Take-Home Midterm of Math 113
Closed-Book Test to Be Taken in Any 4-Hour Period
Between 12:01 a.m. March 18, 2024
And 11:59 p.m. March 20, 2024

(Total Number of Problems = 6)

Please submit the PDF file of your solution to the CANVAS website for Math 113

Instructions. This is a closed book test. That is, the same as you are sitting in an examination hall. You are free to choose any 4-hour period to work on the test problems. It is an honor system. You simply put down your starting time and ending time in your submitted PDF file. The time you use to type up your file in TeX does not count toward the four hours you are allowed to work on the problems. Do not read the problems unless you are ready to start working on them in your chosen 4-hour period.

N.B. Solutions with rigorous details are expected. For problems which can be solved by techniques in some homework problems, complete self-contained solutions are required and homework problems cannot be quoted simply as known facts in the solutions.

Problem 1 *(Evaluation of Definite Integral by Residue Theory).* Evaluate

$$\int_{x=0}^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

and express the answer explicitly as a rational function of π with coefficients in $\mathbb{Q}$.

Hint: Consider the integral of

$$f(z) = \frac{1 - e^{2iz}}{2z^2}$$

over an appropriate sequence of contours and use the double angle formula

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

in trigonometry.

Problem 2 *(Evaluation of Definite Integral by Using Product of Branches of Fractional Powers).* Let $0 < \alpha < 1$. Evaluate the integral

$$\int_{x=0}^{1} \frac{dx}{(x+1)\, x^{\alpha}\, (1-x)^{1-\alpha}}$$

by using the theory of residues. Express your answer as a rational function of π, $2^{1-\alpha}$, and $\sin \pi\alpha$ with coefficients in $\mathbb{Q}$.

Hint: Use the holomorphic function

$$f(z) = \frac{1}{(z+1)\, z^{\alpha}\, (1-z)^{1-\alpha}}$$

on $\mathbb{C} - [0,1]$, where z^{α} and $(1-z)^{1-\alpha}$ are defined respectively as follows.

(i) For $z \in \mathbb{C} - [0,\infty)$ the function z^{α} is defined as $r^{\alpha} e^{i\alpha\theta}$ with $z = re^{i\theta}$ and $0 < \theta < 2\pi$.

(ii) For $z \in \mathbb{C} - [1,\infty)$ (which means that $1-z$ is a point of $\mathbb{C} - (-\infty, 0]$) the function $(1-z)^{1-\alpha}$ is defined as $\rho^{1-\alpha} e^{i(1-\alpha)\varphi}$ with $1 - z = \rho e^{i\varphi}$ and $-\pi < \varphi < \pi$.

Let C_R be the circle of radius R centered at the origin and let Γ_r be composed of the following four pieces: the right-half of the circle $|z-1| = r$, the line-segment joining $1 + ri$ to ri, the left-half of the circle $|z| = r$, and the line-segment joining $-ri$ to $1 - ri$. Use as contour of integration the boundary of the domain enclosed by C_R and Γ_r for $R > 0$ sufficiently large and for $r > 0$ sufficiently small.

Problem 3 *(Computation of Infinite Sum and Derivation of Product Expansion with Use of Logarithmic Derivative).* **(a)** Use the theorem below (labelled as Mittag-Leffler's theorem) to prove that

$$\tan z = 2z \left(\frac{1}{\left(\frac{\pi}{2}\right)^2 - z^2} + \frac{1}{\left(\frac{3\pi}{2}\right)^2 - z^2} + \frac{1}{\left(\frac{5\pi}{2}\right)^2 - z^2} + \cdots \right)$$

(b) Use the result of Part **(a)** and the logarithmic derivative of $\cos z$ to derive an infinite product expansion of $\cos z$.

