^2$ and

$$|g(z)|^2 = \left|\frac{f(z)}{-f(z) + 2A}\right|^2 = \frac{u(z)^2 + v(z)^2}{(u(x) - 2A)^2 + v(z)^2} \leq 1.$$

By Part(a)

$$|g(z)| \leq \left(\frac{|z|}{R}\right)^k$$

for $|z| < R$ and from

$$f(z) = \frac{2Ag(z)}{1 + g(z)}$$

it follows that

$$|f(z)| \leq \left|\frac{2Ag(z)}{1 + g(z)}\right| \leq \frac{2A\left(\frac{|z|}{R}\right)^k}{1 - \left(\frac{|z|}{R}\right)^k} = \frac{2A|z|^k}{R^k - |z|^k}$$

for $|z| < R$ so that

$$\sup_{|z|\leq r} |f(z)| \leq \frac{2A|z|^k}{R^k - |z|^k} \leq \frac{\sup_{|z|\leq r}\left(A|z|^k\right)}{\inf_{|z|\leq r}\left(R^k - |z|^k\right)} = \frac{2Ar^k}{R^k - r^k}.$$

The main step of Part (b) is the estimate

$$|g(z)|^2 = \left|\frac{f(z)}{-f(z)+2A}\right|^2 = \frac{u(z)^2+v(z)^2}{(u(x)-2A)^2+v(z)^2} \leq 1.$$

If no such a kind of estimate is given, **twenty** points will be deducted.

Problem 6 *(Harmonic Functions as Real Parts of Holomorphic Functions).* Let Ω be a simply connected open subset of $\mathbb{C}$ with coordinate $z = x + iy$. Let $u(x, y)$ be a real-valued function on Ω such that u_x, u_y, u_{xx}, u_{yy} exist and are continuous on Ω. Assume that $u_{xx} + u_{yy} = 0$ on Ω. (Here the appearance of one of the two coordinates x, y in the subscript of the notation for a function means the partial derivative of the function with respect to the coordinate.) Prove that u is the real part of a holomorphic function on Ω.

Hint: Let $P_0 \in \Omega$. Construct, with justification from the given assumptions, a function v on Ω whose value at P is equal to the integration of $-u_y dx + u_x dy$ from P_0 to P along a curve composed of horizontal and vertical line segments in Ω. Rigorously prove that $u + iv$ is holomorphic on Ω.

Solution. Let $R = [a, b] \times [c, d]$ be a closed rectangle inside Ω, with $a < b$ and $c < d$. Since both u_x and u_{xx} are continuous, by the fundamental theorem of calculus

$$u_x(b, y) - u_x(a, y) = \int_{x=a}^{b} u_{xx}(x, y)dx$$

and

$$\int_{y=c}^{d} u_x(b, y)dy - \int_{y=c}^{d} u_x(a, y)dy = \int_{y=c}^{d} \left(\int_{x=a}^{b} u_{xx}(x, y)dx\right) dy = \int_R u_{xx}dxdy.$$

Similarly,

$$\int_{x=a}^{b} u_y(x, d)dx - \int_{x=a}^{b} u_y(x, c)dx = \int_R u_{yy}dxdy.$$

Adding up the two equations and using $u_{xx} + u_{yy} = 0$ on Ω, we get

$$\int_{\partial R} (u_x dy - u_y dx) = 0.$$

Let $P_0 \in \Omega$. Let v be the function v on Ω whose value at P is equal to the integration of $-u_y dx + u_x dy$ from P_0 to P along a curve composed of horizontal and vertical line segments in Ω. The function v is well-defined and its definition is independent of the choice of the curve from P_0 to P composed of horizontal and vertical line segments in Ω, because

$$\int_{\partial R}(u_x dy - u_y dx) = 0.$$

for any closed rectangle R contained in the simply connected open subset Ω of $\mathbb{C}$. Fix (x, y) in Ω and choose $h \in \mathbb{R}$ with $|h|$ small enough so that the line segment joining (x, y) to $(x + h, y)$ is contained in Ω. To get the value of $v(x + h, y)$ from the value of $v(x, y)$ we can add the integration of $-u_y dx + u_x dy$ from (x, y) to $(x + h, y)$ along a horizontal line segment so that

$$v(x + h, y) - v(x, y) = \int_{t=x}^{x+h} (-u_y(t, y))dt$$

(where $x = t$ parametrizes the horizontal line segment from (x, y) to $(x + h, y)$), which implies that

$$v_x(x, y) = \lim_{h\to 0} \frac{v(x + h, y) - v(x, y)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{t=x}^{x+h} (-u_y(t, y))dt = -u_y(x, y).$$

Similarly, from

$$v(x, y + h) - v(x, y) = \int_{t=y}^{y+h} u_x(x, t)dt$$

for $h \in \mathbb{R}$ with $|h|$ sufficiently small and from the fundamental theorem of calculus, we conclude that

$$v_y(x, y) = u_x(x, y).$$

Let $f(z) = u_x(x, y) + iv_x(x, y) = u_x(x, y) - iu_y(x, y)$ where $z = x + iy$. Then

$$\int_{\partial R} f(z)dz = \int_{\partial R}(u_x - iu_y)(dx + idy) = \int_{\partial R}(u_x dx + u_y dy) + i \int_{\partial R}(-u_y dx + u_x dy) = 0,$$

because $du = u_x dx + u_y dy$ so that

$$\int_{\partial R}(u_x dx + u_y dy) = \int_{\partial R} du = 0.$$

By Morera's theorem, $f(z)$ is holomorphic on Ω. Let $F(z)$ be a holomorphic function on Ω (obtained by integrating $f(z)dz$ along any path in the simply connected domain Ω originating from the same chosen point) with $F'(z) = f(z)$. Then

$$F_x = F' = f = u_x + iv_x = (u + iv)_x.$$

Moreover,

$$F_y = iF_x = i(u_x + iv_x) = -v_x + iu_x = u_y + iv_y = (u + iv)_y.$$

Hence $u + iv$ differs from F only by a constant and is holomorphic on Ω.

A key step is the verification of the path independence for the integral defining v. The verification is done by the use of the fundamental theorem of calculus, the formulation of an iterated integral as a double integral, and the given equation $u_{xx} + u_{yy} = 0$. The existence of u_{xy}, u_{yx} is not given. The use of $u_{x,y}$, u_{yx} needs to be justified from the assumptions given in the problem. When this key step is not properly done, **twenty-five** points will be deducted.