Theorem of Mittag-Leffler. Let $f(z)$ be a meromorphic function on $\mathbb{C}$ whose poles $\{a_n\}_{1\leq n<\infty}$ are simple with $0<|a_1|\leq|a_2|\leq\cdots$ so that the residue of $f(z)$ at a_n is b_n. Suppose that there is a sequence of closed contours C_n such that the enclosure of C_n includes $a_1,\cdots,a_n$ but no other poles. Assume that the distance R_n from C_n to the origin goes to infinity as $n\to\infty$ and the length C_n is of the order $\mathrm{O}(R_n)$, where $\mathrm{O}(\cdot)$ is the symbol to mean at most of the same order. Assume that for some nonnegative integer p one has $f(z)=\mathrm{o}(R_n^{p+1})$ on C_n as $n\to\infty$, where $\mathrm{o}(\cdot)$ is the symbol to mean higher order. Then

$$f(z)=\sum_{\nu=0}^{p}\frac{z^\nu}{\nu!}f^{(\nu)}(0)+\sum_{n=1}^{\infty}b_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}+\frac{z}{a_n^2}+\cdots+\frac{z^p}{a_n^{p+1}}\right).$$

Problem 4 *(Location of Zeroes by Argument Principle and Rouché's Theorem).* **(a)** By using the argument principle, show that if α,β are positive numbers, the equation

$$z^{2n}+\alpha^2z^{2n-1}+\beta^2=0$$

has $n-1$ roots with positive real parts if n is odd, and n roots with positive real parts if n is even (when the roots are counted with multiplicities).

Hint: Apply the argument principle after computing the net change of argument of

$$z^{2n}+\alpha^2z^{2n-1}+\beta^2$$

while z goes around the boundary of the right half circle of radius $R>0$ centered at 0 in the counterclockwise sense as $R\to\infty$.

(b) Find the number of roots of the equation (counting multiplicities)

$$z^8-4z^5+z^2-1=0$$

in the disk $|z|<1$.

Hint: Apply Rouché's theorem by writing the polynomial as the perturbation of a polynomial of 2 terms by another polynomial of 2 terms.

Problem 5 *(Applications of Maximum Modulus Principle).* **(a)** Let k be a positive integer and let $f(z)$ be a holomorphic function on $|z|<R$ for some $R>0$ such that $f^{(\ell)}(z)$ vanishes at $z=0$ for $0\leq\ell<k$ (where $f^{(\ell)}(z)$ is the ℓ-th derivative of f at z). Let M be the supremum of $|f(z)|$ on $|z|<R$. Prove that

$$|f(z)|\leq\left(\frac{|z|}{R}\right)^kM$$

for $|z| < R$. Moreover, prove that if the equality

$$|f(z)| = \left(\frac{|z|}{R}\right)^k M$$

holds for some z with $0 < |z| < R$, or if $f^{(k)}(0) = \frac{k!M}{R^k}$, then $f(z) = \frac{e^{i\alpha}M}{R^k} z^k$ for some $\alpha \in \mathbb{R}$.

Hint: Consider the holomorphic function

$$h(z) = \begin{cases} \frac{f(z)}{z^k} & \text{for } 0 < |z| < R \\ \\ \frac{f^{(k)}(0)}{k!} & \text{for } z = 0. \end{cases}.$$

Apply the maximum modulus principle to $h(z)$ on $|z| < r$ for $0 < r < R$.

(b) Let k and $f(z)$ be as in Part **(a)**. Let A be the supremum of $\operatorname{Re} f(z)$ on $|z| < R$. Prove that

$$\sup_{|z| \leq r} |f(z)| \leq \frac{2Ar^k}{R^k - r^k}.$$

for $0 < r < R$.

Hint: Apply Part **(a)** to the function

$$g(z) = \frac{f(z)}{-f(z) + 2A}$$

on $|z| < R$.

Problem 6 *(Harmonic Functions as Real Parts of Holomorphic Functions).* Let Ω be a simply connected open subset of $\mathbb{C}$ with coordinate $z = x + iy$. Let $u(x, y)$ be a real-valued function on Ω such that u_x, u_y, u_{xx}, u_{yy} exist and are continuous on Ω. Assume that $u_{xx} + u_{yy} = 0$ on Ω. (Here the appearance of one of the two coordinates x, y in the subscript of the notation for a function means the partial derivative of the function with respect to the coordinate.) Prove that u is the real part of a holomorphic function on Ω.

Hint: Let $P_0 \in \Omega$. Construct, with justification from the given assumptions, a function v on Ω whose value at P is equal to the integration of $-u_y dx + u_x dy$ from P_0 to P along a curve composed of horizontal and vertical line segments in Ω. Rigorously prove that $u + iv$ is holomorphic on Ω.